ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
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PSEUDOSYMMETRIC LOCALLY CONFORMAL KAEHLER MANIFOLDS

Ph.D. THESIS

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Mathematical Engineering Department

Mathematical Engineering Programme

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PSEUDÖ SİMETRİK LOKAL OLARAK KONFORM KAEHLER MANİFOLDLARI

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To my husband and my lovely daughter,

FOREWORD

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TABLE OF CONTENTS

Page

LIST OF FIGURES

Page

PSEUDOSYMMETRIC LOCALLY CONFORMAL KAEHLER MANIFOLDS

SUMMARY

Many particular classes of almost Hermitian manifolds have been intensively studied. Among them, almost Hermitian manifolds whose metric is globally conformal to an almost Kaehler metric have been also encountered. But, obviously, these manifolds have the same topological properties like the almost Kaehler manifolds. Therefore, it is interesting to study almost Hermitian manifolds which are only locally conformal to an almost Kaehler manifold. The notion of a locally conformal Kaehler manifold (l.c.K-manifold) in a Hermitian manifold has been introduced by I. Vaisman in 1976.

After that T. Kashiwada has determined a necessary and sufficient condition that a Hermitian manifold is an l.c.K-manifold by using the tensor equation and introduced the curvature tensor of an l.c.K-manifold with a constant holomorphic sectional curvature (an l.c.K-space form). Furthermore, T. Kashiwada and K. Matsumoto gave some properties about such a manifold. Then we can see a lot of papers about these manifolds and its submanifolds.

Moreover, M. Prvanović found a tensor of Kaehler type for an almost Hermitian manifold and proved that this tensor reduces to the Riemannian curvature tensor R in an almost Kaehler manifold. In addition, the author determined the holomorphic curvature tensor of an l.c.K-manifold and introduced Riemannian curvature tensor in an l.c.K-manifold with a constant holomorphic sectional curvature such that the tensor P is not hybrid.

A Hermitian manifold M with structure $M(J,g)$ is called an l.c.K-manifold if each point $p \in M$ has an open neighborhood U with a differentiable function $\rho : U \to R$ such that $g^* = e^{-2\rho} g \mid_{\mathrm{U}}$ is a Kaehlerian metric on U.

An 2n-dimensional l.c.K-manifold is a Hermitian manifold admitting a global closed 1-form α (Lee form) whose structure (J,g) satisfies

$$\nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki},$$

where ∇ denotes the covariant differentiation with respect to the Hermitian metric g.

An l.c.K-manifold $M(J,g,\alpha)$ is called an l.c.K-space form if it has a constant holomorphic sectional curvature.

A semi-Riemannian manifold (M,g) satisfying the condition $\nabla R = 0$ is said to be locally symmetric. These manifolds are first studied and classified by E. Cartan in the late twenties.

A semi-Riemannian manifold (M,g) satisfying the condition $R \cdot R = 0$ is said to be semisymmetric. E. Cartan studied semi-symmetric manifolds which is a natural generalization of symmetric spaces. R. Deszcz introduced the pseudo-symmetric manifolds in semi-Riemannian manifolds.

A semi-Riemannian manifold (M,g) is said to be pseudosymmetric in the sense of Deszcz if at every point of M the condition

$$R \cdot R = L_R Q(g,R)$$

holds on the set $\mathscr{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)}G \neq 0 \; at \; x\}$, where L_R is some function on $\mathscr{U}_R$.

A semi-Riemannian manifold (M,g), $n \geq 3$, satisfying the condition

$$R \cdot S = L_S Q(g,S)$$

on the set $\mathscr{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \; at \; x\}$ are called Ricci-pseudosymmetric. Every pseudosymmetric manifold is Ricci-pseudosymmetric. The converse statement is not true. The class of Ricci-pseudosymmetric manifolds is an extension of the class of Ricci-semisymmetric $(R \cdot S = 0)$ manifolds as well as of the class of pseudosymmetric manifolds. Evidently, every Ricci-semisymmetric manifolds are Ricci-pseudosymmetric. There exist various examples of Ricci-pseudosymmetric manifolds which are not pseudosymmetric.

In this thesis, some properties of l.c.K-manifolds, l.c.K-space forms and submanifolds of an l.c.K-space form are presented. Furthermore, we state some results on pseudosymmetric and Ricci-pseudosymmetric l.c.K-space forms. Walker type identities on l.c.K-space forms and Roter type l.c.K-space forms are studied. Finally the Bochner curvature tensor on l.c.K-space forms are studied.

In the first chapter, it is mentioned about a review of literature. After that, results obtained in this thesis are summarized.

In the second chapter, we give the fundamental concepts which we will use the next chapters.

In the third chapter, we give a generalization about the results of an l.c.K-space form such that the tensor field P is not hybrid. Moreover, the Sato's form of the holomorphic curvature tensor in almost Hermitian manifolds and l.c.K-manifolds are presented. This chapter contains four sections.

In the first section, we gave the well-known results on l.c.K-manifolds. It is given that the tensor field P is hybrid if and only if the Ricci tensor is hybrid. Then we state that there exists the algebraic curvature tensor satisfying the condition of Kaehler type manifold for any almost Hermitian manifold. This tensor is said to be the holomorphic curvature tensor.

Furthermore, we state the holomorphic curvature tensor in an l.c.K-manifold and the Riemannian curvature tensor in an l.c.K-manifold with a constant holomorphic sectional curvature and the tensor field P is not hybrid.

In the second section, some results on l.c.K-space forms are presented. It is proved that for a 2n-dimensional l.c.K-space form $M(c)$, if the tensor field P is proportional to g and $tr\,P$ is constant, then $M(c)$ is Einstein.

In the third section, we gave the basic definitions of submanifolds of l.c.K-manifolds. The invariant submanifolds of l.c.K-space forms are studied. In the last section, we give the Sato's form of the holomorphic curvature tensor in an almost Hermitian manifold and we determine the Sato's form of the holomorphic curvature tensor in an l.c.K-manifold.

In the fourth chapter, we state some results on pseudosymmetric and Ricci-pseudosymmetric l.c.K-space forms. This chapter contains two sections.

In the first section, we introduced some results of Pseudosymmetric l.c.K-space forms. Moreover, we investigate generalized Einstein metric conditions in an l.c.K-space form. It is proved that for 4-dimensional l.c.K-space forms such that the tensor field P is hybrid and $tr\,P$ is constant,

$$R\cdot C - C\cdot R = [\frac{1}{4}\ (2c + tr\,P)]\ Q(g,R)$$

and for m-dimensional $(m > 4)$ with the tensor P is proportional to g in l.c.K-space forms

$$R\cdot C - C\cdot R \quad = \quad \frac{1}{4(m-1)}\left[(m+2)c + \frac{6(m-2)tr\,P}{m}\right]\ Q(g,R).$$

In addition, we get the results under the assumption $R\cdot R - Q(S,R) = L_1 Q(g,C)$.

In the second section, under the assumption that $R\cdot R - Q(S,R) = L_1 Q(g,C)$ and $R\cdot C = L_2 Q(S,C)$ are satisfied, we obtain the results of Ricci-pseudosymmetric l.c.K-space forms.

In the fifth chapter, Walker type identities on l.c.K-space forms and Roter type l.c.K-space forms are investigated. Moreover Bochner curvature tensor are studied. This chapter contains three sections.

In the first section, we present results on l.c.K-space forms satisfying curvature identities called Walker type identities. It is proved that a 4-dimensional l.c.K-space form such that the tensor field P is hybrid and $tr\,P$ is constant satisfies Walker type identities. For m-dimensional $(m > 4)$ l.c.K-space forms, under the assumption of P is proportional to g, the Walker type identities hold.

In the second section, we introduced the Roter type l.c.K-space forms. If P is hybrid, it is proved $\bar{R}\cdot\bar{R} = Q(\bar{S},\bar{R}) + \bar{L}_1 Q(g,\bar{C})$ in m-dimensional $(m > 4)$ Roter type l.c.K-space forms.

In the last section, firstly, the Bochner curvature tensor in an l.c.K-manifold such that the tensor field P is hybrid is given. Then we present a generalization of the Bochner curvature tensor in an l.c.K-manifold with the tensor field P is not hybrid. Moreover, we state the Bochner curvature tensor in an l.c.K-space form. Furthermore, Walker type identities for Bochner curvature tensor are studied. Next if the condition $B\cdot B = L_B Q(g,B)$ is fulfilled, we proved that pseudosymmetric l.c.K-space forms $(m > 4)$ are Einstein.

PSEUDÖ SİMETRİK LOKAL OLARAK KONFORM KAEHLER MANİFOLDLARI

ÖZET

Hemen hemen Hermitian manifoldlarının belirli sınıfları üzerine yoğun çalışmalar yapılmıştır. Bu, hemen hemen Hermitian manifoldların arasında metriği hemen hemen Kaehler metriğe global olarak konform olanlar daha çok çalışılmıştır. Fakat açıktır ki bu manifoldlar, Kaehler manifoldlarla aynı topolojik özelliklere sahiptirler. Bundan dolayı bir hemen hemen Kaehler manifolduna lokal olarak konform olan hemen hemen Hermitian manifoldlar hakkında çalışmak ilginçtir. Bir Hermitian manifoldda bir lokal olarak konform Kaehler manifold (l.c.K-manifold) kavramı 1976 yılında I. Vaisman tarafından ortaya atılmıştır.

Daha sonra T. Kashiwada tensör denklemini kullanarak bir Hermitian manifoldun l.c.K-manifoldu olması için gerek ve yeter koşulu ispat etmiş ve holomorfik kesitsel eğriliği sabit olan bir l.c.K-manifoldunun (l.c.K-uzay formu) eğrilik tensörünü tanımlamıştır. Ayrıca T. Kashiwada ve K. Matsumoto böyle bir manifoldun bazı özelliklerini vermişlerdir. Dolayısıyla, l.c.K-manifoldlar ve l.c.K-manifoldların alt manifoldları ile ilgili bir çok yayın bulunmaktadır.

İlave olarak, M. Prvanović bir hemen hemen Hermitian manifoldu için Kaehler tipe sahip olan bir tensör tanımlamış ve bir hemen hemen Kaehler manifoldunda bu tensörün Riemann eğrilik tensörüne indirgendiğini ispatlamıştır. Ayrıca, bir l.c.K-manifoldunda holomorfik eğrilik tensörünü vermiş ve P tensörü hibrid olmayacak şekilde bir sabit holomorfik kesitsel eğrilikli l.c.K-manifoldunda Riemann eğrilik tensörünü tanımlamıştır.

Eğer bir Hermitian manifoldunun her $p \in M$ noktasının, $g^* = e^{-2\rho} g \mid_{\mathrm{U}}$ metriği, U kümesinde bir Kaehler metrik olacak şekilde türevlenebilir bir $\rho : U \to R$ fonksiyona sahip olan açık U komşuluğu mevcut ise $M(J,g)$ yapılı Hermitian manifolduna bir l.c.K-manifoldu denir.

2n-boyutlu bir l.c.K-manifold, kompleks yapısı J nin

$$\nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki}$$

eşitliğini sağlayan bir Hermitian manifolddur. Burada ∇, Hermitian metrik g 'ye göre kovaryant türevdir.

Eğer bir l.c.K-manifold $M(J,g,\alpha)$, bir sabit holomorfik kesitsel eğriliğe sahip ise M ye l.c.K-uzay formu denir.

$\nabla R = 0$ koşulunu sağlayan bir (M,g) yarı Riemann manifolduna lokal olarak simetrik denir. Bu manifoldlar ilk kez 1920 li yıllarda E. Cartan tarafından çalışılmış ve sınıflandırılmıştır.

$R \cdot R = 0$ koşulunu sağlayan bir yarı Riemann manifolduna yarı simetrik denir. Simetrik uzayların bir doğal genelleştirilmişi olan yarı Riemann manifoldlar, E. Cartan tarafından çalışılmıştır.

Yarı Rieman manifoldlarda pseudö simetrik manifoldlar R. Deszcz tarafından ortaya atılmıştır.

M'nin her noktasında

$$R \cdot R = L_R Q(g,R)$$

koşulu $\mathscr{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \; at \; x\}$ kümesinde geçerli ise bir (M,g) yarı Riemann manifolduna Deszcz anlamında pseudö simetrik denir. Burada L_R, $\mathscr{U}_R$ 'da bir fonksiyondur.

$\mathscr{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \; at \; x\}$ kümesinde

$$R \cdot S = L_S Q(g,S)$$

koşulunu sağlayan bir yarı Riemann manifolduna, $n \geq 3$, Ricci-pseudö simetrik denir. Her pseudö simetrik manifold bir Ricci-pseudö simetriktir. Ancak bunun tersi doğru değildir. Ricci-pseudö simetrik manifoldlar, pseudö simetrik manifold sınıfı gibi Ricci yarı simetrik $(R \cdot S = 0)$ manifold sınıfının genişletilmişidir. Her Ricci yarı simetrik manifoldlar Ricci-pseudö simetriktir. Pseudö simetrik olmayan, Ricci-pseudö simetrik manifoldlarla ilgili bir çok örnek bulunmaktadır.

Bu tez çalışmasında, l.c.K-manifoldlarının, l.c.K-uzay formlarının ve l.c.K-uzay formlarının alt manifoldlarının bazı özellikleri sunulmuştur. Ayrıca pseudö simetrik ve Ricci-pseudö simetrik l.c.K-uzay formlarla ilgili sonuçlar elde edilmiştir. Buna ilaveten, l.c.K-uzay formlarda Walker tip özdeşlikler ve Roter tip l.c.K-uzay formları çalışılmıştır. Son olarak l.c.K-uzay formlarda Bochner eğrilik tensörü incelenmiştir.

Birinci bölümde, literatür araştırmasına yer verilmiştir. Ayrıca tez çalışmasında elde edilen sonuçlar özetlenmiştir.

İkinci bölümde, tez çalışmasında kullanılan bazı temel kavramlar verilmiştir.

Üçüncü bölümde ilk olarak, P tensörü hibrid olmayacak şekilde, l.c.K-uzay formlarla ilgili genelleştirilmiş bazı sonuçlar verilmiştir. Ayrıca hemen hemen Hermitian manifoldlarda ve l.c.K-manifoldlarda holomorfik eğrilik tensörünün Sato formu sunulmuştur. Bu bölüm dört kısımdan meydana gelmiştir.

İlk kısımda, l.c.K-manifoldlarda bilinen sonuçlara yer verilmiştir. P tensörünün hibrid olması için gerek ve yeter koşulun Ricci tensörünün hibrid olması gerektiği verilmiştir. Ayrıca, bir hemen hemen Hermitian manifold için Kaehler tip özelliğini sağlayan bir eğrilik tensörünün varlığı ifade edilmiştir. Bu tensör holomorfik eğrilik tensörü olarak adlandırılır.

Buna ilaveten, P tensörü hibrid olmamak üzere holomorfik eğrilik tensörü ve sabit holomorfik kesitsel eğrilikli bir l.c.K-manifoldda Riemann eğrilik tensörü sunulmuştur.

İkinci kısımda, l.c.K-uzay formlarla ilgili sonuçlar verilmiştir. Eğer P tensörü g metriğiyle orantılı ve $tr\,P$ sabit ise 2n-boyutlu l.c.K-uzay formunun Einstein olduğu ispatlanmıştır.

Üçüncü kısımda, l.c.K-uzay formlarının alt manifoldları ile ilgili temel tanımlar verilmiştir. L.c.K-uzay formlarının invaryant alt manifoldları çalışılmıştır. Son kısımda ise, bir hemen hemen Hermitian manifoldunun holomorfik eğrilik tensörünün Sato formu verilmiş ve ayrıca bir l.c.K-manifoldda holomorfik eğrilik tensörünün Sato formu elde edilmiştir.

Dördüncü bölümde, pseudö simetrik ve Ricci-pseudö simetrik l.c.K-uzay formlarla ilgili sonuçlar ifade edilmiştir. Bu bölüm iki kısımdan meydana gelmiştir.

İlk kısımda, pseudö simetrik l.c.K-uzay formlarda sonuçlar elde edilmiş tir. Ayrıca, bir l.c.K-uzay formda genelleştirilmiş Einstein metrik koşulları incelenmiştir. P tensörü hibrid ve $tr\,P$ sabit olmak üzere, 4-boyutlu l.c.K-uzay formlarda

$$R \cdot C - C \cdot R = [\frac{1}{4}\ (2c + tr\,P)]\ Q(g,R)$$

ve P tensörü g metriğiyle orantılı olmak üzere, m-boyutlu $(m > 4)$ l.c.K-uzay formlarda

$$R \cdot C - C \cdot R \;=\; \frac{1}{4(m-1)}\left[(m+2)c + \frac{6(m-2)tr\,P}{m}\right] Q(g,R)$$

denklemleri ispatlanmıştır. Ayrıca $R \cdot R - Q(S,R) = L_1 Q(g,C)$ koşulu altında bazı sonuçlar bulunmuştur.

İkinci kısımda ise, $R \cdot R - Q(S,R) = L_1 Q(g,C)$ ve $R \cdot C = L_2 Q(S,C)$ koşulları altında Ricci-pseudö simetrik l.c.K-uzay formlarda sonuçlar elde edilmiştir.

Beşinci bölümde, l.c.K-uzay formlarda Walker tip özdeşlikler ve Roter tip l.c.K-uzay formları incelenmiştir. Ayrıca Bochner eğrilik tensörü üzerinde çalışılmıştır. Bu bölüm üç kısımdan meydana gelmiştir.

Birinci kısımda Walker tip özdeşlikler olarak adlandırılan eğrilik özdeşliklerini sağlayan bir l.c.K-uzay formuyla ilgili sonuçlar sunulmuştur. P tensörü hibrid ve $tr\,P$ sabit olmak üzere 4-boyutlu bir l.c.K-uzay formunun Walker tip özdeşlikleri sağladığı ispatlanmıştır. Ayrıca, P tensörü g metriğiyle orantılı olması şartı altında, m-boyutlu $(m > 4)$ l.c.K-uzay formlarının Walker tip özdeşlikleri sağladığı gösterilmiştir.

İkinci kısımda, Roter tip l.c.K-uzay formları hakkında çalışılmıştır. Eğer P tensörü hibrid ise m-boyutlu $(m > 4)$ Roter tip l.c.K-uzay formlarda $\bar{R} \cdot \bar{R} = Q(\bar{S},\bar{R}) + \bar{L}_1 Q(g,\bar{C})$ olduğu ispatlanmıştır.

Son kısımda ise, P tensörü hibrid olacak şekilde bir l.c.K-manifoldda Bochner eğrilik tensörü verilmiştir. Ayrıca P tensörünün hibrid olmaması durumunda, bir l.c.K-manifoldda Bochner eğrilik tensörünün genelleştirilmişi ispatlanmıştır. Daha sonra, bir l.c.K-uzay formunda Bochner eğrilik tensörü verilmiştir. İlave olarak, Bochner eğrilik tensörü için Walker tip özdeşlikler çalışılmıştır. Son olarak $B \cdot B = L_B Q(g,B)$ şartı altında pseudö simetrik l.c.K-uzay formlarının $(m > 4)$ Einstein oldğu ispatlanmıştır.

1. INTRODUCTION

1.1 Purpose of Thesis

Let M be a real 2n-dimensional Hermitian manifold with structure (J, g), where J is the almost complex structure and g is the Hermitian metric. The manifold M is called *a locally conformal Kaehler manifold* (an l.c.K-manifold) if each point p in M has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that

$$g^* = e^{-2\rho} g \mid_{\mathrm{U}}$$

is a Kaehlerian metric on U. Especially, if we can take $U = M$, then the manifold M is said to be globally conformal Kaehler [1].

The following is essential in l.c.K-manifolds [2].

An 2n-dimensional l.c.K-manifold is a Hermitian manifold admitting a global closed 1-form α (Lee form) whose structure (J, g) satisfies

$$\nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki},$$

where $\beta_i = \alpha^r J_{ri}$ and ∇ denotes the covariant differentiation with respect to the Hermitian metric g.

In an l.c.K-manifold,we have

$$\begin{aligned} R_{hkrs} J_j^r J_i^s &= R_{hkji} + P_{ki} g_{hj} - P_{kj} g_{hi} + P_{hj} g_{ki} - P_{hi} g_{kj} \\ &+ P_{kr} J_i^r J_{hj} - P_{kr} J_j^r J_{hi} + P_{hr} J_j^r J_{ki} - P_{hr} J_i^r J_{kj}. \end{aligned}$$

where

$$P_{ij} = -\nabla_i \alpha_j - \alpha_i \alpha_j + \frac{\|\alpha\|^2}{2} g_{ij}.$$

We note that $P_{ij} = P_{ji}$ and $\|\alpha\|^2 = \alpha_r \alpha^r$.

A 2-plane π in $T_p M$, $p \in M$, is said to be *holomorphic* if $J\pi = \pi$. The manifold M has constant holomorphic sectional curvature if the sectional curvature relative to π does not depend on the holomorphic 2-plane π in $T_p M$.

An l.c.K-manifold $M(J,g,\alpha)$ is called *an l.c.K-space form* if it has a constant holomorphic sectional curvature. Let M(c) be an l.c.K-space form with constant holomorphic sectional curvature c, then the Riemannian curvature tensor R with respect to g can be expressed in the form [2]

$$\begin{aligned} R_{ijhk} &= \frac{c}{4}(g_{ik}g_{jh} - g_{ih}g_{jk} + J_{ik}J_{jh} - J_{ih}J_{jk} - 2J_{ij}J_{hk}) \\ &+ \frac{3}{4}(g_{ik}P_{jh} + g_{jh}P_{ik} - g_{ih}P_{jk} - g_{jk}P_{ih}) \\ &- \frac{1}{4}(\widetilde{P}_{ik}J_{jh} + \widetilde{P}_{jh}J_{ik} - \widetilde{P}_{ih}J_{jk} - \widetilde{P}_{jk}J_{ih} - 2\widetilde{P}_{ij}J_{hk} - 2\widetilde{P}_{hk}J_{ij}), \end{aligned}$$

where the tensor field P is hybrid, i.e. $P_{ir}J_j^r + P_{jr}J_i^r = 0$ and $\widetilde{P}_{ij} = -P_{ir}J_j^r$.

A semi-Riemannian manifold is said to be locally symmetric if the condition $\nabla R = 0$ is satisfied on that manifold. These manifolds are first studied and classified by E. Cartan in the late twenties. A semi-Riemannian manifold is said to be semi-symmetric if the condition $R \cdot R = 0$ is satisfied on that manifold.

A semi-Riemannian manifold is said to be pseudosymmetric if at every point of M the following condition is satisfied:

The tensor $R \cdot R$ and Q(g, R) are linearly dependent. This condition is equivalent to the relation $R \cdot R = L_R Q(g,R)$ where L_R is a function on the set $\mathscr{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \; at \; x\}$. Pseudosymmetric manifolds are a generalization of semisymmetric manifolds.

A semi-Riemannian manifold (M,g) is said to be *Ricci-pseudosymmetric* if at every point of M the condition

$$R \cdot S = L_S Q(g,S)$$

holds on the set $\mathscr{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \; at \; x\}$, where L_S is some function on $\mathscr{U}_S$ [3]. Every pseudosymmetric manifold is Ricci-pseudosymmetric. The converse statement is not true. The class of Ricci-pseudosymmetric manifolds is an extension of the class of Ricci-semisymmetric $(R \cdot S = 0)$ manifolds as well as of the class of pseudosymmetric manifolds. Evidently, every Ricci-semisymmetric is Ricci-pseudosymmetric.

