ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE, ENGINEERING AND TECHNOLOGY

ON CONFORMAL CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

M.Sc THESIS

Sena KAÇAR

Mathematics Department

Mathematical Engineering Programme

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

ON CONFORMAL CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

M.Sc THESIS

Sena KAÇAR
(509121055)

Department of Mathematical Engineering
Mathematical Engineering Programme

Thesis Advisor : Prof. Dr. Fatma Özdemir

DECEMBER 2017

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

YARI SİMETRİK KONNEKSİYONLU RİEMANN UZAYLARININ KONFORM EĞRİLİK TENSÖRLERİ

YÜKSEK LİSANS TEZİ

Sena KAÇAR
(509121055)

Matematik Mühendisliği Anabilim Dalı
Matematik Mühendisliği Programı

Tez Danışmanı : Prof. Dr. Fatma Özdemir

ARALIK 2017

Sena KAÇAR, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509121055 successfully defended the thesis entitled ”ON CONFORMAL CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION”, which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor : **Prof. Dr. Fatma ÖZDEMİR**
İstanbul Technical University

Jury Members : **Prof. Dr. Fatma OZDEMIR**
İstanbul Technical University

Asst. Prof. Bahri GULDOGAN
İstanbul Technical University

Assoc. Prof. Hakan Mete TASTAN
İstanbul University

Date of Submission : **17 Nov 2017**
Date of Defense : **12 Dec 2017**

to my family...

FOREWORD

I want to present thanks to my advisor Prof. Dr. Fatma Özdemir who provides endless knowledge and source to me. Moreover I would like to express my gratitude to my family for their infinite support and love.

DECEMBER 2017 Sena Kaçar

CONTENTS

Page

CONTENTS

Abbreviations

$\mathbf{M}$	: Manifold
$\mathbf{g_{ij}}$	: Metric tensor
$\nabla_{\mathbf{k}}$	: Covariant derivative
$\left\{ {\mathbf{i} \atop \mathbf{jk}} \right\}$	: Connection coefficients of Levi-Civita connections
$\mathbf{R^{i}{}_{jkl}}$	: Curvature tensor of Riemannian manifold
$\mathbf{C^{i}{}_{jkl}}$	: Conformal curvature tensor of Riemannian manifold
$\Gamma^{\mathbf{i}}{}_{\mathbf{jk}}$	: Connection coefficient of semi-symmetric Riemannian manifold
$\mathbf{\bar{R}^{i}{}_{jkl}}$	: Curvature tensor of semi-symmetric Riemannian manifold
$\mathbf{\bar{C}^{i}{}_{jkl}}$	: Conformal curvature tensor of semi-symmetric Riemmannian monifold

ON CONFORMAL CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

SUMMARY

In this thesis, we study on Riemannian manifolds with semi symmetric metric connection, which was introduced in [1-10].

In the first chapter, differentiable manifold, Riemannian manifold, affine connection, Riemannian connection , the covariant derivative of Riemannian manifold with respect to Riemannian connection are given. Then, the curvature tensors of Riemannian manifold are given and their properties of the curvature tensor are mentioned [2,4].

Afterwards, the conformal curvature tensor of Riemannian manifold is obtained and properties of the conformal curvature tensor are given.

In the second chapter, we study on Riemannian manifolds with semi symmetric metric connection.The connection coefficient of Riemannian manifold with semi-symmetric metric connnection is given as

$$\Gamma^{l}{}_{jk} = \left\{ \begin{matrix} l \\ jk \end{matrix} \right\} + \delta_k{}^l \pi_j - \pi^l g_{jk},$$

and the curvature tensor of Rieannian manifold with semi-symmetric metric connection is obtained such that

$$\overline{R}^{h}{}_{ijk} = R^{h}{}_{ijk} + \delta_k{}^h \pi_{ij} - \delta_j{}^h \pi_{ik} + g_{ij} g^{hr} \pi_{rk} - g_{ik} g^{hr} \pi_{rj}.$$

In the third chapter, conformal curvature tensor of Riemannian manifold with semi-symmetric metric connection is given as [10]

$$\overline{C}^{h}{}_{jkl} = \overline{R}^{h}{}_{jkl} + \frac{1}{(n-2)}(\delta^h{}_j \overline{R}_{ik} - \delta^h_i + g_{ik} g^{hl} \overline{R}_{lj} - g_{ik} g^{hl} \overline{R}_{lk}) + \frac{\overline{R}}{(n-1)(n-2)}(\delta^h{}_j \overline{R}_{ik} - \delta^h{}_j g_{ik}).$$

After that, the properties of the tensor $\overline{C}_{mjkl}$ are investigated. In the last part, properties of the conformal curvature tensor of Riemannian manifold with semi-symmetric metric connection are mentioned and we proved that the conformal curvature tensor satisfies the relation

$$\overline{C}_{mijk,l} + \overline{C}_{mikl,j} + \overline{C}_{milj,k} = \frac{1}{n-3}(g_{ik}\overline{C}_{mjl} + g_{mj}\overline{C}_{ikl+g_{il}\overline{C}_{mkj}} + g_{mk}\overline{C}_{ilj} + g_{ij}\overline{C}_{mik} + g_{ml}\overline{C}_{ijk}).$$

YARI SİMETRİK KONNEKSİYONLU RİEMANN UZAYLARININ KONFORM EĞRİLİK TENSÖRLERİ

ÖZET

Bu tezimizde yarı simetrik metrik konneksiyonlu Riemann manifoldları incelenmiştir [1-10].

Tezin birinci bölümünde, diferansiyellenebilir manifold, Riemann manifoldu, affin konneksiyon, Riemann konneksiyonu, Riemann konneksiyonuna göre kovaryant türev tanımları detaylı bir şekilde verilmiştir. Daha sonra, local koordinatlarda Riemann manifoldunun eğrilik, kovaryant eğrilik , Ricci eğrilik tensörleri ve skaler eğrilik tanımları ve bu büyüklüklere ait özellikler verilmiştir [2,4].

Birinci bölümün ikinci kısmında Riemann manifoldlarına ait konform dönüşüm altında konneksiyon katsayısı ve eğrilik tensörü aşağıdaki şekilde elde edilmiştir.

$$C_{lkji} = R_{lkji} - \frac{1}{n-2}(g_{kj}R_{li} - g_{ki}R_{lj} + g_{li}R_{kj} - g_{lj}R_{ki}) + \frac{R}{(n-1)(n-2)}(g_{kj}g_{li} - g_{ki}g_{lj}) \ .$$

Sonrasında ise, bu eğrilik tensörünün özellikleri verilmiştir.

İkinci bölümde, literatürde verilen yarı simetrik metrik konneksiyonlu Riemann manifoldu ile ilgili çalışmalar üzerinde durulmuş ve çalışma boyunca bu manifold RS ile gösterilmiştir. Yarı simetrik metrik konneksiyonlu Riemann manifolduna ait konneksiyon katsayıları ve bu katsayıların tekliği ispat edilmiştir. Riemann maifoldu üzerinde bu konneksiyona göre üretien eğrilik tensörünün hesapları detaylı bir şekilde verilmiştir ve yarı simetrik metrik konneksiyonlu Riemann manifolduna ait konneksiyon katsayıları ve bu katsayıların tekliği ispat edilmiştir. Bu konneksiyon katsayılarının eğrilik tensörünün genel ifadesine yerleştirilmesiyle yarı simetrik metrik konneksiyonlu Riemann manifolduna ait eğrilik tensörü aşağıdakı şekilde elde edilmiştir

$$\overline{R}_{mijk} = R_{mijk} + g_{mk}\pi_{ij} - g_{jm}\pi_{ik} + g_{ij}\pi_{mk} - g_{ik}\pi_{mj}.$$

Devamında ise, bu eğrilik tensörünün özellikleri açıkça verilmiştir.