A semi-Riemannian manifold (M,g) is said to be *Weyl-pseudosymmetric* if at every point of M the condition

$$R \cdot C = L_C Q(g,C)$$

holds on the set $\mathscr{U}_C = \{x \in M \mid C \neq 0 \; at \; x\}$, where L_C is some function on $\mathscr{U}_C$. Every pseudosymmetric manifold is Weyl-pseudosymmetric. The converse statement not true. Every Weyl-semisymmetric manifold $(R \cdot C = 0)$ is Weyl-pseudosymmetric.

This thesis is divided into 6 chapters:

Chapter 2 gives the fundamental concepts which we will use the next chapters.

In chapter 3, we gave the well-known results on l.c.K-manifolds. It is given that the tensor field P is hybrid if and only if the Ricci tensor is hybrid. Then we state that there exists the algebraic curvature tensor satisfying the condition of Kaehler type manifold for any almost Hermitian manifold. This tensor is said to be the holomorphic curvature tensor. Furthermore, we state the holomorphic curvature tensor in an l.c.K-manifold and the Riemannian curvature tensor in an l.c.K-manifold with a constant holomorphic sectional curvature and the tensor field P is not hybrid. It is proved that for a 2n-dimensional l.c.K-space form $M(c)$, if the tensor field P is proportional to g and $tr\, P$ is constant, then $M(c)$ is Einstein. Moreover we gave the basic definitions of submanifolds of l.c.K-manifolds. The invariant submanifolds of l.c.K-space forms are studied. We give The Sato's form of the holomorphic curvature tensor in an almost Hermitian manifold and we determine the Sato's form of the holomorphic curvature tensor in an l.c.K-manifold.

In chapter 4, we introduced some results of Pseudosymmetric l.c.K-space forms. Moreover, we investigate generalized Einstein metric conditions in an l.c.K-space form. Furthermore, under the assumption that $R \cdot R - Q(S,R) = L_1 Q(g,C)$ and $R \cdot C = L_2 Q(S,C)$ are satisfied, we obtain the results of Ricci-pseudosymmetric l.c.K-space forms.

In chapter 5, we present results on l.c.K-space forms satisfying curvature identities called Walker type identities. It is proved that a 4-dimensional l.c.K-space form such that the tensor field P is hybrid and $tr\, P$ is constant satisfies Walker type identities. For m-dimensional $(m > 4)$ l.c.K-space forms, under the assumption of P is proportional to g, the Walker type identities hold. Moreover we introduced the Roter type l.c.K-space forms. If P is hybrid, it is proved $\bar{R}\cdot\bar{R}=Q(\bar{S},\bar{R})+\bar{L}_1Q(g,\bar{C})$ in m-dimensional $(m>4)$ Roter type l.c.K-space forms.

The Bochner curvature tensor in an l.c.K-manifold such that the tensor field P is hybrid is given. Then we present a generalization about the Bochner curvature tensor in an l.c.K-manifold such that the tensor field P is not hybrid. Moreover, we state the Bochner curvature tensor in an l.c.K-space form. Furthermore some properties of the Bochner curvature tensor in an l.c.K-space form are obtained.

Finally, in chapter 6, we give conclusion and recommendations.

1.2 Literature Review

The notion of an l.c.K-manifold in a Hermitian manifold has introduced by I. Vaisman on 1976 [1]. The author gives characterizations of locally conformal almost Kaehler manifolds and some relations between locally conformal Kaehler and globally conformal Kaehler metrics. After that he wrote a series of such manifolds [4] [5] etc. T. Kashiwada has determined a necessary and sufficient condition that a Hermitian manifold is an l.c.K-manifold by using the tensor equation and determined the curvature tensor of an l.c.K-manifold with a constant holomorphic sectional curvature (an l.c.K-space form).

Moreover K. Matsumoto studied some different questions concerning the geometry of l.c.K-manifolds. The author gave some properties about l.c.K-manifolds, l.c.K-space forms and their submanifolds. T. Kashiwada [2] [6] and K. Matsumoto [7] gave some properties about such a manifold.

Furthermore, M. Prvanović found a tensor of Kaehler type for an almost Hermitian manifold and proved that this tensor reduces to the Riemannian curvature tensor R in an almost Kaehler manifold. In addition, the author determined the holomorphic curvature tensor of an l.c.K-manifold and introduced Riemannian curvature tensor in an l.c.K-manifold with a constant holomorphic sectional curvature and the tensor P is not hybrid [8] [9].

E. Cartan studied the semi-symmetric manifolds which is a natural generalization of the symmetric spaces. A fundamental study on Riemannian semisymmetric manifolds has been given by Z. I. Szabó [10] [11] [12].

R. Deszcz introduced the pseudo-symmetric manifolds which is called the pseudosymmetry in the sense of Deszcz which is characterized by the condition $R \cdot R = L_R Q(g,R)$, where L_R is some function and $Q(g,R)$ is the Tachibana tensor [13].

1.3 Hypothesis

In this thesis, some properties of l.c.K-manifolds, l.c.K-space forms and submanifolds of an l.c.K-space form are presented. Furthermore, we state some results on pseudosymmetric and Ricci-pseudosymmetric l.c.K-space forms. Walker type identities on l.c.K-space forms and Roter type l.c.K-space forms are studied. Finally the Bochner curvature tensor on l.c.K-manifolds and l.c.K-space forms are studied.

2. PRELIMINARIES

2.1 Riemannian Manifolds

A Riemannian metric on a smooth manifold M is a (0,2)-tensor field g on M that is symmetric $(i.e., g(X,Y) = g(Y,X))$ and positive definite $(i.e.,\ g(X,X) > 0$ if $X \neq 0)$. A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written by

$$\langle X,Y\rangle = g(X,Y) \qquad for\, X,Y \in T_pM.$$

A manifold together with a given Riemannian metric is called a *Riemannian manifold*.

A *semi-Riemannian metric* on a smooth manifold M is a symmetric (0,2)-tensor field g that is nondegenerate at each point $p \in M$. This means that $g(X,Y) = 0$ for all $Y \in T_pM$ if and only if $X = 0$. A smooth manifold with a semi-Riemannian metric is called *a semi-Riemannian manifold*.

The local components of g on an open set U $\subset$ M are given by

$$g_{ij} = g(\partial_i, \partial_j) = \langle \partial_i, \partial_j \rangle \ ,$$

where $\partial_i = \frac{\partial}{\partial x^i}$ are basis vectors on U.

A connection ∇ on a smooth Riemannian manifold M is a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that

(i) $\nabla_{fX_1+gx_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$,

(ii) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$,

(iii) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$,

where f,g $\in C^\infty(M)$, a,b $\in \mathbb{R}$, X,Y $\in T_pM$. $\nabla_X Y$ is called the *covariant derivative* of Y in the direction of X.

A Riemannian connection (Levi-Civita connection) ∇ on a Riemannian manifold M is a connection such that

(iv) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,

(v) $[X,Y] = \nabla_X Y - \nabla_Y X$

and it is characterized by *the Koszul formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle \\ &+ \langle Y,[Z,X]\rangle - \langle Z,[X,Y]\rangle \ . \end{aligned}$$

Let T be an r-tensor. The covariant derivative ∇T of T is a tensor of order (r+1) given by

$$\begin{aligned} (\nabla T)(X_1,X_2,...,X_r;X) &= (\nabla T)(X_1,...,X_r) \\ &= \nabla_X(T(X_1,...,X_r)) - \sum_{i=1}^{r} T(X_1,...,\nabla_X X_i,...,X_r) \ . \end{aligned}$$

If X,Y $\in T_pM$, then linear operator

$$R(X,Y) : T_pM \to T_pM$$

is called *the curvature operator*. *The Riemannian curvature tensor* R of M is the tensor field of type (0,4) defined by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle = g(R(X,Y)Z,W) \ ,$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ . \qquad (2.1)$$

In terms of local coordinates

$$R_{ijhk} = R^r_{ijh} g_{rk} \ ,$$

where

$$R^k_{ijh} = \partial_i \Gamma^k_{jh} - \partial_j \Gamma^k_{ih} + \Gamma^k_{ir}\Gamma^r_{jh} - \Gamma^k_{jr}\Gamma^r_{ih} \ .$$

The curvature tensor satisfies the following symmetries :

(i) $R(X,Y)Z + R(Y,X)Z = 0$,

(ii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, (First Bianchi identity)

(iii) $R(X,Y,Z,W) = -R(Y,X,Z,W)$,

(iv) $R(X,Y,Z,W) = -R(X,Y,W,Z)$,

(v) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

The total covariant derivative of the curvature tensor satisfies the following identity:

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0. \tag{2.2}$$

The equation (2.2) is called *the second Bianchi identity*.

If $e_1, e_2, ..., e_n$ are local orthonormal vector field, then

$$S(Y,Z) = \sum_{i=1}^{n} g(R(e_i,Y)Z, e_i)$$

defines a (0,2) tensor field with local components

$$S_{ij} = R^r_{rij} = g^{rs} R_{rijs} .$$

The tensor field $S(Y,Z)$ is called a *Ricci tensor*. It is clear that $S_{ij} = S^r_i g_{rj}$ and $S^j_i = g^{jr} S_{ir}$.

The scalar curvature is the function κ defined as the trace of the Ricci tensor:

$$\kappa = \sum_{i=1}^{n} S(e_i,e_i) = tr\, S = S^i_i = g^{ij} S_{ij} . \tag{2.3}$$

The curvature tensor appears also in the Ricci identities:

$$\nabla_i \nabla_j T^h - \nabla_j \nabla_i T^h = R^h_{ijr} T^r , \tag{2.4}$$

$$\nabla_i \nabla_j T_h - \nabla_j \nabla_i T_h = -R^r_{ijh} T_r , \tag{2.5}$$

$$\nabla_i \nabla_j T^s_{hk} - \nabla_j \nabla_i T^s_{hk} = R^s_{ijr} T^r_{hk} - R^r_{ijh} T^s_{rk} - R^r_{ijk} T^s_{hr} . \tag{2.6}$$

A Riemannian manifold is called *an Einstein manifold* if

$$S_{ij} = \lambda g_{ij} ,$$

where λ is constant.

Let M be a Riemannian manifold and $p \in M$. A two dimensional subspace π of the tangent space $T_p M$ is called *a tangent plane* to M at p. π is determined by linearly independent vectors X and Y at p. We define *the sectional curvature* $K(\pi)$ of π spanned by X and Y at p is given by

$$K(X,Y) = K(\pi) = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} . \tag{2.7}$$

In particular, if π is spanned by an orthonormal basis u and v at p , the sectional curvature is given by

$$K(\pi) = R(u,v,v,u).$$

In local components,

$$K(\pi) = \frac{R_{ijhk}X^iY^jY^hX^k}{(g_{ik}g_{jh} - g_{ih}g_{jk})X^iY^jY^hX^k}\ . \tag{2.8}$$

The sectional curvature K of M is a real-valued function on the set of all nondegenerate tangent planes to M.

The famous Theorem of Green can now be stated as follows:

Green's Theorem. In a compact orientable Riemannian manifold M, we have

$$\int_M (\nabla v)d\sigma = 0 \tag{2.9}$$

for any arbitrary vector field v, where $d\sigma$ is the volume element

$$d\sigma = \sqrt{g}dv^1 \wedge dv^2 \wedge \ldots \wedge dv^{2n}.$$

2.2 Submanifolds of Riemannian Manifolds

Let N be an m-dimensional manifold isometrically immersed in a 2n-dimensional manifold M. If the manifold M is covered by a system of coordinate neighborhoods $\{V, v^i\}$ and N is covered by a system of coordinate neighborhoods $\{U, u^\lambda\}$, where here and in the sequel the indices $i, j, h, k, \ldots$ run over the range $1, 2, \ldots, 2n$ and $\nu, \mu, \lambda, \ldots$ run over the range 1, 2, ..., m, then the submanifold N can be locally represented by

$$v^i = v^i(u^\lambda)\ . \tag{2.10}$$

In the following, we shall identify vector fields in N and their image under the differential mapping, that is, if i denotes the immersion of N in M and X is a vector field in N, we identify X and $i_*(X)$.

Thus, if X is a vector field in N and has the local expression $X = u^\lambda \partial_\lambda$, where $\partial_\lambda = \frac{\partial}{\partial u^\lambda}$, then X also has the local expression

$$X = B^i_\lambda u^\lambda \partial_i,$$

where $\partial_i = \frac{\partial}{\partial v^i}$ and $B^i_\lambda = \partial_\lambda v^i = \frac{\partial v^i}{\partial u^\lambda}$.

Suppose that the manifold M is a Riemannian manifold with Riemannian metric g, then the submanifold N is also a Riemannian manifold with Riemannian metric $\widetilde{g}$ is given by

$$\widetilde{g}(X,Y) = g(X,Y) \tag{2.11}$$

for any vector fields X and Y in N. The Riemannian metric $\widetilde{g}$ on N is called the *induced metric* on N.

In local coordinates, it is given by

$$g_{\mu\lambda} = g_{ji} B^j_\mu B^i_\lambda \tag{2.12}$$

with $\widetilde{g} = g_{\mu\lambda} du^\mu du^\lambda$ and $g = g_{ji} dv^j dv^i$.

If a vector ξ_p of M at a point $p \in$ N satisfies

$$g(X_p, \xi_p) = 0$$

for any vector X_p of N at p, then ξ_p is called *a normal vector* of N in M at p.

Let $T^\perp N$ denote the vector bundle of all normal vectors of N in M. The tangent bundle of M , restricted to N, is the direct sum of the tangent bundle TN of N and the normal bundle $T^\perp N$ of N in M, that is

$$TM\,|_{\mathrm{N}} = TN + T^\perp N\,. \tag{2.13}$$

From (2.13), we see that $\nabla_X Y$ can be expressed in the form

$$\nabla_X Y = \widetilde{\nabla}_X Y + h(X,Y) \tag{2.14}$$

where $\widetilde{\nabla}$ is the covariant differentiation defined on the submanifold N with respect to $\widetilde{g}$ and $h(X,Y)$ is a normal vector field on N and is symmetric and bilinear in X and Y. We call h *the second fundamental form* of the submanifold N. The equation (2.14) is called *the Gauss formula*.

Let X and ξ be a vector field and normal vector field on N, respectively. We can decompose $\nabla_X \xi$ as

$$\nabla_X \xi = -A_\xi(X) + \nabla^\perp{}_X \xi\,, \tag{2.15}$$

where $A_\xi(X)$ and $\nabla^\perp{}_X \xi$ are the tangential component and the normal component of $\nabla_X \xi$, respectively. The equation (2.15) is called *the Weingarten's formula*.

A submanifold N is said to be *totally geodesic* if the second fundamental form h vanishes identically, that is, $h = 0$.

Let $\xi_1, ..., \xi_{2n-m}$ be an orthonormal basis of the normal space $T_p^{\perp}(N)$ at a point $p \in N$ and let $A^x = A_{\xi_x}$, then

$$H = \frac{1}{m}(tr\, A^x)\xi_x \tag{2.16}$$

is a normal vector at p which is independent of the choice of the orthonormal basis ξ_x. The vector H is called *the mean curvature vector* at p.

A submanifold N is called a *minimal submanifold* if the mean curvature vector vanishes identically.

The equation of Gauss and Codazzi are respectively given by

$$\begin{aligned} \langle R(X,Y)Z,W\rangle &= \langle \widetilde{R}(X,Y)Z,W\rangle + \langle h(X,Z),h(Y,W)\rangle \\ &- \langle h(X,W),h(Y,Z)\rangle \end{aligned} \tag{2.17}$$

and

$$(R(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z)\ , \tag{2.18}$$

where

$$(\bar{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

2.3 Complex Manifolds

Consider a real 2n-dimensional manifold M of class C^{∞} covered by a system of coordinate neighborhoods v^i, where the indices $i, j, k, ...$ run over the range $1, 2, ..., n, \bar{1}, \bar{2}, ..., \bar{n}$. In each coordinate neighborhoods (v^i) we introduce complex coordinates z^a defined by

$$z^a = v^a + \sqrt{-1}\, v^{\bar{a}}\ , \tag{2.19}$$

where v^a and $v^{\bar{a}}$ are real variables and the indices a, b, ... run over the range $1, 2, ..., n$. We call v^i real coordinates and z^a complex coordinates of a point with respect to these system of coordinates respectively.

M^{2n} is said to admit a complex structure and is called a complex manifold if there exist a system of complex coordinate neighborhoods (z^a) covering the whole manifold M^{2n} such that in the intersection of two coordinate neighborhoods (z^a) and $(z^{\acute{a}})$ we have

$$z^{\acute{a}} = f^{\acute{a}}(z^a) \qquad , \qquad |\ \partial_a z^{\acute{a}}\ | \neq 0\ ,$$

where $f^{\acute{a}}(z^a)$ are analytic functions of complex variables $z^1, z^2, ..., z^n$ and $| \partial_a z^{\acute{a}} |$ denotes the Jacobian determinant, where $\partial_a z^{\acute{a}} = \frac{\partial z^{\acute{a}}}{\partial z^a}$ and

$$\frac{\partial}{\partial z^a} = \frac{1}{2}(\frac{\partial}{\partial v^a} - \sqrt{-1}\,\frac{\partial}{\partial v^{\bar{a}}}),$$

$$\frac{\partial}{\partial \bar{z}^a} = \frac{1}{2}(\frac{\partial}{\partial v^a} + \sqrt{-1}\,\frac{\partial}{\partial v^{\bar{a}}}),$$

where $\bar{z}$ is the conjugate of z defined by

$$\bar{z}^a = v^a - \sqrt{-1}\, v^{\bar{a}}\ . \tag{2.20}$$

Let M be a real differentiable 2n-dimensional manifold. An almost complex structure on M is a tensor field J of type (1,1) on M such that at every point $p \in M$ we have $J^2 = -I$, where I denotes the identity transformation of T_pM. A manifold with an almost complex structure J is called *an almost complex manifold*. Every almost complex manifold is of even dimension and orientable.

We suppose M is an almost complex manifold. Then we define the torsion tensor of J or the Nijenhuis tensor of J by

$$N(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]$$

for any vector fields X and Y. If N vanishes identically, then an almost complex structure is called *a complex structure* and M is called a *complex manifold*.

A Hermitian metric on an almost complex manifold M is a Riemannian metric g such that

$$g(JX,JY) = g(X,Y)$$

for any vector fields X and Y.

An almost complex manifold (resp. a complex manifold) endowed with a Hermitian metric is called an almost Hermitian manifold (resp. a Hermitian manifold).

Every almost complex manifold with a Riemannian metric $\acute{g}$ admits a Hermitian metric. In fact, if we take

$$g(X,Y) = \acute{g}(X,Y) + \acute{g}(JX,JY),$$

it is easily seen that g is a Hermitian metric on M.

A Hermitian manifold M is called a *Kaehler* manifold if the almost complex structure J on M is parallel, that is, $\nabla J = 0$.

The curvature tensor R of Kaehlerian manifold M satisfies

$$R_{ijhk} = J_i^r J_j^s R_{rshk}. \tag{2.21}$$

Let M be an almost Hermitian 2n-dimensional manifold with a Hermitian metric g_{ij} and an almost Hermitian structure J whose components are J_i^j. Since any tangent vector u^i of M and its transform $J(u^i)$ at a point p are mutually orthogonal, they are linearly independent and therefore determine a 2-plane in the tangent space of M at p which is called *a holomorphic plane*. The sectional curvature of M at p with respect to a holomorphic plane is called *the holomorphic sectional curvature* of M at p. If the holomorphic sectional curvature of M at a point p is independent of the holomorphic plane through p, the M is said to have a *constant holomorphic sectional curvature* at p. If the holomorphic sectional curvature of M is constant for all holomorphic planes and all points p, then M is called a *manifold of constant holomorphic sectional curvature* or a *complex space form*.

The holomorphic sectional curvature c of M at p with respect to the holomorphic plane $\pi(U)$ is given by

$$c = -\frac{R_{ijhk}\, J_r^i u^r\, u^j\, J_s^h u^s\, u^k}{g_{rj} u^r\, u^j\, g_{sk} u^s\, u^k}\ . \tag{2.22}$$

The Riemannian curvature tensor R_{ijhk} of a Kaehler manifold with the constant holomorphic sectional curvature c is given by

$$R_{ijhk} = \frac{c}{4}\left(g_{ik}g_{jh} - g_{ih}g_{jk} + J_{ik}J_{jh} - J_{ih}J_{jk} - 2J_{ij}J_{hk}\right). \tag{2.23}$$

2.4 Pseudosymmetrically Related Tensors

In this section, we give the basic definitions, properties and results related with the pseudosymmetric curvature conditions which will be used in the following sections.

Let (M, g) be an $n-$dimensional, $n \geq 3$, semi-Riemannian connected manifold of class C^∞ with Levi-Civita connection ∇. The Ricci operator $\mathscr{S}$ is defined by $g(\mathscr{S}X,Y) = S(X,Y)$, where $X,Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M.

We define the endomorphisms $X \wedge_A Y$, $\mathscr{R}(X,Y)Z$ and $\mathscr{C}(X,Y)$ of $\Xi(M)$ by

$$
\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y \, , && (2.24)\\
\mathscr{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \, , && (2.25)\\
\mathscr{C}(X,Y)Z &= \mathscr{R}(X,Y)Z \\
&- \frac{1}{n-2}(X \wedge_g \mathscr{S}Y + \mathscr{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z \, , && (2.26)
\end{aligned}
$$

respectively, where $X,Y,Z \in \Xi(M)$, A is a symmetric (0,2)-tensor, κ the scalar curvature and $[X,Y]$ is the Lie bracket of vector fields X and Y. In particular we have $(X \wedge_g Y) = X \wedge Y$.

The Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the (0,4)-tensor G of (M, g) are defined by

$$
\begin{aligned}
R(X_1,X_2,X_3,X_4) &= g(\mathscr{R}(X_1,X_2)X_3,X_4) \, ,\\
C(X_1,X_2,X_3,X_4) &= g(\mathscr{C}(X_1,X_2)X_3,X_4) \, ,\\
G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4) \, , && (2.27)
\end{aligned}
$$

respectively, where $X_1,X_2,X_3,X_4 \in \Xi(M)$.

Let $\mathscr{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$. We define the (0,4)-tensor B by $B(X_1,X_2,X_3,X_4) = g(\mathscr{B}(X_1,X_2)X_3,X_4)$. The tensor B is said to be a *generalized curvature tensor* if

$$B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2),$$

$$B(X_1,X_2,X_3,X_4) + B(X_2,X_3,X_1,X_4) + B(X_3,X_1,X_2,X_4) = 0.$$

For a (0,k)-tensor field T, $k \geq 1$, a symmetric (0,2)-tensor field A and a generalized curvature tensor B on (M, g), we define the (0,k+2)-tensor fields $B \cdot T$ and $Q(A,T)$ by

$$
\begin{aligned}
(B \cdot T)(X_1,\ldots,X_k;X,Y) &= -T(\mathscr{B}(X,Y)X_1,X_2,\ldots,X_k)\\
&- \ldots - T(X_1,X_2,\ldots,X_{k-1},\mathscr{B}(X,Y)X_k), && (2.28)
\end{aligned}
$$

$$
\begin{aligned}
Q(A,T)(X_1,\ldots,X_k;X,Y) &= -T((X \wedge_A Y)X_1,X_2,\ldots,X_k)\\
&- \ldots - T(X_1,X_2,\ldots,X_{k-1},(X \wedge_A Y)X_k), && (2.29)
\end{aligned}
$$

respectively, where $X,Y,Z,X_1,X_2,\ldots,X_k \in \Xi(M)$.

Putting in the above formulas $\mathscr{B} = R$ or $\mathscr{B} = \mathscr{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $R \cdot C$, $R \cdot S$, $C \cdot S$, Q (g, R), Q (S, R), Q (g, C), Q (g, S) and Q (S, C) respectively.

For symmetric (0,2)-tensor E and F we define their Kulkarni-Nomizu product $E \wedge F$ by

$$\begin{aligned}(E \wedge F)(X_1,X_2,X_3,X_4) &= E(X_1,X_4)F(X_2,X_3) + E(X_2,X_3)F(X_1,X_4) \\ &- E(X_1,X_3)F(X_2,X_4) - E(X_2,X_4)F(X_1,X_3). \quad (2.30)\end{aligned}$$

For a symmetric (0,2)-tensor E and (0,k)-tensor T, $k \geq 2$, we define their Kulkarni-Nomizu product $E \wedge T$ by [14]

$$\begin{aligned}(E \wedge T)(X_1,X_2,X_3,X_4;Y_3,...,Y_k) &= E(X_1,X_4)T(X_2,X_3,Y_3,...,Y_k) \\ &+ E(X_2,X_3)T(X_1,X_4,Y_3,...,Y_k) \\ &- E(X_1,X_3)T(X_2,X_4,Y_3,...,Y_k) \\ &- E(X_2,X_4)T(X_1,X_3,Y_3,...,Y_k). \quad (2.31)\end{aligned}$$

For symmetric (0, 2)-tensors E and F we have [15]

$$Q(E, E \wedge F) = -Q(F, \bar{E}), \quad (2.32)$$

where $\bar{E} = \frac{1}{2} E \wedge E$. We also have [16]

$$E \wedge Q(E,F) = -Q(F, \bar{E}). \quad (2.33)$$

For a symmetric (0, 2)-tensor E and a skew-symmetric (0, 2)-tensor ω , we define

$$(\omega \bar{\wedge} E)_{ijhk} = \omega_{ik}E_{jh} + \omega_{jh}E_{ik} - \omega_{ih}E_{jk} - \omega_{jk}E_{ih}. \quad (2.34)$$

For skew-symmetric (0, 2)-tensors ω and τ , we define

$$(\omega \bar{\bar{\wedge}} \tau)_{ijhk} = \omega_{ik}\tau_{jh} + \omega_{jh}\tau_{ik} - \omega_{ih}\tau_{jk} - \omega_{jk}\tau_{ih} - 2(\omega_{ij}\tau_{hk} + \omega_{hk}\tau_{ij}). \quad (2.35)$$

Let (M, g) be covered by a system of charts $\{W; x^k\}$. We define by g_{ij} , R_{hijk} , S_{ij} , $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$\begin{aligned}C_{hijk} &= R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\ &+ \frac{\kappa}{(n-1)(n-2)}G_{hijk} , \quad (2.36)\end{aligned}$$

the local components of the metric tensor g, the Riemannian-Christoffel curvature tensor R, the Ricci tensor S, the tensor G and the Weyl tensor C, respectively.

Further, the tensor S^2 is defined by $S^2(X,Y) = S(\mathscr{S}X,Y)$.

The local components of the (0, 6)-tensor fields $R \cdot T$ and $Q(g,T)$, (0,4)-tensor field $T \cdot A$ on M are given by

$$\begin{aligned} (R\cdot T)_{hijklm} &= g^{rs}(T_{rijk}R_{shlm} + T_{hrjk}R_{silm} \\ &+ T_{hirk}R_{sjlm} + T_{hijr}R_{sklm}) \,, \end{aligned} \tag{2.37}$$

$$\begin{aligned} Q(g,T)_{hijklm} &= -g_{hm}T_{lijk} - g_{im}T_{hljk} - g_{jm}T_{hilk} - g_{km}T_{hijl} \\ &+ g_{hl}T_{mijk} + g_{il}T_{hmjk} + g_{jl}T_{himk} + g_{kl}T_{hijm}, \end{aligned} \tag{2.38}$$

$$(T\cdot A)_{hijk} = A^r_h T_{rijk} + A^r_i T_{rhjk}, \tag{2.39}$$

respectively, where A is a symmetric (0,2)-tensor field.

Lemma 2.1 [17]. *Any symmetric (0, 2)-tensor E on a semi-Riemannian manifold (M, g), $n \geq 4$, satisfies*

$$G\cdot G = 0, \qquad \bar{E}\cdot G = 0, \qquad (g\wedge E)\cdot G = 0, \qquad G\cdot \bar{E} = Q(g,\bar{E}),$$

$$G\cdot (g\wedge E) = Q(g, g\wedge E), \quad \bar{E}\cdot\bar{E} = -Q(E^2,\bar{E}), \quad g\wedge Q(g,E) = -Q(E,G),$$

$$(g\wedge E)\cdot E = Q(g,E^2), \quad \bar{E}\cdot(g\wedge E) + (g\wedge E)\cdot\bar{E} = -Q(E^2, g\wedge E),$$

$$(g\wedge E)\cdot(g\wedge E) = -Q(E^2,G), \quad Q(E,G) = -Q(g, g\wedge E),$$

$$Q(E, g\wedge E) = -Q(g,\bar{E}), \quad G\cdot E = Q(g,E), \quad \bar{E}\cdot E = Q(E,E^2).$$

Lemma 2.2 [18]. *Let (M, g), $n \geq 3$, be a semi-Riemannian manifold. If E_1, E_2 and F be symmetric (0, 2)-tensors at $x \in M$. Then at x we have*

$$E_1 \wedge Q(E_2,F) + E_2\wedge Q(E_1,F) = -Q(F, E_1\wedge E_2).$$

In this part we present some basic definitions of pseudosymmetric and Ricci-pseudosymmetric manifolds.

A semi-Riemannian manifold (M, g) satisfying the condition $\nabla R = 0$ is said to be *locally symmetric*. Locally symmetric manifolds form a subclass of the class of manifolds characterized by the condition

$$R\cdot R = 0, \tag{2.40}$$

where $R \cdot R$ is a (0, 6)-tensor field with the local components

$$\begin{aligned}(R\cdot R)_{hijklm} &= \nabla_m \nabla_l R_{hijk} - \nabla_l \nabla_m R_{hijk} \\ &= g^{rs}(R_{rijk}R_{shlm} + R_{hrjk}R_{silm} + R_{hirk}R_{sjlm} + R_{hijr}R_{sklm}). \quad (2.41)\end{aligned}$$

Semi-Riemannian manifolds fulfilling (2.40) are called *semisymmetric*. They are not locally symmetric, in general.

A semi-Riemannian manifold is said to be *Ricci-semisymmetric* if on M we have $R \cdot S = 0$.

A more general class of manifolds than the class of semisymmetric manifolds is the class of pseudosymmetric manifolds.

A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric* in the sense of Deszcz [13] if at every point of M the condition

$$R \cdot R = L_R Q(g,R) \quad (2.42)$$

holds on the set $\mathscr{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \; at \; x\}$, where L_R is some function on $\mathscr{U}_R$.

A semi-Riemannian manifold (M,g) is said to be *Ricci-pseudosymmetric* if at every point of M the condition

$$R \cdot S = L_S Q(g,S) \quad (2.43)$$

holds on the set $\mathscr{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \; at \; x\}$, where L_S is some function on $\mathscr{U}_S$ [3]. Every pseudosymmetric manifold is Ricci-pseudosymmetric. The converse statement is not true. The class of Ricci-pseudosymmetric manifolds is an extension of the class of Ricci-semisymmetric $(R \cdot S = 0)$ manifolds as well as of the class of pseudosymmetric manifolds. Evidently, every Ricci-semisymmetric is Ricci-pseudosymmetric. There exist various examples of Ricci-pseudosymmetric manifolds which are not pseudosymmetric.

A semi-Riemannian manifold (M,g) is said to be *Weyl-pseudosymmetric* if at every point of M the condition

$$R \cdot C = L_C Q(g,C) \quad (2.44)$$

holds on the set $\mathscr{U}_C = \{x \in M \mid C \neq 0 \; at \; x\}$, where L_C is some function on $\mathscr{U}_C$. Every pseudosymmetric manifold is Weyl-pseudosymmetric. The converse statement not true. Every Weyl-semisymmetric manifold ($R \cdot C = 0$) is Weyl-pseudosymmetric.

(2.42), (2.43), (2.44) or other conditions of this kind are called *curvature conditions of pseudosymmetry type*.

The inclusion among the above mentioned classes of manifolds can be summarized in Figure 2.1 [19].

The condition

$$R \cdot C - C \cdot R = L_1 Q(g,C) \tag{2.45}$$

holds on the set $\mathscr{U}_C = \{x \in M \mid C \neq 0 \; at \; x\}$, where L_1 is some function on $\mathscr{U}_C$.

The condition

$$R \cdot C - C \cdot R = L \, Q(g,R) \tag{2.46}$$

holds on the set $\mathscr{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \; at \; x\}$, where L is some function on $\mathscr{U}_R$.

The condition

$$R \cdot C - C \cdot R = \bar{L} \, Q(S,R) \tag{2.47}$$

holds on the set $\mathscr{U}_1 = \{x \in M \mid Q(S,R) \neq 0 \; at \; x\}$, where $\bar{L}$ is some function on $\mathscr{U}_1$.

The condition

$$R \cdot C - C \cdot R = L_2 Q(S,C) \tag{2.48}$$

holds on the set $\mathscr{U}_2 = \{x \in M \mid Q(S,C) \neq 0 \; at \; x\}$, where L_2 is some function on $\mathscr{U}_2$.

The condition

$$R \cdot R - Q(S,R) = L_1 Q(g,C) \tag{2.49}$$

holds on the set $\mathscr{U}_C = \{x \in M \mid C \neq 0 \; at \; x\}$, where L_1 is a certain function on $\mathscr{U}_C$.

This condition

$$R \cdot C = L_2 Q(S,C) \tag{2.50}$$

holds on the set $\mathscr{U}_2 = \{x \in M \mid Q(S,C) \neq 0 \ at \ x\}$, where L_2 is a certain function on $\mathscr{U}_2$.

Semi-Riemannian manifolds satisfying (2.49) and (2.50) or other conditions of this kind, described in [13] are called *manifolds of pseudosymmetry type* and also we note that curvature conditions of pseudosymmetry (2.45)- (2.48) are called *generalized Einstein metric conditions* [20].

We refer to [19] for a survey on results on semi-Riemannian manifolds satisfying such conditions. Very recently manifolds satisfying Einstein metric conditions were investigated in: [21] [22] [23] [24] [25] [26].

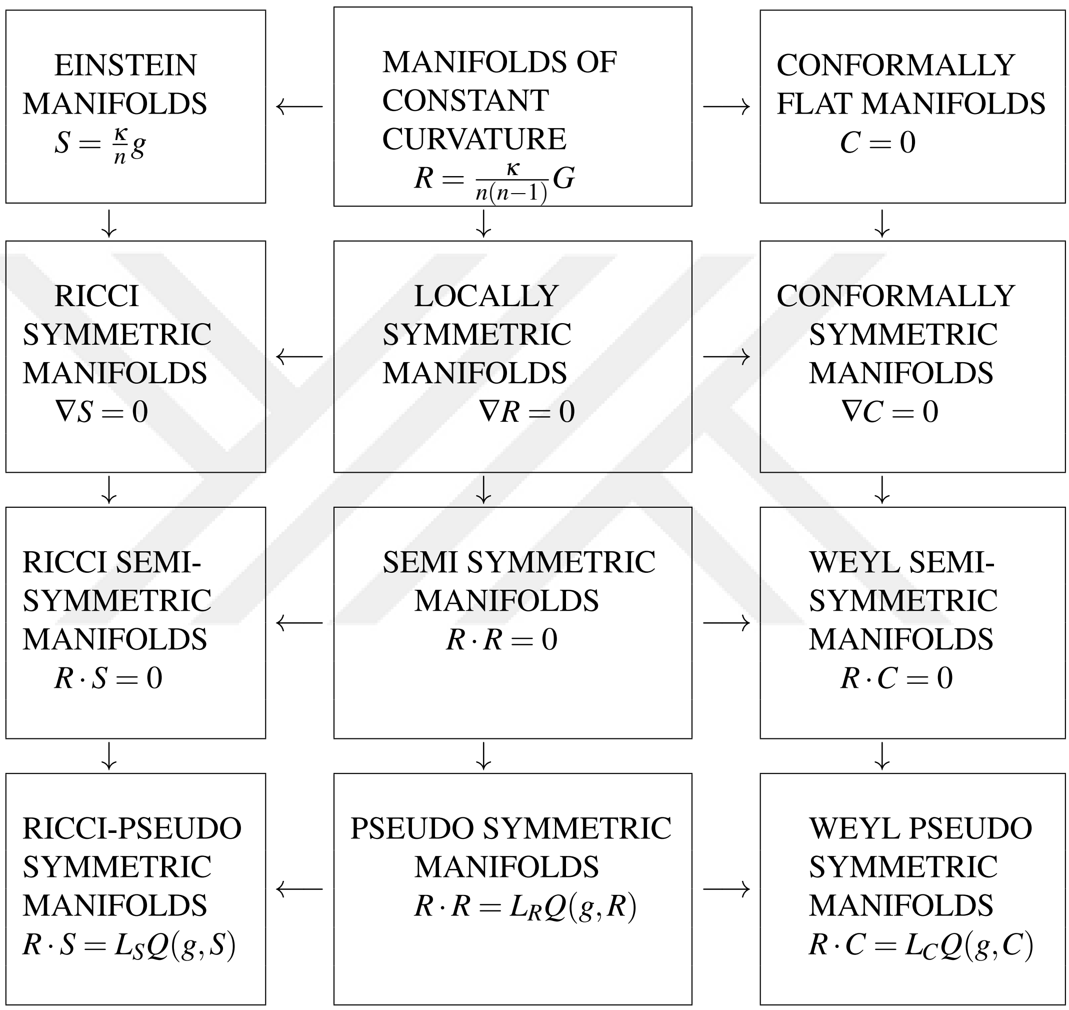

Figure 2.1 : Pseudosymmetric manifolds and some other classes of semi-Riemannian manifolds.

3. LOCALLY CONFORMAL KAEHLER MANIFOLDS

3.1 Locally Conformal Kaehler Manifolds

Let M be a real 2n-dimensional Hermitian manifold with structure (J,g), where J is the almost complex structure and g is the Hermitian metric. The manifold M is called a *locally conformal Kaehler manifold (an l.c.K-manifold)* if each point p in M has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that

$$g^* = e^{-2\rho} g \,|_{\mathrm{U}}$$

is a Kaehlerian metric on U. Especially, if we can take $U = M$, then the manifold M is said to be *globally conformal Kaehler*.

Proposition 3.1 [2]. *A Hermitian manifold $M(J,g)$ is an l.c.K-manifold if and only if*

$$\nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki} \, , \tag{3.1}$$

$$\nabla_i \alpha_j = \nabla_j \alpha_i \qquad (or \quad J_{jr} \nabla_i \beta^r = J_{ir} \nabla_j \beta^r),$$

where α is a global closed 1-form and

$$\beta_i = \alpha^r J_{ri}. \tag{3.2}$$

In l.c.K-manifolds, we have the following formulas [2]:

$$\nabla_j \beta_i = -\beta_j \alpha_i + \beta_i \alpha_j - \|\alpha\|^2 J_{ji} + \nabla_j \alpha^r J_{ri} \, , \tag{3.3}$$

$$\nabla_r \beta^r = 0 \, , \tag{3.4}$$

$$\nabla_k \beta^r J_{ir} = \nabla_i \beta^r J_{kr} \, , \tag{3.5}$$

$$2(n-1)\alpha_i = J_{ri} \nabla_s J^{rs} = J^{sr} \nabla_s J_{ri} \, , \tag{3.6}$$

$$2(n-1)\beta_i = \nabla^r J_{ri} \, , \tag{3.7}$$

$$\alpha^r \nabla_r J_{ij} = \beta^r \nabla_r J_{ij} = 0 \, , \tag{3.8}$$

$$\alpha^r \nabla_j J_{ri} = -\beta_j \alpha_i + \alpha_j \beta_i - \|\alpha\|^2 J_{ji} \tag{3.9}$$

and

$$\beta^r \nabla_j J_{ri} = \beta_j \beta_i + \alpha_j \alpha_i - \|\alpha\|^2 g_{ji} \ , \tag{3.10}$$

where $\|\alpha\|$ denotes the length of the Lee form α with respect to g.

From the above equations we obtain

$$\begin{aligned} \nabla_k \nabla_h J_{ij} - \nabla_h \nabla_k J_{ij} &= P_{kr} J^r_j g_{hi} - P_{kr} J^r_i g_{hj} - P_{hr} J^r_j g_{ki} + P_{hr} J^r_i g_{kj} \\ &- P_{kj} J_{hi} + P_{ki} J_{hj} + P_{hj} J_{ki} - P_{hi} J_{kj}, \end{aligned}$$

where

$$P_{ij} = -\nabla_i \alpha_j - \alpha_i \alpha_j + \frac{\|\alpha\|^2}{2} g_{ij}. \tag{3.11}$$

We note that $P_{ij} = P_{ji}$ and $\|\alpha\|^2 = \alpha_r \alpha^r$.

Using the Ricci identity, we get

$$\begin{aligned} -R_{hkir} J^r_j + R_{hkjr} J^r_i &= P_{kr} J^r_j g_{hi} - P_{kr} J^r_i g_{hj} - P_{hr} J^r_j g_{ki} + P_{hr} J^r_i g_{kj} \\ &- P_{kj} J_{hi} + P_{ki} J_{hj} + P_{hj} J_{ki} - P_{hi} J_{kj} \end{aligned} \tag{3.12}$$

and then

$$\begin{aligned} R_{hkrs} J^r_j J^s_i &= R_{hkji} + P_{ki} g_{hj} - P_{kj} g_{hi} + P_{hj} g_{ki} - P_{hi} g_{kj} \\ &+ P_{kr} J^r_i J_{hj} - P_{kr} J^r_j J_{hi} + P_{hr} J^r_j J_{ki} - P_{hr} J^r_i J_{kj}. \end{aligned} \tag{3.13}$$

The tensor field P_{ij} is hybrid, i.e.,

$$P_{rs} J^r_i J^s_j = P_{ij} \qquad or \qquad P_{ir} J^r_j + P_{jr} J^r_i = 0 \ .$$

Now transvecting (3.12) with g^{ik} we have

$$-S_{hr} J^r_j + R_{hkjr} J^r_i g^{ik} = -(2n-3) P_{hr} J^r_j - P_{jr} J^r_h + (P_{rs} g^{rs}) J_{hj}$$

and so

$$-R_{hkrj} J^{kr} = S_{hr} J^r_j - (2n-3) P_{hr} J^r_j - P_{jr} J^r_h + tr\, P J_{hj}.$$

Using $H_{jh} = R_{hkrj} J^{kr} = -\frac{1}{2} R_{hjkr} J^{kr}$ [27] [28] and $H_{jh} = -H_{hj}$, we get

$$H_{hj} = S_{hr} J^r_j - (2n-3) P_{hr} J^r_j - P_{jr} J^r_h + tr\, P J_{hj}. \tag{3.14}$$

Using the skew-symmetric property of H in (3.14) , we have [2]

$$S_{jr}J_i^r + S_{ir}J_j^r = 2(n-1)(P_{jr}J_i^r + P_{ir}J_j^r) \tag{3.15}$$

which means that, in a real 2n-dimensional (n >1) l.c.K-manifold M, the tensor field P is hybrid if and only if the Ricci tensor is hybrid.

Theorem 3.2 [8]. *For an almost Hermitian manifold $M(J,g)$, the tensor*

$$\begin{aligned}(HR)(X,Y,Z,W) &= \frac{1}{16}\Big[3\Big[R(X,Y,Z,W)+R(JX,JY,Z,W)+R(X,Y,JZ,JW)\\ &+ R(JX,JY,JZ,JW)\Big]-R(X,Z,JW,JY)-R(JX,JZ,W,Y)\\ &- R(X,W,JY,JZ)-R(JX,JW,Y,Z)+R(JX,Z,JW,Y)\\ &+ R(X,JZ,W,JY)+R(JX,W,Y,JZ)+R(X,JW,JY,Z)\Big]\end{aligned} \tag{3.16}$$

is a curvature tensor of Kaehler type.

The tensor (3.16) is said to be the *holomorphic curvature tensor* in an almost Hermitian manifold.

It is easy to see that

$$(HR)(X,Y,Z,W) = -(HR)(Y,X,Z,W)\ , \tag{3.17}$$

$$(HR)(X,Y,Z,W) = -(HR)(X,Y,W,Z)\ , \tag{3.18}$$

$$(HR)(X,Y,Z,W) = (HR)(Z,W,X,Y)\ , \tag{3.19}$$

$$(HR)(X,Y,Z,W)+(HR)(X,Z,W,Y)+(HR)(X,W,Y,Z) = 0\ , \tag{3.20}$$

as well as

$$(HR)(X,Y,JZ,JW) = (HR)(X,Y,Z,W), \tag{3.21}$$

$$(HR)(X,JX,JX,X) = R(X,JX,JX,X). \tag{3.22}$$

With respect to the local coordinates (3.16) reads

$$\begin{aligned}(HR)_{ijhk} &= \frac{1}{16}\Big[3(R_{ijhk}+R_{rshk}J_i^rJ_j^s+R_{ijrs}J_h^rJ_k^s+R_{rspq}J_i^rJ_j^sJ_h^pJ_k^q)\\ &- R_{ihrs}J_k^rJ_j^s-R_{rskj}J_i^rJ_h^s-R_{ikrs}J_j^rJ_h^s-R_{rsjh}J_i^rJ_k^s\\ &+ R_{rhsj}J_i^rJ_k^s+R_{irks}J_h^rJ_j^s+R_{rkjs}J_i^rJ_h^s+R_{irsh}J_k^rJ_j^s\Big].\end{aligned} \tag{3.23}$$

Theorem 3.3 [9]. *The holomorphic curvature tensor of an l.c.K-manifold has the form*

$$\begin{aligned}(HR)_{ijhk} &= R_{ijhk}+\frac{1}{8}\Big[g_{ih}(7P_{jk}-P_{rs}J_j^rJ_k^s)-g_{ik}(7P_{jh}-P_{rs}J_j^rJ_h^s)\\ &+ g_{jk}(7P_{ih}-P_{rs}J_i^rJ_h^s)-g_{jh}(7P_{ik}-P_{rs}J_i^rJ_k^s)\\ &+ J_{ih}(P_{jr}J_k^r-P_{kr}J_j^r)-J_{ik}(P_{jr}J_h^r-P_{hr}J_j^r)\\ &+ J_{jk}(P_{ir}J_h^r-P_{hr}J_i^r)-J_{jh}(P_{ir}J_k^r-P_{kr}J_i^r)\\ &+ 2J_{ij}(P_{hr}J_k^r-P_{kr}J_h^r)+2J_{hk}(P_{ir}J_j^r-P_{jr}J_i^r)\Big]. \qquad (3.24)\end{aligned}$$

Theorem 3.4 [2]. *Let $M(J,g,\alpha)$ be an l.c.K-manifold such that P is hybrid. If the holomorphic sectional curvature at $p\in M$ is constant c, then*

$$\begin{aligned}R_{ijhk} &= \frac{c}{4}(g_{ik}g_{jh}-g_{ih}g_{jk}+J_{ik}J_{jh}-J_{ih}J_{jk}-2J_{ij}J_{hk})\\ &+ \frac{3}{4}(g_{ik}P_{jh}+g_{jh}P_{ik}-g_{ih}P_{jk}-g_{jk}P_{ih})\\ &- \frac{1}{4}(\widetilde{P}_{ik}J_{jh}+\widetilde{P}_{jh}J_{ik}-\widetilde{P}_{ih}J_{jk}-\widetilde{P}_{jk}J_{ih}-2\widetilde{P}_{ij}J_{hk}-2\widetilde{P}_{hk}J_{ij}) \qquad (3.25)\end{aligned}$$

at $p\in M$, where $\widetilde{P}_{ij}=-P_{ir}J_j^r$.