Üçüncü bölümde, [10] dan yararlanarak, yarı simetrik metrik konneksiyonlu Riemann manifoldunun konform dönüşüm altında sahip olduğu konneksiyon katsayısı, aşağıdaki şekilde

$$\Gamma_{jk}^{i\ *} = \Gamma_{jk}^{i} - \delta_j^i q_j + g_{jk} q_j$$

bulunmuştur.

Daha sonra aşağıdaki kısaltmalar kullanılarak,

$$q_{ij} = \nabla_j q_j + q_i q_j - \frac{1}{2} g_{ij} q_s q^s,$$

ve

$$W_{ij} = -q_{ij} + q_{ij} q_j,$$

[10] da elde edilen yeni konneksiyon katsayısına göre eğrilik tensörü ,

$$\overline{R}^{h}_{ijk}{}^{*} = \overline{R}^{h}_{ijk} + \delta^h_k W_{ij} - \delta^h_j W_{ik} + g_{ij} g^{hr} W_{rk} - g_{ik} g^{hr} W_{rj},$$

olarak elde edilir.

Buradan yarı simetrik konneksiyonlu Riemann manifoldunun konform eğrilik tensörü,

$$\overline{C}^{h}_{jkl} = \overline{R}^{h}_{jkl} + \frac{1}{(n-2)}(\delta^h{}_j \overline{R}_{ik} - \delta^h_i + g_{ik} g^{hl} \overline{R}_{lj} - g_{ik} g^{hl} \overline{R}_{lk}) + \frac{\overline{R}}{(n-1)(n-2)}(\delta^h{}_j \overline{R}_{ik} - \delta^h{}_j g_{ik})$$

olarak bulunmuştur.

Sonrasında, tezimizde yarı simetrik metrik konneksiyonlu konform eğrilik tensörünün, Riemann manifoldunun konform eğrilik tensörüne eşit olduğu gösterilmiştir.

Yarı simetrik metrik konneksiyonlu konform eğrilik tensörünün, aşağıdaki özellikleri çıkarılmıştır

$$(\mathbf{1}). \qquad \overline{C}_{mijk} + \overline{C}_{imjk} = 0,$$

$$(\mathbf{2}). \qquad \overline{C}_{mijk} + \overline{C}_{mikj} = 0.$$

Devamında ise, yarı simetrik metrik konneksiyonlu Riemann manifoldunun konform eğrilik tensörünün,

$$R_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik}\, R_{,j} - g_{ij}\, R_{,k}) \ ,$$

$$R_{ikl} = R_{ik,l} - R_{il,k} + \frac{1}{2(n-1)}(g_{il}\, R_{,k} - g_{ik}\, R_{,l}) \ ,$$

$$R_{ilj} = R_{il,j} - R_{ij,l} + \frac{1}{2(n-1)}(g_{ij}\, R_{,l} - g_{il}\, R_{,j}) \ ,$$

kısaltmaları altında, aşağıdaki eşitliğini

$$\overline{C}_{mijk,l} + \overline{C}_{mikl,j} + \overline{C}_{milj,k} = \frac{1}{n-3}(g_{ik}\overline{C}_{mjl} + g_{mj}\overline{C}_{ikl+g_{il}\overline{C}_{mkj}} + g_{mk}\overline{C}_{ilj} + g_{ij}\overline{C}_{mik} + g_{ml}\overline{C}_{ijk}),$$

sağladığı gösterilmiştir.
Daha sonra, yarı simetrik metrik konneksiyonlu Riemann manifoldu için konform rekürantlık koşulu, aşağıdaki şekilde verilmiştir

$$\overline{\nabla}_l \overline{C}_{hijk} = \lambda_l \overline{C}_{hijk}.$$

Buna ek olarak, konform olarak düz olma koşulunun, yarı simetrik metrik konneksiyonlu Riemann manifoldunun konform eğrilik tensörünün sıfıra eşit olmasıyla sağlanacağı söylenmiştir. Ayrıca, RS için Ricci rekürantlık koşulu,

$$\overline{\nabla}_l \overline{R}_{ij} = \lambda_l \overline{R}_{ij},$$

şeklinde verilmiştir. Buna bağlı olarak açıkça görülür ki, yarı simetrik metrik konneksiyonlu Riemann manifoldunun skaler eğriliği şu koşulu sağlar,

$$\overline{\nabla}_l R = \lambda_l R.$$

Yukarıdaki üç koşul kullanılarak, yarı simetrik metrik konneksiyonlu Ricci rekürant bir Riemann manifoldu, konform rekürantlık ya da konform olarak düz olma koşulu sağladığında, yarı simetrik metrik konneksiyonlu Riemann manifoldunun kendisinin de rekürantlığı sağladığı gösterilmiştir.
Daha sonra, ([6], Lemma 2) de verilen yarı simetrik metrik konneksiyonlu Riemann manifoldu,eğer konform olarak düz olma koşulunu sağlarsa,

$$\nabla_l C_{kjih} + \nabla_j C_{lkih} + \nabla_k C_{jlih} = (1-n)(p_l C_{kjih} + p_j C_{lkih} + p_k C_{jlih})$$

ifadesini de sağladığını göstermek için gerekli olan aşağıdaki lemma ispatlanmıştr. [6] da gösterildiği gibi c_j, p_j and B_{hijk} sayıları aşağıdaki denklemleri sağlamaktadır,

$$c_l B_{hijk} + p_h B_{lijk} + p_i B_{hljk} + p_j B_{hilk} + p_k B_{hijl} = 0,$$

$$B_{hijk} + B_{hjki} + B_{hkij} = 0,$$

ve

$$B_{hijk} = B_{jkhi} = -B_{hikj}.$$

Yukarıdaki denklemler sonucu ya her $B_{hijk} = 0$, ya da her $b_j = c_j + 2p_j = 0$ dır.

1 INTRODUCTION

Riemannian spaces endowed with some special connections are being used defining physical spaces, especially in the theories of gravity context, in addition to its wide geometrical richness. This thesis is devoted to study manifolds with a special connection, semi-symmetric metric connection. We present basic definitions and fundamental theorems of curvatures in Riemannian manifolds in the differential geometric frame, and then extend our knowledge to semi-symmetric metric connection case. In literature, calculations of curvature tensor of Riemannian and semi symmetric spaces were studied by many authors [1-10].

In the first chaper, we study on diffetrentiable manifolds. We give the definitions of affine connection, Riemannian metric, Riemannian connection explicitly. Then, we examine these concepts in local coordinates. In the second section of first chapter we work on conformal curvature of Riemannian manifold and how changes the curvature tensor under a conformal mapping.

In the second chapter, we study on Riemannian manifold with semi symmetric metric connection denoted by RS. The uniquennes of the connection coefficient of RS is proved and the curvature tensor with respect to the connection coefficients of RS is obtained [6]. Then, the properties of this curvature tensor are examined in detail.

In the third section, we study conformal curvatures on Riemannian space with semi symmetric metric connection. In [10], the calculations of conformal curvature tensor are given in detail. In this thesis, after a brief calculations of conformal curvature, we give the covariant derivative of conformal curvature tensor of Riemannian manifold with semi symmetric metric connection rigorously. We hope that the relations and results about covariant derivative of conformal curvature will be helpfull for readers.

1.1 Riemannian Manifold

Let M be an n dimensional Haussdorff space and p be any point of M. If there exists an open neighbourhood U of p, and U is a homeomorphism to an n dimensional Euclidean space R^n

, then M is called n dimensional manifold. If there exists a differentiable homeomorphism between U and R^n then, the manifold is called differentiable manifold.