Theorem 3.5 [9]. *An l.c.K-manifold has constant holomorphic sectional curvature if and only if its curvature tensor can be expressed in the form*

$$\begin{aligned}R_{ijhk} &= \frac{c}{4}(g_{ik}g_{jh}-g_{ih}g_{jk}+J_{ik}J_{jh}-J_{ih}J_{jk}-2J_{ij}J_{hk})\\ &+ \frac{1}{8}\Big[g_{ik}(7P_{jh}-P_{rs}J_j^rJ_h^s)-g_{ih}(7P_{jk}-P_{rs}J_j^rJ_k^s)\\ &+ g_{jh}(7P_{ik}-P_{rs}J_i^rJ_k^s)-g_{jk}(7P_{ih}-P_{rs}J_i^rJ_h^s)\\ &+ J_{ik}(P_{jr}J_h^r-P_{hr}J_j^r)-J_{ih}(P_{jr}J_k^r-P_{kr}J_j^r)\\ &+ J_{jh}(P_{ir}J_k^r-P_{kr}J_i^r)-J_{jk}(P_{ir}J_h^r-P_{hr}J_i^r)\\ &- 2J_{ij}(P_{hr}J_k^r-P_{kr}J_h^r)-2J_{hk}(P_{ir}J_j^r-P_{jr}J_i^r)\Big]\ . \qquad (3.26)\end{aligned}$$

If the tensor P is hybrid, the relation (3.24) reduces to

$$\begin{aligned}(HR)_{ijhk} &= R_{ijhk}+\frac{3}{4}(P_{jk}g_{ih}+P_{ih}g_{jk}-P_{jh}g_{ik}-P_{ik}g_{jh})\\ &+ \frac{1}{4}(J_{ih}P_{jr}J_k^r+J_{jk}P_{ir}J_h^r-J_{ik}P_{jr}J_h^r-J_{jh}P_{ir}J_k^r\\ &+ 2J_{hk}P_{ir}J_j^r+2J_{ij}P_{hr}J_k^r). \qquad (3.27)\end{aligned}$$

3.2 Locally Conformal Kaehler Space Forms

In this section, some properties of locally conformal Kaehler space forms are presented.

An l.c.K-manifol M is called *an l.c.K-space form* if the holomorphic sectional curvature of the section $\{X, JX\}$ at each point of M has a constant value. Let M(c) be an l.c.K-space form with constant holomorphic sectional curvature c, then the Riemannian curvature tensor R with respect to g can be expressed in the form (3.26).

Theorem 3.6 [29]. *Let $M(c)$ be a 2n-dimensional l.c.K-space form. If the tensor field P is proportional to g and $tr\, P$ is constant, then $M(c)$ is Einstein.*

Proof. Contracting (3.26) with g^{hk}, we have

$$4S_{ij} = [2(n+1)c + 3\, trP]g_{ij} + (7n-10)P_{ij} - (n+2)P_{rs}J_i^r J_j^s. \tag{3.28}$$

If the tensor field P is proportional to g and $tr\, P$ is constant, then P is written by

$$P_{ij} = \frac{tr\, P}{2n} g_{ij}\, . \tag{3.29}$$

Substituting (3.29) into (3.28), we obtain

$$S_{ij} = \left[\frac{1}{2}(n+1)c + \frac{3(n-1)}{2n} tr\, P\right] g_{ij}\, , \tag{3.30}$$

which means that $M(c)$ is Einstein. ■

Corollary 3.7 [29]. *A real 2n-dimensional Einstein l.c.K-space form $M(c)$ is a complex space form if $tr\, P = 0$.*

Theorem 3.8 [29]. *Let $M(c)$ be a 2n-dimensional l.c.K-space form. If κ is constant and $\|\alpha\|$ is non-zero constant, then*

$$\left[(\nabla_j \nabla_r \alpha_s)\alpha^r + 2(\nabla_j \alpha_s)\|\alpha\|^2\right] J^{js} - (\nabla_j \alpha_r)\beta^r \beta^j = 0. \tag{3.31}$$

Proof. Let $M(c)$ be an l.c.K-space form with constant holomorphic sectional curvature c. Since the scalar curvature $\kappa = n(n+1)c + 3(n-1)\, tr\, P$ is constant, then

$$tr\, P = -\nabla_r \alpha^r + (n-1)\|\alpha\|^2 \tag{3.32}$$

is constant. Now differentiating (3.28), we get

$$\begin{aligned} 4\nabla_k S_{jh} &= (7n-10)\nabla_k P_{jh} - (n+2)\Big[(\nabla_k P_{rs})J_j^r J_h^s + (\nabla_k J_j^r)P_{rs}J_h^s \\ &+ (\nabla_k J_h^s)P_{rs}J_j^r\Big]. \end{aligned} \tag{3.33}$$

Substituting (3.11) into (3.33), using (2.5) and the equality $\nabla_j \alpha_i = \nabla_i \alpha_j$, we have

$$\begin{aligned} 4(\nabla_k S_{jh} - \nabla_j S_{kh}) &= (7n-10)\Big\{R_{kjh}^r \alpha_r + (\nabla_j \alpha_h)\alpha_k - (\nabla_k \alpha_h)\alpha_j \\ &+ \frac{1}{2}[(\nabla_k \|\alpha\|^2)g_{jh} - (\nabla_j \|\alpha\|^2)g_{kh}]\Big\} \\ &- (n+2)\Big[(\nabla_k P_{rs})J_j^r J_h^s - (\nabla_j P_{rs})J_k^r J_h^s + (\nabla_k J_j^r)P_{rs}J_h^s \\ &+ (\nabla_k J_h^s)P_{rs}J_j^r - (\nabla_j J_k^r)P_{rs}J_h^s - (\nabla_j J_h^s)P_{rs}J_k^r\Big]. \end{aligned} \tag{3.34}$$

Contracting (3.34) with g^{jh} and taking into account $2\nabla_r S_j^r = \nabla_j \kappa$ [28], we obtain

$$\begin{aligned} (7n & -10)\Big[S_k^r \alpha_r + (\nabla_j \alpha^j)\alpha_k\Big] - (n+2)\Big[-(\nabla_j P_{rs})J_k^r J_h^s g^{jh} \\ &+ (\nabla_k J_j^r)P_{rs}J_h^s g^{jh} + (\nabla_k J_h^s)P_{rs}J_j^r g^{jh} - (\nabla_j J_k^r)P_{rs}J_h^s g^{jh} \\ &- (\nabla_j J_h^s)P_{rs}J_k^r g^{jh}\Big] = 0, \end{aligned} \tag{3.35}$$

where

$$\nabla_k J_j^r = -\beta_j \delta_k^r + \beta^r g_{kj} - \alpha_j J_k^r + \alpha^r J_{kj} \ . \tag{3.36}$$

Now contracting (3.28) with g^{hr} and transvecting with α_r, we get

$$4S_k^r \alpha_r = [2(n+1)c + 3\, tr\, P]\alpha_k + 6(n-2)P_{kh}\alpha^h \ . \tag{3.37}$$

From (3.32), we get

$$3\, tr\, P\ \alpha_k = -3(\nabla_r \alpha^r)\alpha_k + 3(n-1)\|\alpha\|^2 \alpha_k \tag{3.38}$$

and transvecting P_{kh} with α^h , we obtain

$$P_{kh}\alpha^h = -\frac{1}{2}\nabla_k \|\alpha\|^2 - \frac{1}{2}\|\alpha\|^2 \alpha_k \ . \tag{3.39}$$

Substituting (3.37), (3.38) and (3.39) into (3.35) and transvecting with β^k, we find (3.31). ■

3.3 Submanifolds of Locally Conformal Kaehler Manifolds

Let N be a real m-dimensional manifold isometrically immersed in a real 2n-dimensional l.c.K-manifold M. If the manifold M is covered by a system of coordinate neighborhoods $\{V, v^i\}$ and N is covered by a system of coordinate neighborhoods $\{U, u^\lambda\}$, where here and in the sequel the indices $i, j, h, k, ...$ run over the range $1, 2, ..., 2n$ and $\nu, \mu, \lambda ...$ run over the range 1, 2, ..., m, then the submanifold N can be locally represented by

$$v^i = v^i(u^\lambda)\ . \tag{3.40}$$

In the following, we shall identify vector fields in N and their image under the differential mapping. We put

$$B^i_\lambda = \partial_\lambda v^i = \frac{\partial v^i}{\partial u^\lambda}.$$

Let $g_{\mu\lambda}$ be the induced metric on N, then we have

$$g_{\mu\lambda} = g_{ji} B^j_\mu B^i_\lambda\ . \tag{3.41}$$

Let ξ^i_x be a system of orthogonal normal vectors, where the indices $x, y, z, ...$ run over the range $1, 2, ..., 2n-m$. Then we have

$$g_{ji} B^j_\lambda \xi^i_x = 0. \tag{3.42}$$

In local coordinates, the equation of Gauss and Codazzi are given by

$$R_{ijhk} B^i_\omega B^j_\nu B^h_\mu B^k_\lambda = R_{\omega\nu\mu\lambda} - h^x_{\omega\lambda} h_{\nu\mu x} + h^x_{\omega\mu} h_{\nu\lambda x}\ , \tag{3.43}$$

$$R_{ijhk} B^i_\omega B^j_\nu B^h_\mu \xi^k_x = \nabla_\omega h_{\nu\mu x} - \nabla_\nu h_{\omega\mu x}\ , \tag{3.44}$$

respectively, where $h^x_{\mu\lambda}$ denote the second fundamental tensor.

Now the transformation $J^k_h B^h_\lambda$ of B^h_λ by J^k_h can be written as

$$J^k_h B^h_\lambda = \widetilde{J}^\varepsilon_\lambda B^k_\varepsilon + \widetilde{J}^x_\lambda \xi^k_x\ , \tag{3.45}$$

where $\widetilde{J}^\varepsilon_\lambda$ and $\widetilde{J}^x_\lambda$ are a tensor field of type (1,1) and a normal bundle valued 1-form in N, respectively.

The transformation $J_h^k \xi_y^h$ of ξ_y^h by J_h^k can be written as

$$J_h^k \xi_y^h = -\widetilde{J}_y^{\varepsilon} B_{\varepsilon}^k + \widetilde{J}_y^x \xi_x^k \,, \tag{3.46}$$

where $\widetilde{J}_y^{\varepsilon}$ and $\widetilde{J}_y^x$ are a tangent bundle valued 1-form and a tensor field of type (1,1) of the normal bundle in N, respectively.

A submanifold N is called *invariant* if $JT_pN = T_pN$ for any point $p \in N$, where T_pN denotes the tangent vector space of N at p in N, that is, a real m-dimensional submanifold N of an l.c.K-manifold M is said to be *invariant* if the tangent space at each point of N is invariant under the action of J.

For an invariant submanifold N, we have

$$\widetilde{J}_{\mu}^x = 0 \,. \tag{3.47}$$

Using (3.45) and (3.46) , we have

$$\widetilde{J}_{\mu}^{\varepsilon} \widetilde{J}_{\varepsilon}^{\lambda} = -\delta_{\mu}^{\lambda} \quad , \quad \widetilde{J}_y^x \widetilde{J}_x^z = -\delta_y^z \,.$$

Moreover

$$g_{\varepsilon\gamma} \widetilde{J}_{\mu}^{\varepsilon} \widetilde{J}_{\lambda}^{\gamma} = g_{\mu\lambda} \,.$$

Next, we decompose the Lee vector field α^k as follows

$$\alpha^k = \alpha^{\varepsilon} B_{\varepsilon}^k + \alpha^x \xi_x^k \,, \tag{3.48}$$

where α^{ε} and α^x are the tangential and the normal part of α^k respectively.

For an invariant submanifold N of an l.c.K-manifold M satisfying $\alpha^x = 0$, identically, that is , the Lee vector field α^k is always tangent to N, say $\alpha^k = \alpha^{\varepsilon} B_{\varepsilon}^k$, we have the following :

$$\nabla_{\nu} \widetilde{J}_{\mu\lambda} = -\beta_{\mu} g_{\nu\lambda} + \beta_{\lambda} g_{\nu\mu} - \alpha_{\mu} \widetilde{J}_{\nu\lambda} + \alpha_{\lambda} \widetilde{J}_{\nu\mu} \,, \tag{3.49}$$

where $\beta_{\mu} = -\alpha_{\varepsilon} \widetilde{J}_{\mu}^{\varepsilon}$.

Proposition 3.9 [7]. *An invariant submanifold N of an l.c.K-manifold M in which Lee vector field α^k is tangent to N is an l.c.K-manifold with structure $(\widetilde{J}_{\mu}^{\lambda}, g_{\mu\lambda}, \alpha_{\lambda})$.*

Theorem 3.10 [7]. *An invariant submanifold N of an l.c.K-manifold M is minimal, that is,*

$$trace(h_{\mu x}^{\lambda}) = h_{\mu\lambda x} g^{\mu\lambda} = 0 \tag{3.50}$$

if and only if the Lee vector field α^λ is tangent to N.

Let M(c) be an l.c.K-space form with constant holomorphic sectional curvature c and N be a real m-dimensional invariant submanifold of M(c).

Theorem 3.11. *Let N be a real m-dimensional minimal invariant submanifold of an l.c.K-space form M(c). Then we have*

$$4\,\kappa \leqslant m(m+2)c+6(m-2)\,tr\,p\,, \tag{3.51}$$

where κ is the scalar curvature with respect to $g_{\nu\mu}$. The equality holds if and only if the submanifold N is totally geodesic.

Proof . Transvecting (3.26) with $B^i_\omega B^j_\nu B^h_\mu B^k_\lambda$ and using (3.41), (3.43), (3.45) and in view of Theorem 3.10., we obtain

$$\begin{aligned}
4R_{\omega\nu\mu\lambda} &= c(g_{\mu\nu}g_{\omega\lambda}-g_{\mu\omega}g_{\nu\lambda}+\widetilde{J}_{\nu\mu}\widetilde{J}_{\omega\lambda}-\widetilde{J}_{\omega\mu}\widetilde{J}_{\nu\lambda}-2\widetilde{J}_{\omega\nu}\widetilde{J}_{\mu\lambda})\\
&+\frac{1}{2}\Big\{g_{\omega\lambda}(7p_{\nu\mu}-p_{\varepsilon\gamma}\widetilde{J}^{\varepsilon}_{\nu}\widetilde{J}^{\gamma}_{\mu})-g_{\omega\mu}(7p_{\nu\lambda}-p_{\varepsilon\gamma}\widetilde{J}^{\varepsilon}_{\nu}\widetilde{J}^{\gamma}_{\lambda})\\
&+g_{\nu\mu}(7p_{\omega\lambda}-p_{\varepsilon\gamma}\widetilde{J}^{\varepsilon}_{\omega}\widetilde{J}^{\gamma}_{\lambda})-g_{\nu\lambda}(7P_{\omega\mu}-p_{\varepsilon\gamma}\widetilde{J}^{\varepsilon}_{\omega}\widetilde{J}^{\gamma}_{\mu})\\
&+\widetilde{J}_{\omega\lambda}(p_{\nu\varepsilon}\widetilde{J}^{\varepsilon}_{\mu}-p_{\mu\varepsilon}\widetilde{J}^{\varepsilon}_{\nu})-\widetilde{J}_{\omega\mu}(p_{\nu\varepsilon}\widetilde{J}^{\varepsilon}_{\lambda}-p_{\lambda\varepsilon}\widetilde{J}^{\varepsilon}_{\nu})\\
&+\widetilde{J}_{\nu\mu}(p_{\omega\varepsilon}\widetilde{J}^{\varepsilon}_{\lambda}-p_{\lambda\varepsilon}\widetilde{J}^{\varepsilon}_{\omega})-\widetilde{J}_{\nu\lambda}(p_{\omega\varepsilon}\widetilde{J}^{\varepsilon}_{\mu}-p_{\mu\varepsilon}\widetilde{J}^{\varepsilon}_{\omega})\Big\}\\
&-\widetilde{J}_{\omega\nu}(p_{\mu\varepsilon}\widetilde{J}^{\varepsilon}_{\lambda}-p_{\lambda\varepsilon}\widetilde{J}^{\varepsilon}_{\mu})-\widetilde{J}_{\mu\lambda}(p_{\omega\varepsilon}\widetilde{J}^{\varepsilon}_{\nu}-p_{\nu\varepsilon}\widetilde{J}^{\varepsilon}_{\omega})\\
&+4(h^x_{\omega\lambda}h_{\nu\mu x}-h^x_{\omega\mu}h_{\nu\lambda x})\,,
\end{aligned} \tag{3.52}$$

where

$$p_{\nu\mu}=P_{ji}B^j_\nu B^i_\mu=-\nabla_\nu\alpha_\mu-\alpha_\nu\alpha_\mu+\frac{\|\alpha\|^2}{2}g_{\nu\mu}\,. \tag{3.53}$$

Contracting (3.52) with $g^{\omega\lambda}$ and using (3.50) , we get

$$4S_{\nu\mu}=\{(m+2)c+3\,tr\,p\}g_{\nu\mu}+(\frac{7m}{2}-10)p_{\nu\mu}-(\frac{m}{2}+2)p_{\varepsilon\gamma}\widetilde{J}^{\varepsilon}_{\nu}\widetilde{J}^{\gamma}_{\mu}-4h^x_{\omega\mu}h^{\omega}_{\nu x}\,, \tag{3.54}$$

where $S_{\nu\mu}$ denotes the Ricci tensor with respect to $g_{\nu\mu}$.

Transvecting (3.54) with $g^{\nu\mu}$, we get

$$4\,\kappa=m(m+2)c+6(m-2)\,tr\,p-4h^x_{\omega\mu}h^{\omega\mu}_x\,.$$

and so

$$4(\kappa+\|h\|^2)=m(m+2)c+6(m-2)\,tr\,p\,, \tag{3.55}$$

where $\|h\|$ denotes the length of the second fundamental tensor $h_{\nu\mu x}$. By virtue of (3.55), we have (3.51).

If the equality holds in (3.51), then by using (3.55), we get

$$\|h\| = 0 \qquad and \qquad h_{\nu\mu x} = 0$$

which means that the submanifold N is totally geodesic.
Conversely, if the submanifold N is totally geodesic, then $h_{\nu\mu x} = 0$ and $\|h\| = 0$ and using (3.55) the equality holds in the equation (3.51). ■

Theorem 3.12. *Let N be a real m-dimensional invariant closed minimal submanifold of an l.c.K-space form M(c). Then we have*

$$4\int_N \kappa\, dN \leqslant m(m+2)c\, Vol\, N + 3(m-2)^2 \int_N \|\alpha\|^2 dN\,, \tag{3.56}$$

where dN and Vol N denote the volume element and the volume of N, respectively. The equality holds if and only if the submanifold N is totally geodesic.

Proof. By transvecting (3.53) with $g^{\nu\mu}$, we find

$$tr\, p = -\nabla_\nu \alpha^\nu + \frac{m-2}{2}\|\alpha\|^2 \tag{3.57}$$

and substituting (3.57) into (3.55) , we get

$$4\,\kappa = m(m+2)c - 6(m-2)\nabla_\nu \alpha^\nu + 3(m-2)^2\|\alpha\|^2 - 4\|h\|^2\,. \tag{3.58}$$

Since the submanifold is compact, using Green's Theorem , we have

$$4\int_N \kappa\, dN = m(m+2)c\, Vol\, N + 3(m-2)^2 \int_N \|\alpha\|^2 dN - 4\int_N \|h\|^2 dN \tag{3.59}$$

and so (3.56).

If the submanifold N is totally geodesic, then $\|h\| = 0$ and by virtue of (3.59), we write

$$4\int_N \kappa\, dN = m(m+2)c\, Vol\, N + 3(m-2)^2 \int_N \|\alpha\|^2 dN\,.$$

Hence, the equality holds in (3.56).
Conversely if the equality holds in (3.56), from (3.59), we obtain $\|h\| = 0$, that is, the submanifold N is totally geodesic. ■

Theorem 3.13. *Let N be a minimal invariant submanifold of an l.c.K- space form M(c) such that $R_{\omega\nu\mu\lambda}$ is tangent to N if and only if*

$$h^{\varepsilon}_{\lambda x}\alpha_\varepsilon = 0 \qquad and \qquad h^{\varepsilon}_{\lambda y}\alpha_\varepsilon = 0. \tag{3.60}$$

Proof. Transvecting (3.26) with $B^i_\omega B^j_\nu B^h_\mu \xi^k_x$, we get

$$\begin{aligned} 4R_{ijhk}B^i_\omega B^j_\nu B^h_\mu \xi^k_x &= \frac{1}{8}\Big[g_{\omega\mu}(7h^\varepsilon_{\nu x}\alpha_\varepsilon - h^\varepsilon_{\gamma y}\alpha_\varepsilon \widetilde{J}^\gamma_\nu \widetilde{J}^y_x) - g_{\nu\mu}(7h^\varepsilon_{\omega x}\alpha_\varepsilon - h^\varepsilon_{\gamma y}\alpha_\varepsilon \widetilde{J}^\gamma_\omega \widetilde{J}^y_x) \\ &+ \widetilde{J}_{\omega\mu}(h^\varepsilon_{\nu y}\alpha_\varepsilon \widetilde{J}^y_x - h^\varepsilon_{\gamma x}\alpha_\varepsilon \widetilde{J}^\gamma_\nu) - \widetilde{J}_{\nu\mu}(h^\varepsilon_{\omega y}\alpha_\varepsilon \widetilde{J}^y_x - h^\varepsilon_{\gamma x}\alpha_\varepsilon \widetilde{J}^\gamma_\omega) \\ &- 2\widetilde{J}_{\omega\nu}(-h^\varepsilon_{\mu y}\alpha_\varepsilon \widetilde{J}^y_x - h^\varepsilon_{\gamma x}\alpha_\varepsilon \widetilde{J}^\gamma_\mu)\Big]. \end{aligned} \tag{3.61}$$

Since $R_{\omega\nu\mu\lambda}$ is tangent to N, in view of (3.44), we get $\nabla_\omega h_{\nu\mu x} - \nabla_\nu h_{\omega\mu x} = 0$ and so

$$\begin{aligned} &\tfrac{1}{8}\Big[(7g_{\omega\mu}\delta^\lambda_\nu - 7g_{\nu\mu}\delta^\lambda_\omega - \widetilde{J}_{\omega\mu}\widetilde{J}^\gamma_\nu\delta^\lambda_\gamma + \widetilde{J}_{\nu\mu}\widetilde{J}^\gamma_\omega\delta^\lambda_\gamma + 2\widetilde{J}_{\omega\nu}\widetilde{J}^\gamma_\mu\delta^\lambda_\gamma)h^\varepsilon_{\lambda x}\alpha_\varepsilon \\ &+ (-g_{\omega\mu}\widetilde{J}^\gamma_\nu\widetilde{J}^y_x\delta^\lambda_\gamma + g_{\nu\mu}\widetilde{J}^\gamma_\omega\widetilde{J}^y_x\delta^\lambda_\gamma + \widetilde{J}_{\omega\mu}\widetilde{J}^y_x\delta^\lambda_\nu - \widetilde{J}_{\nu\mu}\widetilde{J}^y_x\delta^\lambda_\omega \\ &+ 2\widetilde{J}_{\omega\nu}\widetilde{J}^y_x\delta^\lambda_\mu)h^\varepsilon_{\lambda y}\alpha_\varepsilon\Big] = 0 \end{aligned} \tag{3.62}$$

and so we get (3.60).

Conversely, if (3.60) holds in (3.62), we have

$$4R_{ijhk}B^i_\omega B^j_\nu B^h_\mu \xi^k_x = 0,$$

and so by using (3.44) we get

$$\nabla_\omega h_{\nu\mu x} - \nabla_\nu h_{\omega\mu x} = 0,$$

that is, $R_{\omega\nu\mu\lambda}$ is tangent to N. ■

3.4 Sato's Form of the Holomorphic Curvature Tensor

In this section, using the Sato's form of the holomorphic curvature tensor in an almost Hermitian manifold we determine the Sato's form of the holomorphic curvature tensor in an l.c.K-manifold.

The curvature tensor of an almost Hermitian manifold of constant holomorphic sectional curvature c is given by [30]

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + J(X,W)J(Y,Z) \\ &- J(X,Z)J(Y,W) - 2J(X,Y)J(Z,W)] \\ &= \frac{1}{96}\Big\{26[G(X,Y,Z,W) - G(Z,W,X,Y)] - 6[G(JX,JY,JZ,JW) \\ &+ G(JZ,JW,JX,JY)] + 13[G(X,Z,Y,W) + G(Y,W,X,Z) \end{aligned}$$

$$
\begin{aligned}
& - \quad G(X,W,Y,Z) - G(Y,Z,X,W)] - 3[G(JX,JZ,JY,JW) \\
& + \quad G(JY,JW,JX,JZ) - G(JX,JW,JY,JZ) - G(JY,JZ,JX,JW)] \\
& + \quad 4[G(X,JY,Z,JW) + G(JX,Y,JZ,W)] + 2[G(X,JZ,Y,JW) \\
& + \quad G(JX,Z,JY,W) - G(X,JW,Y,JZ) - G(JX,W,JY,Z)]\Big\}, \qquad (3.63)
\end{aligned}
$$

where

$$G(X,Y,Z,W) = R(X,Y,Z,W) - R(X,Y,JZ,JW). \qquad (3.64)$$

Theorem 3.14 [29]. *The Sato's form of the holomorphic curvature tensor of an l.c.K-manifold has the form*

$$
\begin{aligned}
(HR)_{ijhk} \quad = \quad & \frac{13}{24}[P_{kj}g_{hi} - P_{ki}g_{hj} + P_{hi}g_{kj} - P_{hj}g_{ki} \\
+ \quad & P_{kr}J^r_jJ_{hi} - P_{kr}J^r_iJ_{hj} + P_{hr}J^r_iJ_{kj} - P_{hr}J^r_jJ_{ki}]. \qquad (3.65)
\end{aligned}
$$

Proof. Substituting (3.64) into (3.63), using (3.16) and the Bianchi identity we obtain

$$(HR)(X,Y,Z,W) = \frac{1}{24}\Big\{13[-R(X,Y,Z,W) + R(JX,JY,Z,W)]\Big\}. \qquad (3.66)$$

The tensor (3.66) is said to be the Sato's form of the holomorphic curvature tensor.

Now substituting (3.13) into (3.66) , we get (3.65). ■

4. PSEUDOSYMMETRIC LOCALLY CONFORMAL KAEHLER SPACE FORMS

4.1 Pseudosymmetric Locally Conformal Kaehler Space Forms

Let $M(c)$ be an $m = 2n$-dimensional l.c.K-space form with constant holomorphic sectional curvature c and the tensor field P is hybrid. The Riemannian curvature tensor R with respect to g is given by (3.25).

Contracting (3.25) with g^{ik}, we get

$$S_{jh} = \frac{1}{4}[(m+2)c + 3\,tr\,P]g_{jh} + \frac{3}{4}(m-4)P_{jh} \tag{4.1}$$

which is the Ricci tensor of an l.c.K-space form.

Proposition 4.1 [7]. *If the tensor field P is hybrid and $tr\,P$ is constant in a 4-dimensional l.c.K-space form M(c), then M(c) is Einstein.*

Theorem 4.2 [7]. *A real m-dimensional $(m \neq 4)$ l.c.K-space form $M(c)$ in which the tensor field P is hybrid and $tr\,P$ is constant is Einstein if and only if the tensor field P is proportional to g.*

In view of (2.37), we have

$$(R \cdot C)_{hijklm} = g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{silm} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}) \,, \tag{4.2}$$

$$(C \cdot R)_{hijklm} = g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{silm} + R_{hirk}C_{sjlm} + R_{hijr}C_{sklm}) \,. \tag{4.3}$$

Using (2.36) in (4.3) we obtain

$$\begin{aligned}
(C \cdot R)_{hijklm} &= g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{silm} + R_{hirk}C_{sjlm} + R_{hijr}C_{sklm}) \\
&= (R \cdot R)_{hijklm} \\
&- \frac{1}{m-2}Q(S,R)_{hijklm} + \frac{\kappa}{(m-1)(m-2)}Q(g,R)_{hijklm} \\
&- \frac{1}{m-2}(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\
&+ g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}),
\end{aligned} \tag{4.4}$$

where

$$A_{hijk} = S_h^s R_{sijk}. \tag{4.5}$$

Applying, in the same way, (2.36) in (4.2) we get

$$\begin{aligned}
(R\cdot C)_{hijklm} &= g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{silm} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}) \\
&= (R\cdot R)_{hijklm} - \frac{1}{m-2}\Big[R_{hklm}S_{ij} - R_{jhlm}S_{ik} + R_{jilm}S_{hk} \\
&- R_{kilm}S_{hj} + R_{ijlm}S_{hk} - R_{hjlm}S_{ik} + R_{khlm}S_{ij} - R_{iklm}S_{hj} \\
&+ g_{ij}S_k^s R_{shlm} + g_{hk}S_j^s R_{silm} + g_{hk}S_i^s R_{sjlm} + g_{ij}S_h^s R_{sklm} \\
&- g_{ik}S_j^s R_{shlm} - g_{hj}S_k^s R_{silm} - g_{ik}S_h^s R_{sjlm} - g_{hj}S_i^s R_{sklm}\Big] \\
&+ \frac{\kappa}{(m-1)(m-2)}\Big[R_{khlm}g_{ij} - R_{jhlm}g_{ik} + R_{jilm}g_{hk} \\
&- R_{kilm}g_{hj} + R_{ijlm}g_{hk} - R_{hjlm}g_{ik} + R_{hklm}g_{ij} - R_{iklm}g_{hj}\Big] \\
&= (R\cdot R)_{hijklm} - \frac{1}{m-2}\Big[g_{ij}(A_{khlm} + A_{hklm}) + g_{hk}(A_{jilm} + A_{ijlm}) \\
&- g_{ik}(A_{jhlm} + A_{hjlm}) - g_{hj}(A_{kilm} + A_{iklm})\Big]
\end{aligned} \tag{4.6}$$

and so

$$\begin{aligned}
(R\cdot C - C\cdot R)_{hijklm} &= \frac{1}{m-2}Q(S,R)_{hijklm} - \frac{\kappa}{(m-1)(m-2)}Q(g,R)_{hijklm} \\
&+ \frac{1}{m-2}(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\
&+ g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}) \\
&- \frac{1}{m-2}\Big[g_{ij}(A_{khlm} + A_{hklm}) + g_{hk}(A_{jilm} + A_{ijlm}) \\
&- g_{ik}(A_{jhlm} + A_{hjlm}) - g_{hj}(A_{kilm} + A_{iklm})\Big].
\end{aligned} \tag{4.7}$$

Theorem 4.3 [31]. *Let M(c) be a 4-dimensional l.c.K-space form such that the tensor field P is hybrid and tr P is constant . Then we have*

$$\begin{aligned}
R\cdot C - C\cdot R &= \left[\frac{1}{4}\,(2c + tr\,P)\right] Q(g,R) \\
&= \left[\frac{1}{4}\,(2c + tr\,P)\right] Q(g,C) \\
&= \frac{1}{3}Q(S,R) \\
&= \frac{1}{3}Q(S,C).
\end{aligned} \tag{4.8}$$

Proof. Using (4.1) and (4.5) for a 4-dimensional l.c.K-space form, we have

$$S_{ij} = \frac{3}{4}\,(2c + tr\,P)g_{ij}\ , \tag{4.9}$$

$$A_{hijk} = \frac{3}{4}\,(2c + tr\,P)\,R_{hijk}\,,$$

$$Q(S,R) = \frac{3}{4}\,(2c + tr\,P)\,Q(g,R),$$

$$\kappa = 3(2c + tr\,P)$$

and so

$$g_{ij}(A_{khlm} + A_{hklm}) = 0, \qquad g_{hk}(A_{jilm} + A_{ijlm}) = 0,$$

$$g_{ik}(A_{jhlm} + A_{hjlm} = 0, \qquad g_{hj}(A_{kilm} + A_{iklm}) = 0.$$

Now the equations (4.6) and (4.4) reduce to

$$(R \cdot C)_{hijklm} = (R \cdot R)_{hijklm} \tag{4.10}$$

and

$$(C \cdot R)_{hijklm} = (R \cdot R)_{hijklm} - \left[\frac{1}{4}\,(2c + tr\,P)\right]\,Q(g,R)_{hijklm}, \tag{4.11}$$

respectively. Hence we get

$$R \cdot C - C \cdot R = \frac{1}{4}\,(2c + tr\,P)\,Q(g,R).$$

Now using (2.36) and (4.9), we get

$$C = R - \frac{1}{4}\,(2c + tr\,P)\,G \tag{4.12}$$

and so $Q(g,R) = Q(g,C)$. This completes the proof. ■

Theorem 4.4 [31]. *Let M(c) be an m-dimensional* $(m > 4)$ *l.c.K-space form. If the tensor field P is hybrid, tr P is constant and P is proportional to g, then we have*

$$\begin{aligned} R \cdot C - C \cdot R &= \frac{1}{4(m-1)}\left[(m+2)c + \frac{6(m-2)tr\,P}{m}\right]\,Q(g,R) \\ &= \frac{1}{4(m-1)}\left[(m+2)c + \frac{6(m-2)tr\,P}{m}\right]\,Q(g,C) \\ &= \frac{1}{m-1}Q(S,R) \\ &= \frac{1}{m-1}Q(S,C). \end{aligned} \tag{4.13}$$

Proof . In view of Theorem 4.2., we have $P = \frac{tr\,P}{m} g$ if and only if M(c) is Einstein. Then the equations (4.6) and (4.4) reduce to

$$(R \cdot C)_{hijklm} = (R \cdot R)_{hijklm} \tag{4.14}$$

and

$$\begin{aligned}(C \cdot R)_{hijklm} &= (R \cdot R)_{hijklm} \\ &- \frac{1}{4(m-1)} \left[(m+2)c + \frac{6(m-2)tr\,P}{m} \right] Q(g,R)_{hijklm},\end{aligned} \tag{4.15}$$

respectively. Thus we have

$$R \cdot C - C \cdot R = \frac{1}{4(m-1)} \left[(m+2)c + \frac{6(m-2)tr\,P}{m} \right] Q(g,R).$$

By using (2.36) we have

$$C = R - \frac{1}{4(m-1)} \left[(m+2)c + \frac{6(m-2)tr\,P}{m} \right] G \tag{4.16}$$

and so $Q(g,R) = Q(g,C)$. This completes the proof. ■

Theorem 4.5. *Let M(c) be an m-dimensional $(m > 4)$ l.c.K-space form and the tensor field P is hybrid. Then we have*

$$\begin{aligned}(m-2)(R \cdot C - C \cdot R)_{hijklm} &= \frac{\alpha(m-2) - \beta\, tr\,P}{m-1} Q(g,R)_{hijklm} \\ &+ \beta \Big\{ Q(P,R)_{hijklm} + \frac{3}{4} Q(g,\bar{P})_{hijklm} \\ &+ \frac{1}{4} \Big[(c\,\widetilde{P}_{hm} - D_{hm})(J \,\overline{\wedge}\, g)_{lijk} + (c\,\widetilde{P}_{im} - D_{im})(J \,\overline{\wedge}\, g)_{hljk} \\ &- (c\,\widetilde{P}_{jm} - D_{jm})(J \,\overline{\wedge}\, g)_{hilk} + (c\,\widetilde{P}_{km} - D_{km})(J \,\overline{\wedge}\, g)_{hijl} \\ &- (c\,\widetilde{P}_{hl} - D_{hl})(J \,\overline{\wedge}\, g)_{mijk} + (c\,\widetilde{P}_{il} - D_{il})(J \,\overline{\wedge}\, g)_{hmjk} \\ &+ (c\,\widetilde{P}_{jl} - D_{jl})(J \,\overline{\wedge}\, g)_{himk} + (c\,\widetilde{P}_{kl} - D_{kl})(J \,\overline{\wedge}\, g)_{hijm} \Big] \\ &+ \frac{1}{8} \Big[g_{hm}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{lijk} + g_{im}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{hljk} + g_{jm}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{hilk} \\ &+ g_{km}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{hijl} - g_{hl}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{mijk} - g_{il}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{hmjk} \\ &- g_{jl}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{himk} - g_{kl}(\widetilde{P} \,\overline{\overline{\wedge}}\, \widetilde{P})_{hijm} \Big] \\ &- \frac{c}{2} [J_{jk}(\widetilde{P} \,\overline{\wedge}\, g)_{hilm} + J_{hi}(\widetilde{P} \,\overline{\wedge}\, g)_{jklm}] \\ &+ \frac{1}{2} [J_{jk}(D \,\overline{\wedge}\, g)_{hilm} + J_{hi}(D \,\overline{\wedge}\, g)_{jklm}] \Big\},\end{aligned} \tag{4.17}$$

where $\alpha = \frac{1}{4}[(m+2)c + 3\,tr\,P]$, $\beta = \frac{3}{4}(m-4)$ and $D_{ij} = P_i^s \widetilde{P}_{sj}$.

Proof. Now substituting (4.1) into (4.6), (4.4) and (4.7), we get

$$\begin{aligned}(R\cdot C)_{hijklm} &= (R\cdot R)_{hijklm} - \frac{\beta}{m-2}\Big[g_{ij}(E_{hklm}+E_{khlm}) + g_{hk}(E_{ijlm}+E_{jilm})\\ &- g_{ik}(E_{hjlm}+E_{jhlm}) - g_{hj}(E_{iklm}+E_{kilm})\Big], \end{aligned} \tag{4.18}$$

$$\begin{aligned}(C\cdot R)_{hijklm} &= (R\cdot R)_{hijklm} - \frac{\alpha}{m-1}Q(g,R)_{hijklm} + \frac{\beta\, tr\, P}{(m-1)(m-2)}Q(g,R)_{hijklm}\\ &- \frac{\beta}{m-2}Q(P,R)_{hijklm} - \frac{\beta}{m-2}(g_{hl}E_{mijk} - g_{hm}E_{lijk} - g_{il}E_{mhjk}\\ &+ g_{im}E_{lhjk} + g_{jl}E_{mkhi} - g_{jm}E_{lkhi} - g_{kl}E_{mjhi} + g_{km}E_{ljhi}) \end{aligned} \tag{4.19}$$

and so

$$\begin{aligned}(m-2)(R\cdot C - C\cdot R)_{hijklm} &= \frac{\alpha(m-2)-\beta\, tr\, P}{m-1}Q(g,R)_{hijklm} + \beta Q(P,R)_{hijklm}\\ &+ \beta\Big[g_{hl}E_{mijk} - g_{hm}E_{lijk} - g_{il}E_{mhjk} + g_{im}E_{lhjk}\\ &+ g_{jl}E_{mkhi} - g_{jm}E_{lkhi} - g_{kl}E_{mjhi} + g_{km}E_{ljhi}\\ &- g_{ij}(E_{khlm}+E_{hklm}) - g_{hk}(E_{jilm}+E_{ijlm})\\ &+ g_{ik}(E_{jhlm}+E_{hjlm}) + g_{hj}(E_{kilm}+E_{iklm})\Big]. \end{aligned} \tag{4.20}$$

Furthermore we have

$$\begin{aligned}E_{hklm} &= P_h^s R_{sklm}\\ &= \frac{c}{4}[P_{hm}g_{kl} - P_{hl}g_{km} + \widetilde{P}_{hm}J_{kl} - \widetilde{P}_{hl}J_{km} - 2\widetilde{P}_{hk}J_{lm}]\\ &+ \frac{3}{4}(P_{hm}P_{kl} + g_{kl}P^2_{hm} - P_{hl}P_{km} - g_{km}P^2_{hl})\\ &- \frac{1}{4}(\widetilde{P}_{hm}\widetilde{P}_{kl} + J_{kl}D_{hm} - \widetilde{P}_{hl}\widetilde{P}_{km} - J_{km}D_{hl} - 2\widetilde{P}_{hk}\widetilde{P}_{lm} - 2J_{lm}D_{hk}), \end{aligned} \tag{4.21}$$

where $P^2_{ij} = P_i^s P_{sj}$. Then we have

$$\begin{aligned}E_{hklm} + E_{khlm} &= P_h^s R_{sklm} + P_k^s R_{shlm}\\ &= \frac{c}{4}\Big[(P_{hm}g_{kl} + P_{km}g_{hl} - P_{kl}g_{hm} - P_{hl}g_{km})\\ &+ (\widetilde{P}_{hm}J_{kl} + \widetilde{P}_{km}J_{hl} - \widetilde{P}_{kl}J_{hm} - \widetilde{P}_{hl}J_{km})\Big]\\ &+ \frac{3}{4}(P^2_{hm}g_{kl} + P^2_{km}g_{hl} - P^2_{kl}g_{hm} - P^2_{hl}g_{km})\\ &- \frac{1}{4}(D_{hm}J_{kl} + D_{km}J_{hl} - D_{kl}J_{hm} - D_{hl}J_{km}). \end{aligned} \tag{4.22}$$

Substituting (4.21), (4.22) into (4.20) and using (2.34), (2.35) and Lemma 2.1., we obtain (4.17). ∎

Theorem 4.6 [31]. *Let M(c) be an m-dimensional* $(m > 4)$ *l.c.K-space form such that the tensor field P is hybrid. If the relation* (2.49) *is fulfilled on* $\mathscr{U}_C \subset M(c)$, *then at every point of* $\mathscr{U}_C$ *we have*

$$P_h^s R_{sijk} + P_j^s R_{sikh} + P_k^s R_{sihj} = 0. \tag{4.23}$$

Proof. The left side of the equation (2.49) in local coordinates takes the form

$$\begin{aligned} & g^{rs}(R_{rijk}R_{shlm} + R_{hrjk}R_{silm} + R_{hirk}R_{sjlm} + R_{hijr}R_{sklm}) \\ - & (S_{lh}R_{mijk} + S_{li}R_{hmjk} + S_{lj}R_{himk} + S_{lk}R_{hijm} - S_{mh}R_{lijk} \\ - & S_{mi}R_{hljk} - S_{mj}R_{hilk} - S_{mk}R_{hijl}) = L_1\, Q(g,C)_{hijklm} \end{aligned}$$

and contracting with g^{ij} we get

$$S_h^s R_{sklm} + S_k^s R_{shlm} = S_l^s R_{skhm} + S_l^s R_{shkm} - S_m^s R_{skhl} - S_m^s R_{shkl} \tag{4.24}$$

and substituting (4.1) into the above equation we have

$$P_h^s R_{sklm} + P_k^s R_{shlm} = P_l^s R_{skhm} + P_l^s R_{shkm} - P_m^s R_{skhl} - P_m^s R_{shkl}. \tag{4.25}$$

Summing (4.25) cyclically in h, l, m, we have

$$\begin{aligned} 3(P_h^s R_{sklm} + P_l^s R_{skmh} + P_m^s R_{skhl}) &= P_h^s(R_{smkl} + R_{slmk}) + P_l^s(R_{shkm} + R_{smhk}) \\ &- P_m^s(R_{slkh} + R_{shlk}), \end{aligned}$$

which yields

$$3(P_h^s R_{sklm} + P_l^s R_{skmh} + P_m^s R_{skhl}) = -P_h^s R_{sklm} - P_l^s R_{skmh} - P_m^s R_{skhl}. \qquad \blacksquare$$

Theorem 4.7. *Let M(c) be a 4-dimensional l.c.K-space form such that the tensor field P is hybrid and tr P is constant . If the relation* (2.49) *is fulfilled on* $\mathscr{U}_C \subset M(c)$, *where* L_1 *is some function on* $\mathscr{U}_C$, *then M(c) is pseudosymmetric with the function* $L_R = L_1 + \frac{3}{4}(2c + tr\, P)$.

Proof. Using (4.9) and (4.12) in (2.49), we have

$$R \cdot R - \frac{3}{4}(2c + tr\, P)Q(g,R) = L_1 Q(g,R)$$

and so

$$R \cdot R = [L_1 + \frac{3}{4}(2c + tr\, P)]Q(g,R). \tag{4.26}$$

This completes the proof. ■

Theorem 4.8. *Let M(c) be an m-dimensional* $(m>4)$ *l.c.K-space form such that the tensor field P is hybrid, tr P is constant and the tensor field P is proportional to g . If the relation* (2.49) *is fulfilled on* $\mathscr{U}_C \subset M(c)$*, where* L_1 *is some function on* $\mathscr{U}_C$*, then M(c) is pseudosymmetric with the function* $L_R = L_1 + \frac{1}{4}[(m+2)\,c + \frac{6(m-2)}{m}\,tr\,P]$.

Proof. In view of Theorem 4.2., we have

$$S = \frac{1}{4}[(m+2)\,c + \frac{6(m-2)}{m}\,tr\,P]\,g \tag{4.27}$$

Substituting (4.27) into (2.49) and using (4.16), we get

$$R\cdot R - \frac{1}{4}[(m+2)\,c + \frac{6(m-2)}{m}tr\,P]Q(g,R) = L_1 Q(g,R)$$

and so

$$R\cdot R = \left[L_1 + \frac{1}{4}[(m+2)\,c + \frac{6(m-2)}{m}tr\,P]\right]Q(g,R). \tag{4.28}$$

This completes the proof. ■

4.2 Ricci-pseudosymmetric Locally Conformal Kaehler Space Forms

In this section, some properties of Ricci-pseudosymmetric l.c.K-space forms are presented. Firstly, we consider Ricci-pseudosymmetric l.c.K-space forms satisfying (2.50). After that, Ricci-pseudosymmetric l.c.K-space forms satisfying (2.49) and (2.50) are studied.