It is given definition of a manifold more explicitly in following.

Let M be a Hausdorff space and Λ be the set indices and for open sets U_i, $\phi_i(U_i)$ be an open subset of R^n under ϕ homeomorphism. If a family of collection $\{(U_i,\phi_i)\}_{i\in\Lambda}$ on M satisfies the following properties

(1) $M = U_{i\in\Lambda} U_i$,

(2) for any pair i and $j \in \Lambda$, the differentiable mapping $\phi_j \circ \phi_i^{-1}$ is such that

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j).$$

(3) the collection $(U_i,\phi_i)_{i\in\Lambda}$ is maximal with respect to the condition (1) and (2).

then the collection $\{(U_i,\phi_i)\}_{i\in\Lambda}$ defines an n-dimensional differentiable structure.

If a function ϕ is a one-to-one, into and continuous function and has continuous inverse, then it is called as a homeomorphism. If a homeomorphism ϕ is differentiable then, it is called differentiable homeomorphism i.e, diffeomorphism.

Let U be an open subset of M and $\phi : U \to R^n$ be a homeomorphism. If ϕ has all partial derivatives of order k and these derivatives are continuous, then the function ϕ is differentiable class of C^k.

If it can be defined a differentiable structure on M class of C^k, then M is called a differentiable manifold class of C^k [5].

Definition 1.1: Let $T(M)$ be a set of fields of tangent vectors class of C^∞ and X, Y and $Z \in T(M)$ and $\forall\ f,g \in C^\infty(M,R)$. For any map ∇ defined as

$$\nabla : T(M) \times T(M) \to T(M),$$

$$(X,Y) \to \nabla_X Y, \quad (X,Y \in T(M))$$

if the following properties are satisfied

$(\mathbf{1})$. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_x Z$,

$(\mathbf{2})$. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,

$(\mathbf{3})$. $\nabla_{gX} Y = g\nabla_X Y$,

$(\mathbf{4})$. $\nabla_X(gY) = (Xg)Y + g\nabla_X Y$,

then ∇ is called an affine connection on M.

Definition 1.2: If there exist a bilinear, symmetric and positive definite and class of C^∞ function denoted by $<,>$ on M, then M is called Riemannian manifold and also the function $<,>$ is called Riemannian metric.

Let $T(M)$ be a tangent space at $p \in$ M and any vector fiels X,Y and $Z \in T(M)$. If the fuction $<,>$ defined as

$$<,>: T_p(M) \times T_p(M) \rightarrow R$$

$$<,>: (X,Y) \rightarrow <X,Y>$$

satisfied the followings

$(\mathbf{1})$. $\quad <X+Y,Z> = <X,Z> + <Y,Z>,$

$\qquad <X,Y+Z> = <X,Y> + <X,Z>,$

$(\mathbf{2})$. $\quad <X,Y> = <Y,X>,$

$(\mathbf{3})$. $\quad <X,X> > 0 \quad X \neq 0,$

(4). If X and Y is class of C^∞ in a neighbourhood U of p, then, the inner product $<X,Y>_p = <X_p,Y_p>$ is class of C^∞ in a neighbourhood U.

Also, the Riemannian metric $<X,Y>$ can be expressed $g(X,Y)$.

Furthermore, if the affine connection ∇, also satisfies the following two property

$(\mathbf{1})$. $\quad \nabla_X Y - \nabla_Y X = [X,Y],$

and

$(\mathbf{2})$. $\quad Z<X,Y> = <\nabla_Z X,Y> + <X,\nabla_Z Y>,$

then ∇ is called Riemannian connection.

We now give the definition of metric tensor in local coordinates since many tensor calculations in our thesis is based on local coordinates.

In 1854, Riemann defined the infinitesimal distance ds between infinitesimally close two points x^i and $x^i + dx^i$ in n-dimensional spaces as

$$ds^2 = g_{ij} dx^i dx^j, \quad (i,j = 1,2,\cdots,n), \tag{1.1}$$

where x^i are coordinate functions and g_{ij} is a non-singular symmetric tensor of rank 2 with type (0,2). g_{ij} is function of x^i's and called Riemannian metric tensor. Such a space and a geometry are characterized by Riemannian metric tensor, also they are called Riemannian space and Riemannian geometry.

Let C be a smooth curve with parameterized by $x^i = x^i(t)$ in M and x^i and, $x^i + dx^i$ be two very close points on the curve C. Any vector U is parallel transported along the curve C by

$$dU^i = -\Gamma_{jk}{}^{i} U^j dx^k, \tag{1.2}$$

here $\Gamma_{jk}{}^{i}$ denotes the connection coefficients of the parallel transport.

We now give the statetment of fundamental theorem for Riemannian manifolds.

Theorem 1.1. For a Riemannian manifold M with its metric tensor g_{ij} defined (1.1) there exists unique connection that is compatible with Riemannian metric tensor g_{ij} of M.

Let x and $\bar{x}$ be coordinate functions on M and, also, $\Gamma_{jk}^{\,i}$ and $\overline{\Gamma}_{jk}^{\,i}$ connection coefficients for corresponding these coordinates, respectively. Under general coordinate transformations, $\Gamma_{jk}^{\,i}$ transforms as

$$\overline{\Gamma}_{ml}^{\,h}\frac{\partial x^i}{\partial \bar{x}^h} = \frac{\partial^2 x^i}{\partial \bar{x}^m \partial \bar{x}^l} + \Gamma_{jk}{}^{i}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^l}, \tag{1.3}$$

therefore, $\Gamma_{ml}^{\,i}$ is not a tensor. The coefficients of the connection $\Gamma_{jk}{}^{i}$ of the Riemannian manifold is sometimes called Levi-Civita.

It is observed that the followings are

$$g^{ih}g_{jh} = \delta^i_j, \qquad \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{1.4}$$

The coefficients of Levi-Civita connection are expressed by $\left\{ {i \atop jk} \right\}$ explicitly, and it is given by [2]

$$\left\{ {i \atop jk} \right\} = g^{ij}[jk,h], \qquad [jk,h] = \frac{1}{2}\left(\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right), \tag{1.5}$$

where $[jk,h]$ is the first kind of Christoffel symbol, and $\left\{ {i \atop jk} \right\}$ is the second kind of Christoffel symbol.

It is obvious that these symbols are symmetric with respect to the indices j and k.

In general, on a manifold M the connection coefficients $\Gamma_{jk}^{\,i}$ can be decomposed of symmetric and anti-symmetric part as

$$\Gamma_{jk}^{\,i} = \Gamma_{(jk)}^{\,i} + \Gamma_{[jk]}^{\,i}. \tag{1.6}$$

Here, the symmetric part is defined by

$$\Gamma_{(jk)}^{\,i} = \frac{1}{2}(\Gamma_{jk}^{\,i} + \Gamma_{kj}^{\,i}), \tag{1.7}$$

and the anti-symmetric part is defined by

$$\Gamma_{[jk]}^{\,i} = \frac{1}{2}(\Gamma_{jk}^{\,i} - \Gamma_{kj}^{\,i})\,. \tag{1.8}$$

In spite of $\Gamma_{jk}^{\,i}$ is not a tensor, their anti-symmetric part $\Gamma_{[jk]}^{\,i}$ is a tensor and it is called torsion tensor of the connection.

It is known that the Riemannian connection $\Gamma_{jk}^{\,i}$ is a symmetric and torsion free connection.

In a Riemannian manifold, components of a contravariant vector field u^i, and any covariant vector field u_i, and any tensor field $U_{ji}^{\;h}$ of rank 3, we have the followings [1]

$$\nabla_j u^i = \frac{\partial u^i}{\partial x^j} + u^h \Gamma^i_{hj}\,, \tag{1.9}$$

$$\nabla_j u_i = \frac{\partial u_i}{\partial x^j} - u_k \Gamma^k_{ij}\,, \tag{1.10}$$

$$\nabla_k U^h_{ji} = \frac{\partial U^a_{ji}}{\partial x^k} + U^a_{ij}\Gamma^h{}_{ka} - U^h{}_{ai}\Gamma^a_{kj} - U^h{}_{ja}\Gamma^a_{ki}. \tag{1.11}$$

From (1.11), it can be seen that the covariant derivative of the Riemannian metric tensor g_{ij} is

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \left\{ \begin{matrix} a \\ kj \end{matrix} \right\} g_{ai} - \left\{ \begin{matrix} a \\ ki \end{matrix} \right\} g_{aj} = 0\,. \tag{1.12}$$

Also, using (1.4) and (1.12), it is seen

$$\nabla_k g^{ij} = 0\,. \tag{1.13}$$

In general, the curvature tensor of any manifold M is given by [4]

$$R^h{}_{kji} = \partial_k \Gamma^h{}_{ji} - \partial_j \Gamma^h{}_{ki} + \Gamma^h{}_{kt}\Gamma^t{}_{ji} - \Gamma^h{}_{jt}\Gamma^t{}_{ki}\,. \tag{1.14}$$

where ∂_k denotes the derivative with respect to coordinate functions of manifold M,

i.e, $\partial_k = \frac{\partial}{x^k}$.

If we use the Levi-Civita connection $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ in (1.14), instead of $\Gamma^i{}_{jk}$, it is obtained that the curvature tensor of Riemannian manifold M is

$$R^{\,h}{}_{kji} = \partial_k \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ki \end{matrix} \right\} + \left\{ \begin{matrix} h \\ kl \end{matrix} \right\} \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} - \left\{ \begin{matrix} h \\ jl \end{matrix} \right\} \left\{ \begin{matrix} l \\ ki \end{matrix} \right\}, \tag{1.15}$$

and $R^{\,h}{}_{kji}$ is called the Riemannian curvature tensor.

Multiplying (1.15) by the metric tensor g_{ah} we get

$$R_{hkji} = R^{\,a}{}_{kji}\, g_{ah}, \tag{1.16}$$

and R_{hkji} is called the covariant curvature tensor of Riemannian manifold. The following properties are valid for the Riemannian covariant curvature tensor R_{khji}

$$(\mathbf{1}). \qquad R_{kjih} = -R_{jkih} \, , \tag{1.17}$$

$$(\mathbf{2}). \qquad R_{kjih} = -R_{kjhi} \, , \tag{1.18}$$

$$(\mathbf{3}) \, . \qquad R_{kjih} = R_{ihkj} \, , \tag{1.19}$$

and from (1.18) and (1.19), it is concluded that

$$(\mathbf{4}). \qquad R_{kkih} = R_{kjhh} = 0 \, . \tag{1.20}$$

In the equation (1.15), if the indices are changed cyclicly and additing obtained equations, we have

$$R^h{}_{ikj} + R^h{}_{jik} + R^h{}_{kji} = 0 \, , \tag{1.21}$$

that is called the first Bianchi identity for the Riemannian curvature tensor of M .

Multiplying the equation (1.21) by g_{hm}, the another representation of the first Bianchi identity is obtained as

$$R_{mikj} + R_{mjik} + R_{mkji} = 0 \, . \tag{1.22}$$

Another identity holds for the covariant derivative of the Riemannian curvature tensor

$$\nabla_l R^h{}_{kji} + \nabla_i R^h{}_{klj} + \nabla_j R^h{}_{kil} = 0 \, , \tag{1.23}$$

that is called the second Bianchi identity. Multiplying (1.23) by metric tensor g_{hm}, the second Bianchi identity can be written in the covariant form as

$$R_{mkji,l} + R_{mklj,i} + R_{mkil,j} = 0 \, .$$

Contracting the indices h and i in $R^h{}_{kji}$ in (1.15) we have

$$R_{kj} = R^i{}_{kji}, \tag{1.24}$$

and the R_{kj} is called the Ricci curvature tensor. Also, we have

$$R_{kj} = R_{akj}{}^a = g^{ab} R_{akjb} = g^{ba} R_{jbak} = g^{ba} R_{bjka} = R_{jk}, \tag{1.25}$$

which shows that the Ricci curvature tensor is symmetric.
Multiplying R_{kj} by g^{jk}, we get

$$R = g^{jk} R_{kj} \, , \tag{1.26}$$

which is called the scalar curvature R of the Riemnannian manifold.

1.2 The Conformal Curvature Of Riemannian Manifold

In this section, first we review the conformal curvature tensor of Riemannian manifold with respect to Levi-Civita connection which is studied in [6], then we examine properties of this tensor in detail.

Let M and N be two n dimensional Riemannian manifolds, p be a positive differentiable function on M, g and g^* be two metric tensors defined on M and N, respectively.
Now, we assume that the metric tensor g^* is obtained by using a one to one differentiable mapping p from M into N such that

$$g^*_{ij} = p^2 g_{ij} \, . \tag{1.27}$$

Especially, this transformation is called conformal mapping.
Also, from (1.27), we have

$$g^{*ij} = p^{-2} g^{ij} \, . \tag{1.28}$$

Under conformal mapping for the metric tensor g^*_{ij}, connection coefficients are

$$\begin{Bmatrix} h \\ ji \end{Bmatrix}^* = \frac{1}{2} g^{*th} \left(\frac{\partial g^*{}_{it}}{\partial x^j} + \frac{\partial g^*{}_{jt}}{\partial x^i} - \frac{\partial g^*_{ji}}{\partial x^t} \right). \tag{1.29}$$

Substituting (1.27) and (1.28) in (1.29), the connection coefficients of Riemannian manifold transform into

$$\begin{Bmatrix} h \\ ji \end{Bmatrix}^* = \begin{Bmatrix} h \\ ji \end{Bmatrix} + \left(\delta_i^h p_j + \delta_j^h p_i - g_{ji} p^h \right), \tag{1.30}$$

where $p_i = \partial p / \partial x^i$ and $p^h = g^{hm} p_m$.
The relation (1.30) shows that how connection coefficients change under conformal transformations.
Now, we examine how the curvature tensor of Riemannian manifold is changed by using g^*_{ij} and $\begin{Bmatrix} h \\ ji \end{Bmatrix}^*$. If we use (1.30) in (1.15), the curvature tensor of Riemannian manifold is obtained as

$$R^{*h}{}_{kji} = \partial_k \begin{Bmatrix} h \\ ji \end{Bmatrix}^* - \partial_j \begin{Bmatrix} h \\ ki \end{Bmatrix}^* + \begin{Bmatrix} h \\ kl \end{Bmatrix}^* \begin{Bmatrix} l \\ ji \end{Bmatrix}^* - \begin{Bmatrix} h \\ jl \end{Bmatrix}^* \begin{Bmatrix} l \\ ki \end{Bmatrix}^* . \tag{1.31}$$

Substituting (1.30) in (1.31), we get

$$R^{*h}{}_{kji} = \partial_k \left[\left\{ \begin{array}{c} h \\ ji \end{array} \right\} + (\delta_j{}^h p_i + \delta_i{}^h p_j - g_{ji} p^h) \right] - \partial_j \left[\left\{ \begin{array}{c} h \\ ki \end{array} \right\} + (\delta_k{}^h p_i + \delta_i{}^h p_k - g_{ki} p^h) \right]$$
$$+ \left[\left\{ \begin{array}{c} h \\ kl \end{array} \right\} + (\delta_k{}^h p_l + \delta_l{}^h p_k - g_{kl} p^h) \right] \left[\left\{ \begin{array}{c} l \\ ji \end{array} \right\} + (\delta_j{}^l p_i + \delta_i{}^l p_j - g_{ji} p^l) \right]$$
$$- \left[\left\{ \begin{array}{c} h \\ jl \end{array} \right\} + (\delta_j{}^h p_l + \delta_l{}^h p_j - g_{jl} p^h) \right] \left[\left\{ \begin{array}{c} l \\ ki \end{array} \right\} + (\delta_k{}^l p_i + \delta_i{}^l p_k - g_{ki} p^l) \right]. \tag{1.32}$$