Theorem 4.9. *Let M(c) be an m-dimensional* $(m>4)$ *Ricci-pseudosymmetric l.c.K-space form. If the tensor field P is hybrid and the condition* (2.50) *is fulfilled for* $L_2 \neq 0$ *at* $x \in \mathscr{U}_s \cap \mathscr{U}_2 \subset M(c)$*, then*

$$P_h^r R_{rijk} + P_j^r R_{rikh} + P_k^r R_{rihj} = 0, \tag{4.29}$$

$$P_h^r C_{rijk} + P_j^r C_{rikh} + P_k^r C_{rihj} = 0, \tag{4.30}$$

$$C\cdot P = 0, \tag{4.31}$$

$$P_{ij}^2 = \frac{-2\alpha + a}{\beta}\,P_{ij} + \frac{-\alpha^2 + a\,\alpha + b}{\beta^2}\,g_{ij}, \tag{4.32}$$

where $\alpha = \frac{1}{4}[(m+2)c + 3\,tr\,P]$, $\beta = \frac{3}{4}(m-4)$, $a = (m-2)L_s + \frac{m\alpha+\beta\,tr\,P}{m-1}$ *and*
$b = \frac{m\alpha^2+2\,\alpha\,\beta\,tr\,P+\beta^2\,tr(P^2)}{m} - \frac{m\alpha+\beta\,tr\,P}{m}\Big[(m-2)L_s + \frac{m\alpha+\beta\,tr\,P}{m-1}\Big]$.

Proof. In local coordinates (2.43) takes the form

$$S^r_h R_{rijk} + S^r_i R_{rhjk} = L_S(g_{hj}S_{ik} - g_{hk}S_{ij} + g_{ij}S_{hk} - g_{ik}S_{hj}). \tag{4.33}$$

Summing cyclically (4.33) in h, j, k we obtain

$$S^r_h R_{rijk} + S^r_j R_{rikh} + S^r_k R_{rihj} = 0. \tag{4.34}$$

Now substituting (4.1) into the above equality we have

$$\begin{aligned} S^r_h R_{rijk} + S^r_j R_{rikh} + S^r_k R_{rihj} &= \frac{1}{4}[(m+2)c + 3\,tr\,P](R_{hijk} + R_{jikh} + R_{kihj}) \\ &+ \frac{3}{4}(m-4)(P^r_h R_{rijk} + P^r_j R_{rikh} + P^r_k R_{rihj}). \end{aligned}$$

Using the Bianchi identity we obtain (4.29).

Now applying (4.34) in (2.36) we get

$$S^r_h C_{rijk} + S^r_j C_{rikh} + S^r_k C_{rihj} = 0 \tag{4.35}$$

and using (4.1) we have (4.30).

The relation (2.50) in local coordinates takes the form

$$\begin{aligned} &g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{silm} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}) \\ =\; & L_2\Big(S_{hl}C_{mijk} - S_{hm}C_{lijk} + S_{il}C_{hmjk} - S_{im}C_{hljk} + S_{jl}C_{himk} \\ -\; & S_{jm}C_{hilk} + S_{kl}C_{hijm} - S_{km}C_{hijl}\Big) \end{aligned} \tag{4.36}$$

and contracting (4.36) with g^{hk} we get

$$0 = L_2\Big[S^r_l(C_{rijm} + C_{rjim}) + S^r_m(C_{rilj} + C_{rjli})\Big] \tag{4.37}$$

and by the assumption $L_2 \neq 0$, we obtain

$$S^r_l C_{rijm} + S^r_m C_{rilj} + S^r_l C_{rjim} + S^r_m C_{rjli} = 0.$$

Using (4.35) we have

$$(C\cdot S)_{ijlm} = S^r_i C_{rjlm} + S^r_j C_{rilm} = 0. \tag{4.38}$$

Now substituting (4.1) into (4.37) we have

$$L_2\Big\{\tfrac{1}{4}[(m+2)c+3\,trP]\Big[(C_{lijm}+C_{ljim})-(C_{mijl}+C_{mjil})\Big]$$

$$+\quad \frac{3}{4}(m-4)\Big[P^r_l(C_{rijm}+C_{rjim})-P^r_m(C_{rijl}+C_{rjil})\Big]\Big\}=0$$

and so

$$P^r_l C_{rijm}+P^r_m C_{rilj}+P^r_l C_{rjim}+P^r_m C_{rjli}=0.$$

Now applying (4.30) we have

$$(C\cdot P)_{ijlm}=P^r_i C_{rjlm}+P^r_j C_{rilm}=0.$$

In view of (2.36) we have

$$C\cdot P=R\cdot S-\frac{1}{m-2}Q(g,S^2)+\frac{\kappa}{(m-1)(m-2)}Q(g,S).$$

Applying (2.43) and (4.38) we get

$$Q\Big(g,S^2-\Big[(m-2)L_s+\frac{\kappa}{m-1}\Big]S\Big)=0.$$

Using (Lemma 2.4(i) of [16]) we obtain

$$S^2=\Big[(m-2)L_s+\frac{\kappa}{m-1}\Big]S+\lambda g, \qquad \lambda\in\mathbb{R}. \tag{4.39}$$

Now Substituting (4.1) into (4.39), we obtain (4.32). This completes the proof. ■

Theorem 4.10. *Let M(c) be an m-dimensional* $(m>4)$ *Ricci-pseudosymmetric l.c.K-space form. If the tensor field P is hybrid and the condition* (2.50) *holds for* $L_2\neq 0$ *at* $x\in\mathscr{U}_S\cap\mathscr{U}_2\subset M$*, then*

$$\Big(mL_S-(m\alpha+tr\,P\,\beta)L_2\Big)P^r_l C_{rijk}=\Big(tr\,P\,L_S-(\alpha\,tr\,P+\beta\,tr(P^2))L_2\Big)C_{lijk} \tag{4.40}$$

at x, where $\alpha=\frac{1}{4}[(m+2)c+3\,tr\,P]$ *and* $\beta=\frac{3}{4}(m-4)$.

Moreover, if $L_S=\frac{\kappa}{m}L_2$ *at x, then*

$$tr(P^2)=\frac{1}{\beta}(\alpha\,tr\,P-\frac{\kappa\,tr\,P}{m}). \tag{4.41}$$

Proof. In view of (4.39) and (4.35) or (4.38) we get

$$S^2_{hr}C^r_{ijk}+S^2_{jr}C^r_{ikh}+S^2_{kr}C^r_{ihj}=0, \tag{4.42}$$

$$C \cdot S^2 = 0. \tag{4.43}$$

Transvecting (2.50) with S_p^m we get

$$\begin{aligned}
& g^{rs}[S_p^m R_{mlhs} C_{rijk} + S_p^m R_{mlis} C_{hrjk} + S_p^m R_{mljs} C_{hirk} + S_p^m R_{mlks} C_{hijr}] \\
= \; & L_2 \Big(S_{hl} S_p^r C_{rijk} + S_{il} S_p^r C_{hrjk} + S_{jl} S_p^r C_{hirk} + S_{kl} S_p^r C_{hijr} \\
- \; & S_{hp}^2 C_{lijk} - S_{ip}^2 C_{hljk} - S_{jp}^2 C_{hilk} - S_{kp}^2 C_{hijl} \Big).
\end{aligned}$$

Now symmetrization in p, l, we have

$$\begin{aligned}
& g^{rs} \Big[(S_p^m R_{mlhs} + S_l^m R_{mphs}) C_{rijk} + (S_p^m R_{mlis} + S_l^m R_{mpis}) C_{hrjk} \\
+ \; & (S_p^m R_{mljs} + S_l^m R_{mpjs}) C_{hirk} + (S_p^m R_{mlks} + S_l^m R_{mpks}) C_{hijr} \Big] \\
= \; & L_2 \Big[(S_{hl} S_p^r + S_{hp} S_l^r) C_{rijk} + (S_{il} S_p^r + S_{ip} S_l^r) C_{hrjk} + \\
+ \; & (S_{jl} S_p^r + S_{jp} S_l^r) C_{hirk} + (S_{kl} S_p^r + S_{kp} S_l^r) C_{hijr} \\
- \; & S_{ph}^2 C_{lijk} - S_{lh}^2 C_{pijk} - S_{pi}^2 C_{hljk} - S_{li}^2 C_{hpjk} \\
- \; & S_{pj}^2 C_{hilk} - S_{lj}^2 C_{hipk} - S_{pk}^2 C_{hijl} - S_{lk}^2 C_{hijp} \Big].
\end{aligned}$$

In view of (2.43) we get

$$\begin{aligned}
& L_S \Big[g_{ph} S_l^r C_{rijk} + g_{lh} S_p^r C_{rijk} - g_{pi} S_l^r C_{rhjk} - g_{li} S_p^r C_{rhjk} \\
+ \; & g_{pj} S_l^r C_{rkhi} + g_{lj} S_p^r C_{rkhi} - g_{pk} S_l^r C_{rjhi} - g_{lk} S_p^r C_{rjhi} \\
- \; & S_{lh} C_{pijk} - S_{ph} C_{lijk} - S_{li} C_{hpjk} - S_{pi} C_{hljk} \\
- \; & S_{lj} C_{hipk} - S_{pj} C_{hilk} - S_{lk} C_{hijp} - S_{pk} C_{hijl} \Big] \\
= \; & L_2 \Big[(S_{hl} S_p^r + S_{hp} S_l^r) C_{rijk} + (S_{il} S_p^r + S_{ip} S_l^r) C_{hrjk} \\
+ \; & (S_{jl} S_p^r + S_{jp} S_l^r) C_{hirk} + (S_{kl} S_p^r + S_{kp} S_l^r) C_{hijr} \\
- \; & S_{ph}^2 C_{lijk} - S_{lh}^2 C_{pijk} - S_{pi}^2 C_{hljk} - S_{li}^2 C_{hpjk} \\
- \; & S_{pj}^2 C_{hilk} - S_{lj}^2 C_{hipk} - S_{pk}^2 C_{hijl} - S_{lk}^2 C_{hijp} \Big].
\end{aligned} \tag{4.44}$$

Contracting (4.44) with g^{hp} and using (4.35), (4.38), (4.42) and (4.43) we obtain

$$L_S (m S_l^r C_{rijk} - \kappa C_{lijk}) = L_2 (\kappa S_l^r C_{rijk} - tr(S^2) C_{lijk}). \tag{4.45}$$

In view of (4.1) we have

$$tr(S^2) = S_{ir} S^{ir} = m\alpha^2 + 2\, \alpha\, \beta\; tr\, P + \beta^2\; tr(P^2) \tag{4.46}$$

and

$$\kappa = m\alpha + \beta\ tr\ P. \tag{4.47}$$

Applying (4.46) and (4.47) in (4.45) we obtain (4.40).

Finally, if $L_S = \frac{\kappa}{m} L_2$, then (4.40), in view of $C \neq 0$ and $L_2 \neq 0$ at x, yields

$tr(P^2) = \frac{1}{\beta}(\alpha\, trP - \frac{\kappa\, trP}{m}).$ ■

The following proposition is based on ([32], Lemma 3.1.).

Theorem 4.11. *Let M(c) be an m-dimensional* $(m > 4)$ *Ricci-pseudosymmetric l.c.K-space form. If the tensor field P is hybrid and the conditions* (2.49) *and* (2.50) *hold for* $L_2 \neq 0$ *at* $x \in \mathscr{U}_S \cap \mathscr{U}_2$ *then*

$$\begin{aligned} (m-1)L_S(R \cdot P) &= \Big[\alpha(\alpha - mL_S) - \beta\ trPL_S\Big] Q(g,P) \\ &+ Q(\alpha\beta g + \beta^2 P, P^2), \end{aligned} \tag{4.48}$$

where $\alpha = \frac{1}{4}[(m+2)c + 3\ trP]$ *and* $\beta = \frac{3}{4}(m-4)$.

Proof. Contracting (4.33) with g^{hk} we find

$$T_{ij} = S^{rs}R_{rijs} = S^2_{ij} - mL_S S_{ij} + \kappa L_S g_{ij}. \tag{4.49}$$

Applying the operation $R\cdot$ to the equation (4.49), we obtain

$$(R \cdot S)_{rshk} R^{r\ \ s}_{ij} + S^{rs}(R \cdot R)_{rijshk} = (R \cdot S^2)_{ijhk} - mL_S(R \cdot S)_{ijhk}. \tag{4.50}$$

In view of (2.43), (2.49) and $S^{rs}C_{rijs} = 0$, which follows immediately from (4.38), the left hand side of this identity is equal to

$$\begin{aligned} &L_S(S^r_k R_{rjih} + S^r_k R_{rijh} - S^r_h R_{rjik} - S^r_h R_{rijk}) \\ + \quad &S^2_{hr} R^r_{jik} + S^2_{hr} R^r_{ijk} - S^2_{kr} R^r_{jih} - S^2_{kr} R^r_{ijh} \\ + \quad &S_{ih} T_{jk} - S_{ik} T_{jh} + S_{jh} T_{ik} - S_{jk} T_{ih} \\ - \quad &L_1(S^r_k C_{rjih} + S^r_k C_{rijh} - S^r_h C_{rjik} - S^r_h C_{rijk}). \end{aligned}$$

Using twice (4.35) and next (4.38) we can easily see that the expression in the last brackets vanishes. Moreover in view of (4.34), we have

$$S^r_k R_{rjih} - S^r_h R_{rjik} = -S^r_i R_{rjhk}$$

and using $S^2_{hr}R^r_{ijk} + S^2_{jr}R^r_{ikh} + S^2_{kr}R^r_{ihj} = 0$, we get

$$S^2_{hr}R^r_{jik} - S^2_{kr}R^r_{jik} = -S^2_{ir}R^r_{jkh} = S^2_{ir}R^r_{jhk}.$$

Taking into account all these identities one can easily see that the left hand side of (4.50) can be written as follows:

$$-L_S(R \cdot S)_{ijhk} + (R \cdot S^2)_{ijhk} + S_{ih}T_{jk} - S_{ik}T_{jh} + S_{jh}T_{ik} - S_{jk}T_{ih}.$$

Substituting this expression into (4.50) we obtain

$$(m-1)L_S(R \cdot S)_{ijhk} = S_{ik}T_{jh} - S_{ih}T_{jk} + S_{jk}T_{ih} - S_{jh}T_{ik}. \tag{4.51}$$

In local coordinates, (5.12) takes the form

$$P^r_h R_{rijk} + P^r_i R_{rhjk} = L_S(g_{hj}P_{ik} - g_{hk}P_{ij} + g_{ij}P_{hk} - g_{ik}P_{hj}) \tag{4.52}$$

and contracting (4.52) with g^{hk} and using (4.1), we have

$$P^{rs}R_{rijs} = \alpha P_{ij} + \beta P^2_{ij} - mL_S P_{ij} + trP\, L_S g_{ij} \tag{4.53}$$

and

$$\begin{aligned} T_{jh} = S^{rs}R_{rjhs} &= \alpha g^{rs}R_{rjhs} + \beta P^{rs}R_{rjhs} \\ &= \alpha S_{jh} + \beta P^{rs}R_{rjhs}. \end{aligned}$$

Using above equation we have

$$\begin{aligned} (m-1)L_S(R \cdot S)_{ijhk} &= \beta(P^{rs}R_{rjhs}S_{ik} - P^{rs}R_{rjks}S_{ih} \\ &+ P^{rs}R_{rihs}S_{jk} - P^{rs}R_{riks}S_{jh}) \end{aligned}$$

and so

$$\begin{aligned} (m-1)L_S(R \cdot P)_{ijhk} &= P^{rs}R_{rjhs}S_{ik} - P^{rs}R_{rjks}S_{ih} \\ &+ P^{rs}R_{rihs}S_{jk} - P^{rs}R_{riks}S_{jh}. \end{aligned} \tag{4.54}$$

Applying (4.53) and (4.1) in (4.54), we find (4.48). ■

Theorem 4.12. *Let M(c) be an m-dimensional* $(m > 4)$ *Ricci-pseudosymmetric l.c.K-space form such that the tensor field P is hybrid. If the conditions* (2.49) *and* (2.50) *hold then on* $\mathscr{U}_S \cap \mathscr{U}_2$ *we have*

$$\left[\alpha(L_2-1)-L_1\right] Q(g,R)+\beta(L_2-1)\, Q(P,R)$$

$$=\frac{\beta}{m-2}\, Q(g,\widetilde{U})\left[L_2(\beta-\alpha+\frac{\kappa}{m-1})-L_1-L_S\right] \tag{4.55}$$

and

$$\begin{aligned}
&(L_2-1)(tr\, P\, R_{mijk}+P_i^r R_{rmjk})+\tfrac{m-1}{\beta}\left[\alpha(L_2-1)-L_1\right]R_{mijk}\\
&= \beta(L_2-1)(P_{mk}P_{ij}-P_{mj}P_{ik})\\
&+ \left[\alpha(L_2-1)+(m-1)\tau\right](g_{ij}P_{mk}-g_{ik}P_{mj})\\
&+ \left[\alpha(L_2-1)-L_1+\tau\right](g_{mk}P_{ij}-g_{mj}P_{ik})\\
&+ \left[\frac{\alpha}{\beta}(\alpha(L_2-1)-L_1)-trP\,\tau\right](g_{mj}g_{ik}-g_{mk}g_{ij}),
\end{aligned} \tag{4.56}$$

where $\alpha=\frac{1}{4}[(m+2)c+3\, tr\, P]$, $\beta=\frac{3}{4}(m-4)$ *and* $\tau=\frac{1}{m-1}\left[L_2(\beta-\alpha+\frac{\kappa}{m-1})-L_1-L_S\right]$.

Proof. The Weyl curvature tensor C can also be presented in the following form:

$$C=R-\frac{1}{m-2}U+\frac{\kappa}{(m-1)(m-2)}G,$$

where

$$U_{hijk}=g_{hk}S_{ij}-g_{hj}S_{ik}+g_{ij}S_{hk}-g_{ik}S_{hj}. \tag{4.57}$$

Applying the operation $R\cdot$ and in view of (2.43) we get

$$\begin{aligned}
(R\cdot U)_{hijklm} &= g_{hk}(R\cdot S)_{ijlm}-g_{hj}(R\cdot S)_{iklm}+g_{ij}(R\cdot S)_{hklm}-g_{ik}(R\cdot S)_{hjlm}\\
&= L_S\Big(g_{hk}Q(g,S)_{ijlm}-g_{hj}Q(g,S)_{iklm}\\
&+ g_{ij}Q(g,S)_{hklm}-g_{ik}Q(g,S)_{hjlm}\Big)\\
&= -L_S\Big(S_{il}G_{hmjk}+S_{jl}G_{himk}+S_{kl}G_{hijm}+S_{hl}G_{mijk}\\
&- S_{im}G_{hljk}-S_{jm}G_{hilk}-S_{km}G_{hijl}-S_{hm}G_{lijk}\Big)\\
&= -L_SQ(S,G)_{hijklm}\\
&= L_SQ(g,U)_{hijklm}.
\end{aligned} \tag{4.58}$$

Substituting (4.1) into (4.57) we have

$$U_{hijk}=2\alpha G_{hijk}+\beta\widetilde{U}_{hijk}, \tag{4.59}$$

where

$$\widetilde{U}_{hijk} = g_{hk}P_{ij} - g_{hj}P_{ik} + g_{ij}P_{hk} - g_{ik}P_{hj}. \tag{4.60}$$

Using (4.59) we get

$$(R\cdot U)_{hijklm} = \beta(R\cdot\widetilde{U})_{hijklm} \tag{4.61}$$

and

$$(R\cdot\widetilde{U})_{hijklm} = -L_S Q(P,G)_{hijklm} = L_S Q(g,\widetilde{U})_{hijklm}. \tag{4.62}$$

Moreover, using (2.36) we obtain $R\cdot C = R\cdot R - \frac{1}{m-2}R\cdot U$. Substituting (4.58), (2.49) and (2.50) into (4.62) we get

$$L_2 Q(S,C) = Q(S,R) + L_1 Q(g,C) - \frac{1}{m-2}R\cdot U \tag{4.63}$$

and using (4.1) and (4.61) we have

$$\begin{aligned} L_2(\alpha Q(g,C) + \beta Q(P,C)) &= \alpha Q(g,R) + \beta Q(P,R) + L_1 Q(g,C) \\ &- \frac{\beta}{m-2}(R\cdot\widetilde{U}). \end{aligned} \tag{4.64}$$

After straightforward calculations, we get

$$Q(g,C) = Q(g,R) - \frac{\beta}{m-2}Q(g,\widetilde{U}) \tag{4.65}$$

and

$$\begin{aligned} Q(P,C) &= Q(P,R) + \frac{1}{m-2}Q(P,U) + \frac{\kappa}{(m-1)(m-2)}Q(P,G) \\ &= Q(P,R) + \frac{2\alpha-\beta}{m-2}Q(g,\widetilde{U}) - \frac{\kappa}{(m-1)(m-2)}Q(g,\widetilde{U}). \end{aligned} \tag{4.66}$$

Substituting (4.65) and (4.66) into (4.64) we get (4.55).

Using (2.38) and (4.1), we obtain

$$g^{hl}Q(g,R)_{hijklm} = (m-1)R_{mijk} - \alpha G_{mijk} + \beta(g_{jm}P_{ik} - g_{km}P_{ij}),$$

$$\begin{aligned} g^{hl}Q(P,R)_{hijklm} &= tr\,P\,R_{mijk} + P^r_i R_{rmjk} + \alpha(g_{ik}P_{mj} - g_{ij}P_{mk}) \\ &+ \beta(P_{ik}P_{mj} - P_{ij}P_{mk}) \end{aligned}$$

and

$$\begin{aligned} g^{hl}Q(g,\widetilde{U})_{hijklm} &= (m-1)(g_{ij}P_{mk} - g_{ik}P_{mj}) + tr\,P(g_{ik}g_{mj} - g_{ij}g_{mk}) \\ &+ g_{mk}P_{ij} - g_{mj}P_{ik}. \end{aligned}$$

Contracting (4.55) with g^{hl} and using the above relations we get (4.56), which completes the proof. ■

5. CURVATURE PROPERTIES OF LOCALLY CONFORMAL KAEHLER SPACE FORMS

5.1 Walker Type Identities On Locally Conformal Kaehler Space Forms

In this section, we present results on l.c.K-space forms satisfying curvature identities named Walker type identities.

Lemma 5.1 [33]. *For a symmetric (0,2)-tensor A and a generalized curvature tensor B on a semi-Riemannian manifold (M,g)*, $n \geq 3$, *we have*

$$Q(A,B)_{hijklm} + Q(A,B)_{jklmhi} + Q(A,B)_{lmhijk} = 0. \tag{5.1}$$

It is well-known that the following identity

$$(R \cdot R)_{hijklm} + (R \cdot R)_{jklmhi} + (R \cdot R)_{lmhijk} = 0 \tag{5.2}$$

holds on any semi-Riemannian manifold. The equation (5.2) is called *the Walker type identity*.