Arranging terms in (1.32), we have

$$R^{*h}{}_{kji} = R^h{}_{kji} + \delta_j^h \left(\partial_k p_i - \left\{ \begin{array}{c} l \\ ki \end{array} \right\} p_l - p_i p_k + \frac{1}{2} p_l p^l g_{ki} \right) - \delta_k^h \left(\partial_j p_i - \left\{ \begin{array}{c} l \\ ji \end{array} \right\} p_l - p_i p_j + \frac{1}{2} p_l p^l g_{ji} \right)$$

$$- g_{ji} \left(\partial_k p^h + \left\{ \begin{array}{c} h \\ kl \end{array} \right\} p^l + \frac{1}{2} \delta_k^h p_l p^l - p_k p^h \right) + g_{ki} \left(\partial_j p_h - \left\{ \begin{array}{c} h \\ jl \end{array} \right\} p_l + \frac{1}{2} \delta_j^h p_l p^l - p_j p^h \right).$$

Then, we reach

$$R^{*h}{}_{kji} = R^h{}_{kji} + \delta_i^h p_{kj} - \delta_j^h p_{ki} - g_{ij} p_k{}^h + g_{kj} p_i{}^h, \tag{1.33}$$

and for the simplicity in calculations we define

$$p_{ji} = \nabla_j p_i - p_j p_i + \frac{1}{2} p_l p^l g_{ji}, \tag{1.34}$$

and

$$p_k^h = p_{kl}\, g^{lh}. \tag{1.35}$$

Equating the indices h and i, it is obtained

$$R^*{}_{kj} = R_{kj} + (n-2) p_{kj} + p_h{}^h g_{kj}\ . \tag{1.36}$$

After some calculations , we reach

$$R^{*h}{}_{kji} = R^h{}_{kji} - \delta_k^h (L^*{}_{ji} - L_{ij}) + \delta_j^h (L^*{}_{ki} - L_{ik}) - p_k{}^h g_{ji} + p_j{}^h g_{ki}, \tag{1.37}$$

where

$$L_{ji} = \frac{1}{n-2} R_{ji} - \frac{R g_{ji}}{2(n-1)(n-2)}\ . \tag{1.38}$$

Thus, we arrive

$$R^{*h}{}_{kji} + \delta_k^h L_{ji}{}^* - \delta_j^h L_{ki}{}^* + L_{kt}{}^k g^{th} g_{ji} - L_{jt}{}^* g^{th} g_{ki} = R^h_{kji} + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_{kt} g^{th} g_{ji} - L_{jt} g^{th} g_{ji}. \tag{1.39}$$

Arranging (1.39), then we find that the invariant part under the conformal mapping of the Riemannian curvature tensor is

$$C^{h}{}_{kji}{}^{*} = R^{*h}{}_{kji} + \delta_k^h L^*_{ji} - \delta_j^h L_{ki}{}^{*} + L_{kt}{}^{*} g^{th} g_{ji} - L^*_{jt} g^{th} g_{ki} \tag{1.40}$$

and

$$C^{h}{}_{kji} = R^{h}{}_{kji} + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_{kt} g^{th} g_{ji} - L_{jt} g^{th} g_{ki}. \tag{1.41}$$

Consequently, it is obtained

$$C^{*h}{}_{kji} = C^{h}{}_{kji}. \tag{1.42}$$

The tensor $C^{h}{}_{kji}$ is called as the conformal curvature tensor.
Multiplying (1.41) by the metric tensor g_{hl}, we have

$$C_{lkji} = R_{lkji} - \frac{1}{n-2}(g_{kj}R_{li} - g_{ki}R_{lj} + g_{li}R_{kj} - g_{lj}R_{ki}) + \frac{R}{(n-1)(n-2)}(g_{kj}g_{li} - g_{ki}g_{lj}), \tag{1.43}$$

that is called the conformal covariant curvature tensor of Riemannian manifold.
For conformal curvature tensor C_{lkji}, the following properties hold

$$(\mathbf{1}). \qquad C_{lkji} + C_{lkij} = 0, \tag{1.44}$$

$$(\mathbf{2}). \qquad C_{lkji} + C_{klij} = 0, \tag{1.45}$$

and

$$(\mathbf{3}). \qquad C^{k}_{kji} = 0. \tag{1.46}$$

Also, it is seen that the conformal curvature tensor satisfies the following property similar to the first Bianchi identity of the Riemannian curvature tensor

$$C^{h}_{ijk} + C^{h}_{jki} + C^{h}_{kij} = 0. \tag{1.47}$$

2 RIEMANNIAN MANIFOLDS WITH A SEMI-SYMMETRIC METRIC CONNECTION

Riemannian manifolds are enriched by introducing several connections on them. In this chapter, we examine a Riemannian manifold with a special connection, semi-symmetric metric connection, and study its curvature tensors and their properties.

2.1 Semi-Symmetric Metric Connection

In 1924, Friedmann and Schouten, introduced the idea of the semi-symmetric linear connection for differentiable manifolds without the condition of the metricity . Later, in 1932 Hayden defined a metric connection with torsion on Riemannian manifold. In literature, many author K. Yano, T. Imai, Z. Nakao, K. Amur, S.S. Pujar studied on semi-symmetric metric spaces [7].

Let M be a Riemannian manifold with metric tensor g, $\overline{\nabla}$ be any linear connection, and ∇ be Levi-Civita connection on M. The torsion tensor for any non-symmetric linear connection is defined as

$$T^h{}_{ij} = \Gamma^h{}_{ij} - \Gamma^h{}_{ji}. \tag{2.1}$$

In this thesis, we consider torsion tensor, in local coordinates , for $\overline{\nabla}$

$$T^h{}_{ij} = \delta^h_j \pi_i - \delta^h_i \pi_j, \tag{2.2}$$

where π is a one form.

Then, the connection $\overline{\nabla}$ is called semi-symmetric connection on M. Furthermore, if, $\overline{\nabla}$ satisfies the property $\overline{\nabla} g = 0$, then $\overline{\nabla}$ is also defined by semi symmetric metric connection.

Here, a Riemannian manifold M with endoved connection $\overline{\nabla}$ is called Riemannian manifold with semi-symmetric metric connection and it will be denoted by RS in this text.

Also, Yano studied spaces with semi symmetric metric connection and some properties of semi symmetric metric connection [8].

Let v^h be a contravariant vector field, w_i be a covariant vector field, f be a scalar and $S_{ji}^{\ h}$ be a general tensor field on any manifold M with torsion, then the following Ricci's identities are satisfied.

$$\nabla_k \nabla_j v^h - \nabla_j \nabla_k v^h = R_{kji}^{\ \ \ h} v^i - 2T_{kj}^{\ \ t} \nabla_t v^h, \tag{2.3}$$

and

$$\nabla_k \nabla_j w_i - \nabla_j \nabla_k w_i = -R_{kji}{}^{h} w_h - 2T_{kj}{}^{t} \nabla_t w_i, \tag{2.4}$$

and

$$\nabla_j \nabla_i f - \nabla_i \nabla_j f = -2T_{ji}{}^{h} \nabla_h f, \tag{2.5}$$

and

$$\nabla_l \nabla_k S_{ji}{}^{h} - \nabla_k \nabla_l S_{ji}{}^{h} = R_{lkt}{}^{h} S_{ji}{}^{t} - R_{lkj}{}^{t} S_{ti}{}^{h} - R_{lki}{}^{t} S_{jt}{}^{h} - 2T_{lk}{}^{t} \nabla_t S_{ji}{}^{h}, \tag{2.6}$$

where $R_{kji}{}^{h}$ is the Riemannian curvature tensor, and $T_{ij}{}^{h}$ is the torsion tensor of the manifold M [2,3]. In Riemannian manifold it shoul be noted that for the above given Ricci identities,(2.3-2.6), the torsion tensor $T_{ji}{}^{h}$ is identical zero.

There are many application areas of semi symmetric metric connection in gravitational, and mathematical physics and engineering problems.

Now, we examine the existence and uniquennes of the semi-symmetric metric connection $\overline{\nabla}$ in detail.

In the following proof, we use Yano and Imai's notation and proof techniques for the uniquennes of the connection of RS.

Theorem 2.1.1: In a Riemannian manifold with semi-symmetric metric connection the connection coefficients of a semi-symmetric metric connection are determined uniquely as

$$\Gamma^{h}{}_{kj} = \left\{ \begin{array}{c} h \\ kj \end{array} \right\} + \delta_j^{h} \pi_k - \pi^{h} g_{kj} \ , \tag{2.7}$$

where $\left\{ \begin{array}{c} h \\ jk \end{array} \right\}$ denotes the Riemannian connection coefficient, and π_j is a 1-form defined in (2.2).

Proof: Let ∇ be derivative with respect to the Riemannian connection, and $\overline{\nabla}$ be covariant derivative with semi-symmetric metric connection then, we have the covariant derivative of the metric tensor with respect to $\overline{\nabla}$

$$\overline{\nabla}_k g_{ij} = \partial_k g_{ij} - \Gamma^{h}{}_{ki} g_{hj} - g_{ih} \Gamma^{h}{}_{kj} = 0 \ , \tag{2.8}$$

and the covariant derivative of the Riemannian metric g_{ij} with respect to Levi-Civita connection

$$\nabla_k g_{ij} = \partial_k g_{ij} - \left\{ \begin{array}{c} h \\ ki \end{array} \right\} g_{hj} - g_{ih} \left\{ \begin{array}{c} h \\ jk \end{array} \right\} = 0 \ . \tag{2.9}$$

Subtracting (2.9) from (2.8) , we obtain

$$g_{hj} \left[\Gamma^{h}{}_{ki} - \left\{ \begin{array}{c} h \\ ki \end{array} \right\} \right] + g_{ih} \left[\Gamma^{h}{}_{kj} - \left\{ \begin{array}{c} h \\ kj \end{array} \right\} \right] = 0 \ , \tag{2.10}$$

If we define $U_{ki}{}^{h}$ such that

$$U_{ki}{}^{h} = \Gamma^{h}{}_{ki} - \left\{ \begin{matrix} h \\ ki \end{matrix} \right\}, \tag{2.11}$$

and substituting (2.11) into (2.10) then,we find

$$g_{hj} U_{ki}{}^{h} + g_{ih} U_{kj}{}^{h} = 0 \,. \tag{2.12}$$

From the equation (2.1) , we have

$$T_{ij}{}^{h} = U_{ij}{}^{h} - U_{ji}{}^{h} \,, \tag{2.13}$$

and

$$U_{ij}{}^{h} = T_{ij}{}^{h} + U_{ji}{}^{h} \,. \tag{2.14}$$

Substituting (2.14) into (2.12) , the equation (2.12) becomes

$$(T_{ki}{}^{h} + U_{ik}{}^{h}) g_{hj} + (T_{kj}{}^{h} + U_{jk}{}^{h}) g_{ih} = 0. \tag{2.15}$$

If we change the indices of (2.12) cyclicly, we get

$$U_{ki}{}^{h} g_{hj} + U_{kj}{}^{h} g_{ih} = 0 \,, \tag{2.16}$$

and

$$U_{jk}{}^{h} g_{hi} + U_{ji}{}^{h} g_{kh} = 0 \,, \tag{2.17}$$

and

$$U_{ij}{}^{h} g_{hk} + U_{ik}{}^{h} g_{jh} = 0 \,, \tag{2.18}$$

summing up (2.16) and (2.18) and subtracting (2.17), we get

$$(U_{ki}{}^{h} + U_{ik}{}^{h}) g_{jh} + (U_{kj}{}^{h} - U_{jk}{}^{h}) g_{ih} + (U_{ij}{}^{h} - U_{ji}{}^{h}) g_{kh} = 0 \,.$$

Using (2.14), we have

$$(T_{ki}{}^{h} + 2U_{ik}{}^{h}) g_{jh} + T_{kj}{}^{h} g_{ih} + T_{ij}{}^{h} g_{kh} = 0. \tag{2.19}$$

Also, using (2.1), we have

$$T_{ji}{}^{h} = \delta_{j}{}^{h} \pi_{i} - \delta_{i}{}^{h} \pi_{j} \,, \tag{2.20}$$

and

$$T^{h}{}_{ji} = \delta_{i}{}^{h} \pi_{j} - \pi^{h} g_{ij}. \tag{2.21}$$

Multiplying (2.19) by g^{jl} , and using (2.21) and (2.22) the (2.19) becomes

$$T_{ki}{}^{l} - T^{l}{}_{ki} - T^{l}{}_{ik} + 2U_{ik}{}^{l} = 0, \tag{2.22}$$

and

$$2U_{ik}{}^{l} = T^{l}{}_{ki} + T^{l}{}_{ik} + T_{ki}{}^{l} . \tag{2.23}$$

Then, from (2.23), we obtain

$$U_{ik}{}^{l} = \frac{1}{2}(T^{l}{}_{ki} + T^{l}{}_{ik} + T_{ki}{}^{l}) , \tag{2.24}$$

Using (2.24), (2.20) and (2.21) in (2.11), then we uniquelly determine the connection coefficient of the semi-symmetric metric connection as

$$\Gamma^{h}{}_{jk} = \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} + \delta_{k}{}^{h}\pi_{j} - \pi^{h} g_{jk} .$$

2.2 The Curvature Of Riemannian Manifold with Semi-Symmetric Metric Connection

In [6] it is examined the curvature tensor of RS and its properties. In this section, we review these concepts and give the calculation of the curvatures of this manifold.

Lemma 2.2.1: Let ∇_k denote the covariant derivative with respect to the Riemannian connection and $\overline{\nabla}_k$ be the covariant derivative of the connection for the RS. Then, we have

$$\overline{\nabla}_k g_{ij} = 0. \tag{2.25}$$

Proof: Using (2.7) and (2.8) , we have

$$\begin{aligned}
\overline{\nabla}_k g_{ij} &= \partial_k g_{ij} - g_{hj}\Gamma^h{}_{ik} - g_{ih}\Gamma^h_{jk} \\
&= \partial_k g_{ij} - g_{hj}\left[\left\{ {h \atop ik} \right\} + \delta_k{}^h \pi_i - g_{ik}\pi^h\right] - g_{ih}\left[\left\{ {h \atop jk} \right\} + \delta_k{}^h \pi_j - g_{jk} - g_{jk}\pi^h\right] \\
&= \partial_k g_{ij} - g_{hj}\left\{ {h \atop ik} \right\} - g_{ih}\left\{ {h \atop jk} \right\} \\
&= \nabla_k g_{ij}
\end{aligned}$$

from (1.12), $\nabla_k g_{ij} = 0$ then, we find $\overline{\nabla}_k g_{ij} = 0$.