On any semi-Riemannian manifold (M,g), $n \geq 4$, the following three identities are equivalent to each other [34]:

$$(R \cdot C)_{hijklm} + (R \cdot C)_{jklmhi} + (R \cdot C)_{lmhijk} = 0, \tag{5.3}$$

$$(C \cdot R)_{hijklm} + (C \cdot R)_{jklmhi} + (C \cdot R)_{lmhijk} = 0 \tag{5.4}$$

and

$$(R \cdot C - C \cdot R)_{hijklm} + (R \cdot C - C \cdot R)_{jklmhi} + (R \cdot C - C \cdot R)_{lmhijk} = 0. \tag{5.5}$$

The equations (5.3) - (5.5) are called *the Walker type identities*. We also can consider the following Walker type identity

$$(C \cdot C)_{hijklm} + (C \cdot C)_{jklmhi} + (C \cdot C)_{lmhijk} = 0. \tag{5.6}$$

Theorem 5.2 [35]. *Let M(c) be a 4-dimensional l.c.K-space form such that the tensor field P is hybrid and tr P is constant. Then the Walker type identities* (5.3) - (5.5) *and* (5.6) *hold on M(c).*

Proof. In view of Theorem 4.3., we have

$$R\cdot C - C\cdot R \;=\; \left[\frac{1}{4}\,(2c+tr\,P)\right]\,Q(g,R)$$

and using (5.1) we get (5.5) (equivalently (5.3) and (5.4)) .

Further, we note that (2.36) turns into $C = R - \frac{2c+trP}{4}G$. This gives

$$\begin{aligned} C\cdot C &= C\cdot(R-\frac{2c+trP}{4}G) = C\cdot R \\ &= (R-\frac{2c+trP}{4}G)\cdot R = R\cdot R-\frac{2c+trP}{4}Q(g,R). \end{aligned} \tag{5.7}$$

Now using (5.1) and (5.2) we complete the proof. ∎

Theorem 5.3 [35]. *Let M(c) be an m-dimensional* $(m>4)$ *l.c.K-space form such that tr P is constant, the tensor field P is hybrid and is proportional to g, then the Walker type identities* (5.3) - (5.5) *and* (5.6) *hold on M(c).*

Proof. In view of Theorem 4.4. and (5.1) we get (5.5) (equivalently (5.3) and (5.4)) .

Using (4.16), we get

$$\begin{aligned} C\cdot C &= C\cdot\left(R-\frac{1}{4(m-1)}\left[(m+2)c+\frac{6(m-2)}{m}tr\,P\right]G\right) \\ &= C\cdot R \\ &= \left(R-\frac{1}{4(m-1)}\left[(m+2)c+\frac{6(m-2)}{m}tr\,P\right]G\right)\cdot R \\ &= R\cdot R-\frac{1}{4(m-1)}\left[(m+2)c+\frac{6(m-2)}{m}tr\,P\right]Q(g,R). \end{aligned}$$

Using (5.1) and (5.2), we get the result. ∎

Lemma 5.4 [35]. *Let M(c) be an m-dimensional* $(m>4)$ *l.c.K-space form such that the tensor field P is hybrid. Then, we have*

$$\begin{aligned} &(m-2)\left[(R\cdot C)_{hijklm}+(R\cdot C)_{jklmhi}+(R\cdot C)_{lmhijk}\right] \\ &= -\beta\left[(g\wedge(R\cdot P))_{hijklm}+(g\wedge(R\cdot P))_{jklmhi}+(g\wedge(R\cdot P))_{lmhijk}\right]. \end{aligned} \tag{5.8}$$

Proof. Substituting (4.1) into (2.36), we obtain

$$C = R-\frac{\beta}{m-2}(g\wedge P)-\frac{\alpha(m-2)-\beta\,trP}{(m-1)(m-2)}G, \tag{5.9}$$

where $\alpha = \frac{1}{4}[(m+2)c + 3\, trP]$ and $\beta = \frac{3}{4}(m-4)$ and so

$$R \cdot C = R \cdot R - \frac{\beta}{m-2}\, g \wedge (R \cdot P). \tag{5.10}$$

Using (5.2) the proof is completed. ■

Corollary 5.5 [35]. *If one of the Walker type identities* (5.3) - (5.5) *holds on an m-dimensional* $(m > 4)$ *l.c.K-space form M(c) and the tensor field P is hybrid, then on M(c) we have*

$$(g \wedge (R \cdot P))_{hijklm} + (g \wedge (R \cdot P))_{jklmhi} + (g \wedge (R \cdot P))_{lmhijk} = 0. \tag{5.11}$$

Theorem 5.6 [35]. *Let M(c) be an m-dimensional* $(m > 4)$ *Ricci-pseudosymmetric l.c.K-space form M(c) such that the tensor field P is hybrid. Then the Walker type identities* (5.3) - (5.5) *hold on* $\mathscr{U}_s \subset M$.

Proof. In view of (2.43) and (4.1), m-dimensional $(m > 4)$ Ricci-pseudosymmetric l.c.K-space forms satisfy

$$R \cdot P = L_S Q(g, P). \tag{5.12}$$

Using (5.12) in (5.8), we obtain the following identity on $\mathscr{U}_S$

$$(m-2)\Big[(R \cdot C)_{hijklm} + (R \cdot C)_{jklmhi} + (R \cdot C)_{lmhijk}\Big]$$
$$= -\beta L_S\Big[(g \wedge Q(g,P))_{hijklm} + (g \wedge Q(g,P))_{jklmhi} + (g \wedge Q(g,P))_{lmhijk}\Big].$$

Making use of (2.33) and (5.1), we obtain on $\mathscr{U}_S$

$$\begin{aligned} (m-2)\Big[(R \cdot C)_{hijklm} + (R \cdot C)_{jklmhi} + (R \cdot C)_{lmhijk}\Big] &\\ = \ \beta L_S\Big[Q(P,G)_{hijklm} + Q(P,G)_{jklmhi} + Q(P,G)_{lmhijk}\Big] &\\ = \ 0. & \end{aligned} \tag{5.13}$$

Hence (5.3) (equivalently (5.4), (5.5)) holds on M(c). ■

5.2 Roter Type Locally Conformal Kaehler Space Forms

Let B be a generalized curvature tensor on a semi-Riemannian manifold (M,g), $n \geq 4$. We denote by Ric(B), Weyl(B) and $\kappa(B)$ its Ricci tensor, the Weyl tensor and the scalar curvature tensor, respectively. The subset $\mathscr{U}_B$, $\mathscr{U}_{Ric(B)}$ and $\mathscr{U}_{Weyl(B)}$ are defined in the same manner as the subsets $\mathscr{U}_R$, $\mathscr{U}_S$, and $\mathscr{U}_C$, respectively.

A generalized curvature tensor B on a semi-Riemannian manifold (M,g), $n \geq 4$, is called *Roter type tensor* if

$$B = \frac{\phi}{2} Ric(B) \wedge Ric(B) + \mu g \wedge Ric(B) + \eta\, G, \tag{5.14}$$

on $\mathscr{U}_{Ric(B)} \cap \mathscr{U}_{Weyl(B)}$, where ϕ, μ and η are some functions on that set. Manifolds admitting Roter type tensors were investigated in [36] [23] [17].

A semi-Riemannian manifold (M,g), $n \geq 4$, with the curvature tensor R satisfying (5.14) on $\mathscr{U}_S \cap \mathscr{U}_C \subset M$, i.e.

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta\, G, \tag{5.15}$$

where ϕ, μ and η are some functions on $\mathscr{U}_S \cap \mathscr{U}_c$, is called a *Roter type manifold* [37].

In local coordinates (5.15) takes the form

$$R_{hijk} = \frac{\phi}{2}\Big(2S_{hk}S_{ij} - 2S_{hj}S_{ik}\Big) + \mu\Big(g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj}\Big) + \eta\, G_{hijk}.$$

Substituting (4.1) into the above equation we have

$$\begin{aligned} \bar{R}_{hijk} &= \phi\Big[\alpha^2 G_{hijk} + \alpha\beta (g \wedge P)_{hijk} + \beta^2 (P_{hk}P_{ij} - P_{hj}P_{ik})\Big] \\ &+ \mu\Big[2\alpha G_{hijk} + \beta (g \wedge P)_{hijk}\Big] \\ &+ \eta\, G_{hijk} \end{aligned}$$

and so

$$\bar{R} = x\,(P \wedge P) + y\,(g \wedge P) + z\,G, \tag{5.16}$$

where $x = \frac{\phi\beta^2}{2}$, $y = \phi\alpha\beta + \mu\beta$ and $z = \phi\alpha^2 + 2\alpha\mu + \eta$.

An m-dimensional l.c.K-space form M(c), $(m > 4)$, with the curvature tensor $\bar{R}$ satisfying (5.16) on $\mathscr{U}_S \cap \mathscr{U}_C$ is called a *Roter type l.c.K-space form.*

Using (4.9) for 4-dimensional Roter type l.c.K-space forms, we get

$$\bar{R} = \Big[\eta + \frac{9}{16}\,\phi\,(2c + tr\,P)^2 + \frac{3}{2}\,\mu\,(2c + tr\,P)\Big]\,G. \tag{5.17}$$

Lemma 5.7 [38]. *Let M(c) be an m-dimensional $(m > 4)$ Roter type l.c.K-space form such that the tensor field P is hybrid. If at $x \in \mathscr{U}_S \cap \mathscr{U}_C$ the Weyl curvature tensor $\bar{C}$ is nonzero, then ϕ is nonzero at x.*

Proof. We suppose that ϕ vanishes at x. Now (5.16) reduces to

$$\bar{R} = \mu\beta\,(g \wedge P) + (2\alpha\mu + \eta)G. \tag{5.18}$$

Contracting the above equation with g^{hk}, we have

$$\bar{S}_{ij} = \mu\beta\Big[(m-2)P_{ij} + tr\,P\,g_{ij}\Big] + (2\alpha\mu + \eta)(m-1)g_{ij} \tag{5.19}$$

and so

$$\bar{\kappa} = (m-1)\Big[m(2\alpha\mu + \eta) + 2\beta\mu\,tr\,P\Big]. \tag{5.20}$$

Substituting (5.18), (5.19) and (5.20) into (2.36) we obtain $\bar{C} = 0$, a contradiction. ■

Lemma 5.8 [38]. *Let M(c) be an m-dimensional $(m > 4)$ Roter type l.c.K-space form such that the tensor field P is hybrid. If ϕ is nonzero at a point $x \in \mathscr{U}_S \cap \mathscr{U}_C$, then we have*

$$P^2 = \frac{1}{2x}\Big\{[2x\,tr\,P + y(m-2)]P + [y\,tr\,P\; + z(m-1)]g - \bar{S}\Big\}, \tag{5.21}$$

$$\bar{R}\cdot P = (m-2)\Big(\frac{y^2}{2x} - y\Big)Q(g,P) - Q(P,\bar{S}) - \frac{y}{2x}Q(g,\bar{S}). \tag{5.22}$$

Proof. Contracting (5.16) with g^{hk}, we have

$$\bar{S}_{ij} = 2x(tr\,P\,P_{ij} - P^2_{ij}) + y\Big[(m-2)P_{ij} + tr\,P\,g_{ij}\Big] + z(m-1)g_{ij} \tag{5.23}$$

and so we get (5.21). In view of (5.16) we have

$$\begin{aligned} P^r_h\bar{R}_{rijk} &= 2x(P^2_{hk}P_{ij} - P^2_{hj}P_{ik}) + y(P_{hk}P_{ij} + P^2_{hk}g_{ij} - P_{hj}P_{ik} - P^2_{hj}g_{ik}) \\ &+ z(P_{hk}g_{ij} - P_{hj}g_{ik}) \end{aligned} \tag{5.24}$$

and using (2.39) we get

$$(\bar{R}\cdot P) = 2x\,Q(P,P^2) + y\,Q(g,P^2) + z\,Q(g,P).$$

Substituting (5.21) into the above equation we obtain (5.22). ■

Theorem 5.9 [38]. *Let M(c) be an m-dimensional* $(m > 4)$ *Roter type l.c.K-space form such that the tensor field P is hybrid. Then at a point* $x \in \mathcal{U}_S \cap \mathcal{U}_C$ *at which* ϕ *is nonzero we have*

$$\bar{R} \cdot \bar{R} = Q(\bar{S}, \bar{R}) + \bar{L}_1 Q(g, \bar{C}), \tag{5.25}$$

where

$$\bar{L}_1 = (m-2)\left[\frac{y^2}{2x} - z\right]. \tag{5.26}$$

Proof. Applying (5.16) into (2.37) we get

$$\begin{aligned}(\bar{R} \cdot \bar{R})_{hijklm} &= 2x\Big[P_{ij}(\bar{E}_{hklm} + \bar{E}_{khlm}) + P_{hk}(\bar{E}_{ijlm} + \bar{E}_{jilm}) - P_{ik}(\bar{E}_{jhlm} + \bar{E}_{hjlm}) \\ &- P_{hj}(\bar{E}_{iklm} + \bar{E}_{kilm})\Big] + y\Big[g_{ij}(\bar{E}_{hklm} + \bar{E}_{khlm}) + g_{hk}(\bar{E}_{ijlm} + \bar{E}_{jilm}) \\ &- g_{ik}(\bar{E}_{jhlm} + \bar{E}_{hjlm}) - g_{hj}(\bar{E}_{iklm} + \bar{E}_{kilm})\Big],\end{aligned} \tag{5.27}$$

where

$$\begin{aligned}\bar{E}_{mijk} &= P^s_m \bar{R}_{sijk} = 2x\,(P^2_{mk}P_{ij} - P^2_{mj}P_{ik}) \\ &+ y\,(P_{mk}P_{ij} - P_{mj}P_{ik} + P^2_{mk}g_{ij} - P^2_{mj}g_{ik}) \\ &+ z\,(P_{mk}g_{ij} - P_{mj}g_{ik}),\end{aligned} \tag{5.28}$$

$$\begin{aligned}\bar{E}_{hklm} + \bar{E}_{khlm} &= 2x\,(P^2_{hm}P_{kl} - P^2_{hl}P_{km} + P^2_{km}P_{hl} - P^2_{kl}P_{hm}) \\ &+ y\,(P^2_{hm}g_{kl} - P^2_{hl}g_{km} + P^2_{km}g_{hl} - P^2_{kl}g_{hm}) \\ &+ z\,(P_{hm}g_{kl} - P_{hl}g_{km} + P_{km}g_{hl} - P_{kl}g_{hm}) \\ &= 2x\,Q(P, P^2)_{hklm} + y\,Q(g, P^2)_{hklm} + z\,Q(g, P)_{hklm}.\end{aligned} \tag{5.29}$$

Now (5.27) in view of Lemma 2.1, Lemma 2.2, (2.31) and (5.29) yields

$$\begin{aligned}\bar{R} \cdot \bar{R} &= z\,Q(g, R) - Q(2x\,P^2, 2x\,\bar{P}) - Q\Big(2x\,P^2, y\,(g \wedge P)\Big) \\ &- \frac{y^2}{2x}\,Q(2x\,P^2, G),\end{aligned}$$

where $\bar{P} = \frac{1}{2} P \wedge P$. Substituting (5.21) into the above equation we have

$$\begin{aligned}
\bar{R} \cdot \bar{R} &= z\, Q(g,R) - Q\Big([y\, tr\, P + (m-1)z]g, 2x\, \bar{P}\Big) \\
&+ Q(\bar{S}, 2x\, \bar{P}) - Q\Big([2x\, tr\, P + (m-2)y]\, P, y\, (g \wedge P)\Big) \\
&- Q\Big([y\, tr\, P + (m-1)z]g, y\, (g \wedge P)\Big) \\
&+ Q(\bar{S}, y\, (g \wedge P)) - \frac{y^2}{2x}\, Q\Big([2x\, tr\, P + (m-2)y]P, G\Big) \\
&+ \frac{y^2}{2x}\, Q(\bar{S}, G) + Q(\bar{S}, zG) - Q(\bar{S}, zG)
\end{aligned}$$

and so we get

$$\begin{aligned}
\bar{R} \cdot \bar{R} &= Q(\bar{S}, \bar{R}) + z\, Q(g, \bar{R}) - \Big(\frac{y^2}{2x} - z\Big) Q(g, g \wedge \bar{S}) \\
&- z\, (m-1) Q(g, 2x\, \bar{P}) + \frac{(m-2)y^2}{2x}\, Q(g, 2x\, \bar{P}) \\
&- z\, (m-1) Q(g, y(g \wedge P)) + \frac{(m-2)y^2}{2x}\, Q(g, y(g \wedge P)) \\
&= Q(\bar{S}, \bar{R}) + (m-2)\Big(\frac{y^2}{2x} - z\Big) Q(g, \bar{C}).
\end{aligned}$$

This completes the proof. ■

Theorem 5.10 [38]. *Let M(c) be an m-dimensional* $(m > 4)$ *Roter type l.c.K-space form such that the tensor field P is hybrid. Then at a point* $x \in \mathscr{U}_S \cap \mathscr{U}_C$ *at which* ϕ *is nonzero we have*

$$\begin{aligned}
(m-2)(\bar{R} \cdot \bar{C} - \bar{C} \cdot \bar{R}) &= \Big[2(y\, tr\, P + z\, (m-1)) - \frac{\kappa}{m-1}\Big] Q(g, \bar{R}) \\
&+ \Big[2x\, tr\, P + y\, (m-2)\Big] Q(P, \bar{R}) - 2x\, Q(P^2, \bar{R}) \\
&+ 2x\, \Big[2x\, tr\, P + y\, (m-1)\Big] \Big[P \wedge Q(g, P^2) - g \wedge Q(P, P^2)\Big] \\
&+ y\, \Big[2x\, tr\, P + y\, (m-2)\Big] Q(g, \bar{P}) \\
&+ (2x)^2\, \Big[P \wedge Q(g, N) - g \wedge Q(P, N)\Big], \qquad (5.30)
\end{aligned}$$

where $N_{ij} = P_i^s P_{sj}^2$.

Proof. Substituting (5.23) into (4.7), we get

$$\begin{aligned}
(m-2)(\bar{R} \cdot \bar{C} - \bar{C} \cdot \bar{R})_{hijklm} &= \Big\{2[y\, tr\, P + z\, (m-1)] - \frac{\kappa}{m-1}\Big\} Q(g, \bar{R})_{hijklm} \\
&+ \Big[2x\, tr\, P + y\, (m-2)\Big] Q(P, \bar{R})_{hijklm} - 2x\, Q(P^2, \bar{R})_{hijklm}
\end{aligned}$$

$$
\begin{aligned}
+ & \left[2x\, tr\, P + y\,(m-2)\right]\left[g_{lh}\bar{E}_{mijk} - g_{mh}\bar{E}_{lijk} - g_{li}\bar{E}_{mhjk}\right. \\
+ & \; g_{mi}\bar{E}_{lhjk} + g_{lj}\bar{E}_{mkhi} - g_{mj}\bar{E}_{lkhi} - g_{kl}\bar{E}_{mjhi} + g_{km}\bar{E}_{ljhi} \\
- & \; g_{ij}(\bar{E}_{khlm} + \bar{E}_{hklm}) - g_{hk}(\bar{E}_{jilm} + \bar{E}_{ijlm}) + g_{ik}(\bar{E}_{jhlm} + \bar{E}_{hjlm}) \\
+ & \left. g_{hj}(\bar{E}_{kilm} + \bar{E}_{iklm})\right] - 2x\left[g_{lh}\bar{F}_{mijk} - g_{mh}\bar{F}_{lijk} - g_{li}\bar{F}_{mhjk}\right. \\
+ & \; g_{mi}\bar{F}_{lhjk} + g_{lj}\bar{F}_{mkhi} - g_{mj}\bar{F}_{lkhi} - g_{kl}\bar{F}_{mjhi} + g_{km}\bar{F}_{ljhi} \\
- & \; g_{ij}(\bar{F}_{khlm} + \bar{F}_{hklm}) - g_{hk}(\bar{F}_{jilm} + \bar{F}_{ijlm}) + g_{ik}(\bar{F}_{jhlm} + \bar{F}_{hjlm}) \\
+ & \left. g_{hj}(\bar{F}_{kilm} + \bar{F}_{iklm})\right], \qquad (5.31)
\end{aligned}
$$

where $\bar{F}_{mijk} = P^2_{mr} g^{rs} \bar{R}_{sijk}$.

Using (5.28), we get

$$
\begin{aligned}
& g_{lh}\bar{E}_{mijk} - g_{mh}\bar{E}_{lijk} - g_{li}\bar{E}_{mhjk} + g_{mi}\bar{E}_{lhjk} + g_{lj}\bar{E}_{mkhi} \\
- & \; g_{mj}\bar{E}_{lkhi} - g_{kl}\bar{E}_{mjhi} + g_{km}\bar{E}_{ljhi} \\
= & \; 2x\Big(P \wedge Q(g,P^2)\Big)_{hijklm} + y\Big(Q(g,\bar{P})_{hijklm} - Q(P^2,G)_{hijklm}\Big) \\
- & \; z\, Q(P,G)_{hijklm}. \qquad (5.32)
\end{aligned}
$$

The equation (5.16) implies

$$
\begin{aligned}
\bar{F}_{mijk} &= 2x\,(N_{mk}P_{ij} - N_{mj}P_{ik}) \\
&+ y\,(P^2_{mk}P_{ij} - P^2_{mj}P_{ik} + N_{mk}g_{ij} - N_{mj}g_{ik}) \\
&+ z\,(P^2_{mk}g_{ij} - P^2_{mj}g_{ik}), \qquad (5.33)
\end{aligned}
$$

$$
\begin{aligned}
\bar{F}_{hklm} + \bar{F}_{khlm} &= 2x\, Q(P,N)_{hklm} + y\,\Big(Q(P,P^2)_{hklm} + Q(g,N)_{hklm}\Big) \\
&+ z\, Q(g,P^2)_{hklm}. \qquad (5.34)
\end{aligned}
$$

Using (5.33), we get

$$
\begin{aligned}
& g_{lh}\bar{F}_{mijk} - g_{mh}\bar{F}_{lijk} - g_{li}\bar{F}_{mhjk} + g_{mi}\bar{F}_{lhjk} + g_{lj}\bar{F}_{mkhi} \\
- & \; g_{mj}\bar{F}_{lkhi} - g_{kl}\bar{F}_{mjhi} + g_{km}\bar{F}_{ljhi} \\
= & \; 2x\,\Big(P \wedge Q(g,N)\Big)_{hijklm} + y\,\Big((P \wedge Q(g,P^2))_{hijklm} - Q(N,G)_{hijklm}\Big) \\
- & \; z\, Q(P^2,G)_{hijklm}. \qquad (5.35)
\end{aligned}
$$

Substituting (5.29), (5.32), (5.34) and (5.35) into (5.31) and using (2.32), (2.33) we obtain (5.30). ■

5.3 Bochner Curvature Tensor On Locally Conformal Kaehler Space Forms

In this section, the Bochner curvature tensor in l.c.K-manifolds and l.c.K-space forms are presented. Moreover, some properties of the Bochner curvature tensor in an l.c.K-space form are obtained.

The Bochner curvature tensor in a Kaehler manifold $M^m(J,g)$ is defined by [39]

$$\begin{aligned} B &= R - \frac{1}{m+4}(S \wedge g + \widetilde{S} \overline{\overline{\wedge}} J) \\ &+ \frac{\kappa}{2(m+2)(m+4)}(g \wedge g + J \overline{\overline{\wedge}} J), \end{aligned} \tag{5.36}$$

where $\widetilde{S}_{ij} = S_{ir}J^r_j$.