Next, we derive the curvature tensor of RS, and steps of calculations.

Theorem 2.2.1: The curvature tensor of Riemannian manifold with semi-symmetric connection is given

$$\overline{R}^h{}_{ijk} = R^h{}_{ijk} + \delta_k{}^h \pi_{ij} - \delta_j{}^h \pi_{ik} + g_{ij} g^{hr} \pi_{rk} - g_{ik} g^{hr} \pi_{rj}, \tag{2.26}$$

where

$$\pi_{ij} = \nabla_j \pi_i - \pi_i \pi_j + \frac{1}{2} g_{ij} \pi_s \pi^s. \tag{2.27}$$

and ∇ denotes the Levi-Civita connection, and $R^h{}_{ijk}$ is the Riemannian curvature tensor [2].

Proof: If we substitute (2.7) in general definition of the curvature tensor stated in (1.14) , then we obtain

$$\begin{aligned}
\overline{R}^h{}_{ijk} = \partial_j\left(\left\{ {h \atop ik} \right\} + \delta_k{}^h \pi_i - g_{ik}\pi^h\right) - \partial_k\left(\left\{ {h \atop ij} \right\} + \delta_j{}^h \pi_i - g_{ij}\pi^h\right) \\
+\left(\left\{ {h \atop sj} \right\} + \delta_j^h \pi_s - g_{sj}\pi^h\right)\left(\left\{ {s \atop ik} \right\} + \delta_k^s \pi_i - g_{ik}\pi^s\right)
\end{aligned}$$

$$-\left(\left\{ \begin{array}{c} h \\ sk \end{array} \right\} + \delta_k^h \pi_s - g_{sk}\pi^h\right)\left(\left\{ \begin{array}{c} s \\ ij \end{array} \right\} + \delta_j^s \pi_i - g_{ij}\pi^s\right). \tag{2.28}$$

After some computations, curvature tensor of RS becomes

$$\begin{aligned}
\overline{R}^h{}_{ijk} = R^h{}_{ijk} &+ \delta_k{}^h(\partial_j \pi_i) - (\partial_j g_{ik})\pi^h) - g_{ik}(\partial_j \pi^h) \\
&-\delta_j{}^h(\partial_k \pi_i) - (\partial_k g_{ij})\pi^h) - g_{ij}(\partial_k \pi^h) \\
&+\left\{ \begin{array}{c} h \\ sj \end{array} \right\}(\delta_k{}^s \pi_i - g_{ik}\pi^s) - \left\{ \begin{array}{c} h \\ sk \end{array} \right\}(\delta_j{}^s \pi_i - g_{ij}\pi^s) \\
&+\delta_j{}^h \pi_s\left(\left\{ \begin{array}{c} s \\ ik \end{array} \right\} + \delta_k{}^s \pi_i - g_{ik}\pi^s\right) \\
&-g_{sj}\pi^h\left(\left\{ \begin{array}{c} s \\ ik \end{array} \right\} + \delta_k{}^s \pi_i - g_{ik}\pi^s\right) \\
&-\delta_k{}^h \pi_s\left(\left\{ \begin{array}{c} s \\ ij \end{array} \right\} + \delta_j{}^s \pi_i - g_{ij}\pi^s\right) \\
&+g_{sk}\pi^h\left(\left\{ \begin{array}{c} s \\ ij \end{array} \right\} + \delta_j{}^s \pi_i - g_{ij}\pi^s\right).
\end{aligned} \tag{2.29}$$

Arranging (2.29), we reach

$$\begin{aligned}
\overline{R}^h{}_{ijk} = R^h{}_{ijk} &+ \delta_k^h(\nabla_j \pi_i - \pi_j \pi_i + \frac{1}{2} g_{ij}\pi^s \pi_s) \\
&-\delta_j^h(\nabla_k \pi_i - \pi_k \pi_i + \frac{1}{2} g_{ik}\pi^s \pi_s) \\
&-g_{ik}g^{hr}(\nabla_j \pi_r - \pi_j \pi_r + \frac{1}{2} g_{ik}\pi^s \pi_s) \\
&+g_{ij}g^{hr}(\nabla_k \pi_r - \pi_k \pi_r + \frac{1}{2} g_{kr}\pi^s \pi_s),
\end{aligned} \tag{2.30}$$

also, using (2.27), we obtain the curvature tensor of RS

$$\overline{R}^h{}_{ijk} = R^h{}_{ijk} + \delta_k{}^h \pi_{ij} - \delta_j{}^h \pi_{ik} + g_{ij}g^{hr}\pi_{rk} - + g_{ik}g^{hr}\pi_{rj},$$

where $R^h{}_{ijk}$ denotes the curvature tensor of Riemannian manifold.
Since $\overline{R}_{mijk} = g_{mh}\,\overline{R}^h_{ijk}$, if we multiply both sides of the equation (2.26) by g_{mh} we get the covariant curvature tensor of RS

$$\overline{R}_{mijk} = R_{mijk} + g_{mk}\pi_{ij} - g_{jm}\pi_{ik} + g_{ij}\pi_{mk} - g_{ik}\pi_{mj}. \tag{2.31}$$

Contracting the indices h and k in (2.26), it is obtained that the Ricci curvature tensor $\overline{R}_{ij}$ in the form

$$\overline{R}_{ij} = R\ {}_{ij} + \delta^h{}_h \pi_{ij} - \delta^h{}_j \pi_{ih} + g_{ij}g^{rh}\pi_{rh} - g_{ih}g^{hr}\pi_{rj},$$

or

$$\overline{R}_{ij} = R_{ij} + (n-2)\pi_{ij} + g_{ij}\pi \tag{2.32}$$

where $\pi = g^{mk}\pi_{mk}$ and R_{ij} represents the Ricci tensor of Riemannian manifold.

The symmetric part of the Ricci tensor $\overline{R}_{ij}$ is

$$\overline{R}_{(ij)} = \frac{1}{2}(\overline{R}_{ij} + \overline{R}_{ji}), \tag{2.33}$$

and the anti-symmetric part of the Ricci tensor $\overline{R}_{ij}$ is

$$\overline{R}_{[ij]} = \frac{1}{2}(\overline{R}_{ij} - \overline{R}_{ji}). \tag{2.34}$$

On the other hand , for the anti-symmetric part of any tensor π_i, it is used the notation

$$2\nabla_{[j}\pi_{i]} = \nabla_j\pi_i - \nabla_i\pi_j. \tag{2.35}$$

From (2.27), it can be seen that

$$2\nabla_{[j}\pi_{i]} = \pi_{ij} - \pi_{ji}\ . \tag{2.36}$$

Using (2.32) in (2.33), we get

$$\overline{R}_{(ij)} = R_{ij} + \frac{1}{2}(n-2)(\nabla_j\pi_i + \nabla_i\pi_j) + g_{ij}\pi$$

and

$$\overline{R}_{[ij]} = R_{[ij]} + (n-2)\nabla_{[j}\pi_{i]}$$

Multiplying the (2.32) by g^{ij} , we have the scalar curvature for RS as the form

$$\overline{R} = R + 2(n-1)\pi \tag{2.37}$$

where R denotes the scalar curvature of Riemannian manifold.

Next, we examine the properties of the curvature tensor of RS and the first Bianchi identity.

Theorem 2.2.2: The curvature tensor of semi-symmetric metric Riemannian manifold has the following properties

$$(\mathbf{1}).\qquad \overline{R}_{mijk} + \overline{R}_{mikj} = 0\ , \tag{2.38}$$

$$(\mathbf{2}).\qquad \overline{R}_{mijk} + \overline{R}_{imjk} = 0\ , \tag{2.39}$$

$$(\mathbf{3}).\qquad \overline{R}^h_{ijk} + \overline{R}^h_{jki} + \overline{R}^h_{kij} = 2(\delta^h_i\nabla_{[k}\pi_{j]} + \delta^h_j\nabla_{[i}\pi_{k]} + \delta^h_k\nabla_{[j}\pi_{i]})\ . \tag{2.40}$$

which is called the first Bianchi identity.