Using

$$R_{rstq}J^r_iJ^s_jJ^t_hJ^q_k = R_{ijhk}, \tag{5.37}$$

the Bochner curvature tensor in a Kaehler manifold has been generalized into an almost Hermitian manifold which is given by [40]

$$B = R - (T \wedge g + \widetilde{T} \overline{\overline{\wedge}} J) + \frac{\bar{\kappa} - \kappa}{8m(m-2)}(3\, g \wedge g - J \overline{\overline{\wedge}} J), \tag{5.38}$$

where

$$T = \frac{1}{4(m+4)}\Big(S + 3\bar{S} - \frac{\kappa + 3\bar{\kappa}}{2(m+2)} g\Big), \tag{5.39}$$

$$\widetilde{T}_{ij} = T_{ir}J^r_j, \quad Z_{ijhk} = R_{ijrs}J^r_hJ^s_k - R_{ijhk}, \quad \bar{S}_{ij} = S_{ij} + Z_{ij}, \quad \bar{\kappa} = \bar{S}_{rs}g^{rs}.$$

Theorem 5.11. *In an m-dimensional l.c.K-manifold the Bochner curvature tensor is*

$$\begin{aligned} B_{ijhk} &= R - \frac{1}{m+4}\Big[(S \wedge g) + (\widetilde{S} \overline{\overline{\wedge}} J)\Big] \\ &+ \frac{3(m-3)}{4(m+4)}(P \wedge g) + \frac{3(m-4)}{4(m+4)}(K \overline{\overline{\wedge}} J) \\ &+ \frac{2m\kappa - 3(m^2+2m+8)\, tr\, P}{4m(m+2)(m+4)}(g \wedge g) \\ &+ \frac{-2m\kappa + (m^2-6m+8)\, tr\, P}{4m(m+2)(m+4)}(J \overline{\overline{\wedge}} J) \\ &+ \frac{3}{2(m+4)}(\widehat{K} \wedge g), \end{aligned} \tag{5.40}$$

where $K_{ij} = P_{ir}J^r_j$ and $\widehat{K}_{ij} = P_{rs}J^r_iJ^s_j$.

Proof. In l.c.K-manifolds we have [2]

$$
\begin{aligned}
Z_{ijhk} &= R_{ijrs}J_h^rJ_k^s - R_{ijhk} \\
&= P_{jk}g_{ih} - P_{jh}g_{ik} + P_{ih}g_{jk} - P_{ik}g_{jh} \\
&+ P_{jr}J_k^rJ_{ih} - P_{jr}J_h^rJ_{ik} + P_{ir}J_h^rJ_{jk} - P_{ir}J_k^rJ_{jh}. \qquad (5.41)
\end{aligned}
$$

Contracting the above equation with g^{ik} we get

$$Z_{jh} = -(m-3)P_{jh} - tr\, P\, g_{jh} + P_{rs}J_j^rJ_h^s\,. \qquad (5.42)$$

In view of (5.39) and (5.42) we get

$$
\begin{aligned}
(T\wedge g)+(\widetilde{T}\,\overline{\overline{\wedge}}\,J) &= \frac{1}{m+4}\Big[(S\wedge g)+(\widetilde{S}\,\overline{\overline{\wedge}}\,J)\Big] \\
&- \frac{3(m-3)}{4(m+4)}(P\wedge g) - \frac{3(m-4)}{4(m+4)}(K\,\overline{\overline{\wedge}}\,J) \\
&- \left(\frac{\kappa}{2(m+2)(m+4)} + \frac{3\,tr\,P}{4(m+4)} - \frac{3(m-2)\,tr\,P}{4(m+2)(m+4)}\right)(g\wedge g) \\
&- \left(\frac{-\kappa}{2(m+2)(m+4)} - \frac{3\,tr\,P}{4(m+4)} + \frac{3(m-2)\,tr\,P}{4(m+2)(m+4)}\right)(J\,\overline{\overline{\wedge}}\,J) \\
&- \frac{3}{2(m+4)}(\widehat{K}\wedge g) \qquad (5.43)
\end{aligned}
$$

and we also have

$$\frac{\acute{\kappa}-\kappa}{8m(m-2)}(3\,g\wedge g - J\,\overline{\overline{\wedge}}\,J) = -\frac{tr\,P}{4m}(3\,g\wedge g - J\,\overline{\overline{\wedge}}\,J). \qquad (5.44)$$

Substituting (5.43) and (5.44) into (5.38) we obtain (5.40). ■

Theorem 5.12. *Let M be an m-dimensional l.c.K-manifold such that the tensor field P is hybrid. Then the Bochner curvature tensor is given by*

$$
\begin{aligned}
B &= R - \frac{1}{m+4}\Big((S\wedge g)+(\widetilde{S}\,\overline{\overline{\wedge}}\,J)\Big) \\
&+ \frac{3(m-4)}{4(m+4)}\Big((P\wedge g)-(\widetilde{P}\,\overline{\overline{\wedge}}\,J)\Big) \\
&+ \frac{2m\kappa - 3(m^2+2m+8)\,tr\,P}{4m(m+2)(m+4)}(g\wedge g) \\
&+ \frac{-2m\kappa + (m^2-6m+8)\,tr\,P}{4m(m+2)(m+4)}(J\,\overline{\overline{\wedge}}\,J). \qquad (5.45)
\end{aligned}
$$

Proof. Contracting (5.41) with g^{ik} we get

$$Z_{jh} = -(m-4)P_{jh} - tr\,P\,g_{jh}. \qquad (5.46)$$

In view of (5.39) and (5.46) we get

$$
\begin{aligned}
(T\wedge g)+(\widetilde{T}\,\overline{\overline{\wedge}}\,J) &= \frac{1}{m+4}\Big((S\wedge g)+(\widetilde{S}\,\overline{\overline{\wedge}}\,J)\Big)\\
&- \frac{3(m-4)}{4(m+4)}\Big((P\wedge g)-(\widetilde{P}\,\overline{\overline{\wedge}}\,J)\Big)\\
&- \left(\frac{\kappa}{2(m+2)(m+4)}+\frac{3\,tr\,P}{4(m+4)}-\frac{3(m-2)\,tr\,P}{4(m+2)(m+4)}\right)(g\wedge g)\\
&- \left(\frac{-\kappa}{2(m+2)(m+4)}-\frac{3\,tr\,P}{4(m+4)}+\frac{3(m-2)\,tr\,P}{4(m+2)(m+4)}\right)(J\,\overline{\overline{\wedge}}\,J).
\end{aligned}
\tag{5.47}
$$

Substituting (5.44) and (5.47) into (5.38) we obtain (5.45). ■

Theorem 5.13. *Let M(c) be an m-dimensional l.c.K-space form such that the tensor field P is hybrid. Then the Bochner curvature tensor is given by*

$$B=R+\lambda(g\wedge g)+\gamma\,(J\,\overline{\overline{\wedge}}\,J), \tag{5.48}$$

where

$$\lambda=-\frac{\alpha}{m+4}+\frac{2m\kappa-3(m^2+2m+8)\,tr\,P}{4m(m+2)(m+4)},$$

$$\gamma=\frac{\alpha}{m+4}+\frac{-2m\kappa+(m^2-6m+8)\,tr\,P}{4m(m+2)(m+4)}$$

and $\alpha=\frac{1}{4}[(m+2)c+3\,tr\,P]$.

Proof. Using (4.1) into the (5.45), we get

$$
\begin{aligned}
B_{ijhk} &= R_{ijhk}+\Big[-\frac{\alpha}{m+4}+\frac{2m\kappa-3(m^2+2m+8)\,tr\,P}{4m(m+2)(m+4)}\Big](2g_{ik}g_{jh}-2g_{ih}g_{jk})\\
&+ \Big[\frac{\alpha}{m+4}+\frac{-2m\kappa+(m^2-6m+8)\,tr\,P}{4m(m+2)(m+4)}\Big](2J_{ik}J_{jh}-2J_{ih}J_{jk}-4J_{ij}J_{hk}).
\end{aligned}
$$

Using (2.30) we obtain (5.48). ■

Theorem 5.14. *Let M(c) be an m-dimensional l.c.K-space form such that the tensor field P is hybrid. Then we have*

$$R\cdot B=R\cdot R+2\gamma\,T \tag{5.49}$$

and

$$B\cdot R=R\cdot R+2\lambda\;Q(g,R)+2\gamma\,\bar{T}, \tag{5.50}$$

where

$$\begin{aligned} T_{hijklm} &= J_{ij}(V_{hklm} - V_{khlm}) + J_{hk}(V_{ijlm} - V_{jilm}) \\ &+ J_{ik}(V_{jhlm} - V_{hjlm}) + J_{hj}(V_{kilm} - V_{iklm}) \\ &+ 2J_{jk}(V_{ihlm} - V_{hilm}) + 2J_{hi}(V_{kjlm} - V_{jklm}), \end{aligned} \tag{5.51}$$

$$\begin{aligned} \bar{T}_{hijklm} &= \Big(J_{lh}V_{mijk} + J_{hm}V_{lijk} + J_{il}V_{mhjk} - J_{im}V_{lhjk} \\ &- J_{jl}V_{mkhi} + J_{jm}V_{lkhi} + J_{kl}V_{mjhi} - J_{km}V_{ljhi} \Big) \\ &+ 2J_{lm}(V_{hijk} - V_{ihjk} + V_{jkhi} - V_{kjhi}) \end{aligned} \tag{5.52}$$

and $V_{ijlm} = J_i^s R_{sjlm}$.

Proof. Using (2.37) and in view of (5.48), we get

$$\begin{aligned} (R \cdot B)_{hijklm} &= g^{rs}(B_{rijk}R_{shlm} + B_{hrjk}R_{silm} + B_{hirk}R_{sjlm} + B_{hijr}R_{sklm}) \\ &= (R \cdot R)_{hijklm} + 2\gamma\, T_{hijklm} \end{aligned} \tag{5.53}$$

and

$$\begin{aligned} (B \cdot R)_{hijklm} &= g^{rs}(R_{rijk}B_{shlm} + R_{hrjk}B_{silm} + R_{hirk}B_{sjlm} + R_{hijr}B_{sklm}) \\ &= (R \cdot R)_{hijklm} + 2\lambda Q(g,R)_{hijklm} + 2\gamma\, \bar{T}_{hijklm}. \end{aligned} \tag{5.54}$$

This completes the proof. ■

Theorem 5.15. *Let M(c) be an m-dimensional l.c.K-space form such that the tensor field P is hybrid. Then the following three equalities are equivalent :*

$$(R \cdot B)_{hijklm} + (R \cdot B)_{jklmhi} + (R \cdot B)_{lmhijk} = 0, \tag{5.55}$$

$$(B \cdot R)_{hijklm} + (B \cdot R)_{jklmhi} + (B \cdot R)_{lmhijk} = 0 \tag{5.56}$$

and

$$(R \cdot B - B \cdot R)_{hijklm} + (R \cdot B - B \cdot R)_{jklmhi} + (R \cdot B - B \cdot R)_{lmhijk} = 0 \tag{5.57}$$

on M(c).

Proof. We set

$$\begin{aligned}
\mathscr{A}_{hijklm} &= J_{ij}(V_{hklm}-V_{khlm})+J_{hk}(V_{ijlm}-V_{jilm})\\
&+ J_{ik}(V_{jhlm}-V_{hjlm})+J_{hj}(V_{kilm}-V_{iklm})\\
&+ J_{kl}(V_{jmhi}-V_{mjhi})+J_{jm}(V_{klhi}-V_{lkhi})\\
&+ J_{km}(V_{ljhi}-V_{jlhi})+J_{jl}(V_{mkhi}-V_{kmhi})\\
&+ J_{mh}(V_{lijk}-V_{iljk})+J_{li}(V_{mhjk}-V_{hmjk})\\
&+ J_{mi}(V_{hljk}-V_{lhjk})+J_{lh}(V_{imjk}-V_{mijk})\\
&+ 2J_{jk}(V_{ihlm}-V_{hilm})+2J_{hi}(V_{kjlm}-V_{jklm})\\
&+ 2J_{lm}(V_{kjhi}-V_{jkhi})+2J_{jk}(V_{mlhi}-V_{lmhi})\\
&+ 2J_{hi}(V_{mljk}-V_{lmjk})+2J_{lm}(V_{ihjk}-V_{hijk}). \qquad (5.58)
\end{aligned}$$

Symmetrizing (5.49) with respect to the pairs (h,i), (j,k) and (l,m) and applying (5.2) we obtain

$$\begin{aligned}
(R\cdot B)_{hijklm}+(R\cdot B)_{jklmhi}+(R\cdot B)_{lmhijk} &= 2\gamma\,(T_{hijklm}+T_{jklmhi}+T_{lmhijk})\\
&= 2\gamma\,\mathscr{A}_{hijklm}. \qquad (5.59)
\end{aligned}$$

In the same way, using (5.50) and applying (5.1) and (5.2) we have

$$\begin{aligned}
(B\cdot R)_{hijklm}+(B\cdot R)_{jklmhi}+(B\cdot R)_{lmhijk} &= -2\gamma\,(\bar{T}_{hijklm}+\bar{T}_{jklmhi}+\bar{T}_{lmhijk})\\
&= -2\gamma\,\mathscr{A}_{hijklm}. \qquad (5.60)
\end{aligned}$$

From (5.59) and (5.60) we get

$$(R\cdot B-B\cdot R)_{hijklm}+(R\cdot B-B\cdot R)_{jklmhi}+(R\cdot B-B\cdot R)_{lmhijk}=4\gamma\,\mathscr{A}_{hijklm}. \qquad (5.61)$$

This completes the proof. ■

Theorem 5.16. *Let M(c) be an m-dimensional l.c.K-space form such that the tensor field P is hybrid. Then we have*

$$\begin{aligned}
(B\cdot B)_{hijklm} &= 2(R\cdot B)_{hijklm}-(R\cdot R)_{hijklm}\\
&+ 2(\lambda-\gamma)\Big[J_{ij}(J\barwedge g)_{hklm}+J_{ik}(J\barwedge g)_{jhlm}\\
&+ J_{hk}(J\barwedge g)_{ijlm}+J_{hj}(J\barwedge g)_{kilm}+2J_{jk}(J\barwedge g)_{ihlm}\\
&+ 2J_{hi}(J\barwedge g)_{kjlm}\Big]. \qquad (5.62)
\end{aligned}$$

Proof. In view of (2.37) and using (5.48), we get

$$
\begin{aligned}
(B\cdot B)_{hijklm} &= g^{rs}(B_{rijk}B_{shlm}+B_{hrjk}B_{silm}+B_{hirk}B_{sjlm}+B_{hijr}B_{sklm})\\
&= (R\cdot B)_{hijklm}+2\gamma\Big[J_{ij}(\widetilde{V}_{hklm}-\widetilde{V}_{khlm})+J_{hk}(\widetilde{V}_{ijlm}-\widetilde{V}_{jilm})\\
&+ J_{ik}(\widetilde{V}_{jhlm}-\widetilde{V}_{hjlm})+J_{hj}(\widetilde{V}_{kilm}-\widetilde{V}_{iklm})\\
&+ 2J_{jk}(\widetilde{V}_{ihlm}-\widetilde{V}_{hilm})+2J_{hi}(\widetilde{V}_{kjlm}-\widetilde{V}_{jklm})\Big], \qquad (5.63)
\end{aligned}
$$

where $\widetilde{V}_{ijlm}=J_i^sB_{sjlm}$.

Furthermore in view of (5.48) and using (2.34) we have

$$
\begin{aligned}
\widetilde{V}_{hklm}-\widetilde{V}_{khlm} &= J_h^s\Big[R_{sklm}+2\lambda(g_{sm}g_{kl}-g_{sl}g_{km})+2\,\gamma(J_{sm}J_{kl}-J_{sl}J_{km}-2J_{sk}J_{lm})\Big]\\
&- J_k^s\Big[R_{shlm}+2\lambda(g_{sm}g_{hl}-g_{sl}g_{hm})+2\,\gamma(J_{sm}J_{hl}-J_{sl}J_{hm}-2J_{sh}J_{lm})\Big]\\
&= V_{hklm}-V_{khlm}+2(\lambda-\gamma)(J\,\overline{\wedge}\,g)_{hklm}. \qquad (5.64)
\end{aligned}
$$

Substituting (5.64) into (5.63), we get

$$
\begin{aligned}
(B\cdot B)_{hijklm} &= (R\cdot B)_{hijklm}+2\,\gamma\Big[J_{ij}(V_{hklm}-V_{khlm})+J_{hk}(V_{ijlm}-V_{jilm})\\
&+ J_{ik}(V_{jhlm}-V_{hjlm})+J_{hj}(V_{kilm}-V_{iklm})+2J_{jk}(V_{ihlm}-V_{hilm})\\
&+ 2J_{hi}(V_{kjlm}-V_{jklm})\Big]+2(\lambda-\gamma)\Big[J_{ij}(J\,\overline{\wedge}\,g)_{hklm}J_{ik}(J\,\overline{\wedge}\,g)_{jhlm}\\
&+ J_{hk}(J\,\overline{\wedge}\,g)_{ijlm}+J_{hj}(J\,\overline{\wedge}\,g)_{kilm}+2J_{jk}(J\,\overline{\wedge}\,g)_{ihlm}\\
&+ 2J_{hi}(J\,\overline{\wedge}\,g)_{kjlm}\Big]. \qquad (5.65)
\end{aligned}
$$

Using (5.49), we obtain (5.62). ■

Theorem 5.17. *Let M(c) be an m-dimensional* $(m>4)$ *pseudosymmetric l.c.K-space form such that the tensor field P is hybrid and tr P is constant. If the condition*

$$B\cdot B=L_BQ(g,B) \qquad (5.66)$$

is fulfilled on $\mathscr{U}_B=\{x\in M(c)\mid B\neq 0 \text{ at } x\}$*, where* L_B *is a function on* $\mathscr{U}_B$*, then M(c) is Einstein.*

Proof. Using (5.62) we have

$$
\begin{aligned}
L_BQ(g,B)_{hijklm} &= 2(R\cdot B)_{hijklm}-(R\cdot R)_{hijklm}\\
&+ 2(\lambda-\gamma)\Big[J_{ij}(J\,\overline{\wedge}\,g)_{hklm}+J_{ik}(J\,\overline{\wedge}\,g)_{jhlm}\\
&+ J_{hk}(J\,\overline{\wedge}\,g)_{ijlm}+J_{hj}(J\,\overline{\wedge}\,g)_{kilm}+2J_{jk}(J\,\overline{\wedge}\,g)_{ihlm}\\
&+ 2J_{hi}(J\,\overline{\wedge}\,g)_{kjlm}\Big]. \qquad (5.67)
\end{aligned}
$$

Now in view of (2.42) and (5.49) we get

$$\begin{aligned} L_B Q(g,B)_{hijklm} &= L_R Q(g,R)_{hijklm} + 4\gamma\, T_{hijklm} \\ &+ 2(\lambda-\gamma)\Big[J_{ij}(J\overline{\wedge} g)_{hklm} + J_{ik}(J\overline{\wedge} g)_{jhlm} \\ &+ J_{hk}(J\overline{\wedge} g)_{ijlm} + J_{hj}(J\overline{\wedge} g)_{kilm} + 2J_{jk}(J\overline{\wedge} g)_{ihlm} \\ &+ 2J_{hi}(J\overline{\wedge} g)_{kjlm}\Big]. \end{aligned} \tag{5.68}$$

Contracting (5.68) with g^{hm} and g^{ij} we get

$$L_B(-mS_{lk} + \kappa g_{lk}) = L_R(-mS_{lk} + \kappa g_{lk}). \tag{5.69}$$

Substituting (4.1) into the above equation we get

$$(L_R - L_B)(-m\alpha + \kappa)g_{lk} = (L_R - L_B)(-m\alpha + \kappa)P_{lk},$$

where $\alpha = \frac{1}{4}[(m+2)c + 3\, tr\, P]$.

Using the fact that $L_R \neq L_B$ we obtain

$$P_{lk} = \frac{tr\, P}{m} g_{lk}$$

which means that the tensor field P is proportional to g and in view of Theorem 4.2., M(c) is Einstein. ■

6. CONCLUSIONS AND RECOMMENDATIONS

Let M be a real 2n-dimensional Hermitian manifold with structure (J, g), where J is the almost complex structure and g is the Hermitian metric. The manifold M is called *a locally conformal Kaehler manifold* (an l.c.K-manifold) if each point p in M has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that $g^* = e^{-2\rho} g \mid_{\mathrm{U}}$ is a Kaehlerian metric on U.

An 2n-dimensional l.c.K-manifold is a Hermitian manifold admitting a global closed 1-form α (Lee form) whose structure (J, g) satisfies $\nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki}$, where $\beta_i = \alpha^r J_{ri}$ and ∇ denotes the covariant differentiation with respect to the Hermitian metric g.

An l.c.K-manifold $M(J, g, \alpha)$ is called *an l.c.K-space form* if it has a constant holomorphic sectional curvature. We give a generalization about the results of an l.c.K-space form and invariant submanifolds of l.c.K-space forms with the tensor field P is not hybrid.

It is proved that for a 2n-dimensional l.c.K-space form $M(c)$, if the tensor field P is proportional to g and $tr\,P$ is constant, then $M(c)$ is Einstein. The Sato's form of the holomorphic curvature tensor in an l.c.K-manifold are presented.

Some results on pseudosymmetric and Ricci-pseudosymmetric l.c.K-space forms are obtained. It is proved that for 4-dimensional l.c.K-space forms such that the tensor field P is hybrid and $tr\,P$ is constant, $R \cdot C - C \cdot R = [\frac{1}{4}\,(2c + tr\,P)]\, Q(g, R)$ and for m-dimensional $(m > 4)$ with the tensor P is proportional to g in l.c.K-space forms $R \cdot C - C \cdot R = \frac{1}{4(m-1)} \left[(m+2)c + \frac{6(m-2) tr\,P}{m} \right] Q(g, R)$.

Furthermore, we present results on l.c.K-space forms satisfying curvature identities called Walker type identities. It is proved that a 4-dimensional l.c.K-space form such that the tensor field P is hybrid and $tr\,P$ is constant satisfies Walker type identities. We introduced the Roter type l.c.K-space forms. If P is hybrid, it is proved $\bar{R} \cdot \bar{R} = Q(\bar{S}, \bar{R}) + \bar{L}_1 Q(g, \bar{C})$ in m-dimensional $(m > 4)$ Roter type l.c.K-space forms.

Moreover, we present a generalization about the Bochner curvature tensor in an l.c.K-manifold with the tensor field P is not hybrid. Moreover, we state the Bochner curvature tensor in an l.c.K-space form. Furthermore, Walker type identities for Bochner curvature tensor are studied.

In the future, we aim to study the Bochner pseudosymmetry in l.c.K-manifolds, the l.c.K-space forms which satisfy some properties of the Bochner curvature tensor and some properties of Roter type l.c.K-space forms.

Furthermore, some properties of pseudosymmetric and Ricci-pseudosymmetric l.c.K-space forms will be studied. Moreover, we are going to work hypersurfaces of l.c.K-manifolds and l.c.K-space forms. Later on, as a natural extension, we are going to study pseudosymmetric hypersurfaces of l.c.K-manifolds in the sense of Deszcz.

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▪ **Mutlu, P.**, Şentürk, Z., 2015. On Locally Conformal Kaehler Space Forms, *Filomat*, 29(3), 593-597.

▪ **Mutlu, P.**, Şentürk, Z., Walker Type Identities on Locally Conformal Kaehler Space Forms, *Rend. Sem. Mat. Univ. e Politec. Torino*. (accepted)

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