Proof:

(1). Interchancing the indices j and k on the equation (2.31), we have

$$\overline{R}_{mikj} = R_{mikj} + g_{mj}\pi_{ik} - g_{mk}\pi_{ij} + g_{ik}\pi_{mj} - g_{ij}\pi_{mk}, \tag{2.41}$$

and summing up (2.31) and (2.41), and from (1.18), we obtain

$$\overline{R}_{mijk} + \overline{R}_{mikj} = R_{mijk} + R_{mikj} = 0.$$

(**2**). Interchancing the indices i and m in the equation (2.31), we get

$$\overline{R}_{imjk} = R_{imjk} + g_{ik}\pi_{mj} - g_{ij}\pi_{mk} + g_{mj}\pi_{ik} - g_{mk}\pi_{ij}, \tag{2.42}$$

and using (2.31), (2.42), and (1.17), it is seen that covariant curvature tensor $\overline{R}_{mijk}$ is anti symmetric with respect to the first two indices, i.e,

$$\overline{R}_{mijk} + \overline{R}_{imjk} = R_{mijk} + R_{imjk} = 0. \tag{2.43}$$

(3). Using (2.26), interchancing the indices i,j and k cyclicly, we get

$$\overline{R}^{h}{}_{ijk} = R^{h}{}_{ijk} + \delta^{h}_{k}\pi_{ij} - \delta^{h}_{j}\pi_{ik} + g_{ij}g^{rh}\pi_{rk} - g_{ik}g^{hr}\pi_{rj}, \tag{2.44}$$

and

$$\overline{R}^{h}{}_{jki} = R^{h}{}_{jki} + \delta^{h}_{i}\pi_{jk} - \delta^{h}_{k}\pi_{ji} + g_{jk}g^{rh}\pi_{ri} - g_{ji}g^{hr}\pi_{rk}, \tag{2.45}$$

and

$$\overline{R}^{h}{}_{kij} = R^{h}{}_{kij} + \delta^{h}{}_{j}\pi_{ki} - \delta^{h}{}_{i}\pi_{kj} + g_{ki}g^{rh}\pi_{rj} - g_{kj}g^{hr}\pi_{ri}. \tag{2.46}$$

From (2.44), (2.45), and (2.46), we reach the firsth Bianchi identitiy for RS

$$\overline{R}^{h}{}_{ijk} + \overline{R}^{h}{}_{jki} + \overline{R}^{h}{}_{kji} = R^{h}{}_{ijk} + R^{h}{}_{jki} + R^{h}{}_{kij} + \delta^{h}{}_{i}(\pi_{jk} - \pi_{kj}) + \delta^{h}{}_{j}(\pi_{ki} - \pi_{ik}) + \delta^{h}{}_{k}(\pi_{ij} - \pi_{ji}). \tag{2.47}$$

In the right hand side of the equation (2.47), using (2.36) and the first Bianchi identity for Riemannian manifold given in (1.21), we have the following relation.

$$\overline{R}^{h}{}_{ijk} + \overline{R}^{h}{}_{jki} + \overline{R}^{h}{}_{kij} = 2(\delta^{h}{}_{i}\nabla_{[k}\delta_{j]} + \pi^{h}{}_{j}\nabla_{[i}\pi_{k]} + \pi^{h}{}_{k}\nabla_{[j}\delta_{i]}) \ .$$

3 THE CONFORMAL CURVATURE OF RIEMANNIAN MANIFOLD WITH SEMI-SYMMETRIC METRIC CONNECTION

In this chapter, we examine the conformal curvature tensor with respect to the semi-symmetric metric connection and its some properties are presented.

Let M and $\overline{M}$ be Riemannian manifolds with semi symmetric metric connections and $\Gamma^{h}{}_{ij}$, and $\overline{\Gamma}^{h}{}_{ij}$ be connection coefficients and $R^{h}{}_{ijk}$ be Riemannian curvatute tensor and $\overline{R}^{h}{}_{ijk}$ be the curvature of Riemannian manifold with semi symmetric metric connection of M and $\overline{M}$, respectively.We consider a conformal mapping σ from M to $\overline{M}$ and examine how the connection coefficients and curvatures of these connections are under conformal mapping σ and we review their relationship between them.

In [10], It is shown that connection coefficients for semi-symmetric metric connection given (2.7), transforms into

$$\Gamma^{i}{}_{jk}{}^{*} = \Gamma^{i}{}_{jk} - \delta^{i}_{k} q_j + g_{ik} q^{i}, \tag{3.1}$$

under conformal mapping σ and then

the curvature tensor of Riemannian manifold with semi-symmetric metric connection becomes

$$\overline{R}^{h}_{ijk}{}^{*} = \overline{R}^{h}_{ijk} + \delta^{h}_{k} W_{ij} - \delta^{h}_{j} W_{ik} + g_{ij} g^{hr} W_{rk} - g_{ik} g^{hr} W_{rj}, \tag{3.2}$$

where $\overline{R}^{h}{}_{ijk}{}^{*}$ denotes the transformed curvature tensor and $\overline{R}^{h}{}_{ijk}$ denotes the curvature of the semi symmetric metric connection.

Here

$$q_{ij} = \nabla_j q_i + q_i q_j - \frac{1}{2} g_{ij} g^{rs} q_r q_s, \tag{3.3}$$

and

$$W_{ij} = -q_{ij} + q_i q_j. \tag{3.4}$$

In [10], under the conformal mapping, the invariant part of the curvature tensor of Riemannian manifold with semi symmetric metric connection is given as

$$\overline{C}^h{}_{ijk} = \overline{R}^h{}_{ijk} + \frac{1}{n-2}\left(\delta^h_j \overline{R}_{ik} - \delta^h_k \overline{R}_{ij} + g_{ik} g^{hl} \overline{R}_{lj} - g_{ij} g^{hl} \overline{R}_{lk}\right) + \frac{\overline{R}}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}), \tag{3.5}$$

which is called conformal curvature tensor of these manifolds.
Multiplying (3.5) by g_{hm} it is obtained the covariant conformal curvature tensor as

$$\overline{C}_{mijk} = \overline{R}_{mijk} + \frac{1}{n-2}\left(g_{jm}\overline{R}_{ik} - g_{km}\overline{R}_{ij} + g_{ik}\overline{R}_{mj} - g_{ij}\overline{R}_{mk}\right) + \frac{\overline{R}}{(n-1)(n-2)}(g_{km}g_{ij} - g_{jm}g_{ik}). \tag{3.6}$$

By using an alternative method , we study the conformal curvature tensor of RS and restate its definition [6], and we give the calculation of the following lemma in explicitly to be helpfull characterizing the spaces with respect to their curvature invariants.

Lemma 3.1: The conformal curvature tensor with respect to Riemannian connection is equal to the conformal curvature tensor with respect to semi symmetric metric connection.

Proof: Using

$$\overline{C}_{mijk} = \overline{R}_{mijk} + \frac{1}{n-2}\left(g_{jm}\overline{R}_{ik} - g_{km}\overline{R}_{ij} + g_{ik}\overline{R}_{mj} - g_{ij}\overline{R}_{mk}\right) + \frac{\overline{R}}{(n-1)(n-2)}(g_{km}g_{ij} - g_{jm}g_{ik})$$

and substituting (2.31), (2.32) and (2.37) in (3.6), we get

$$\begin{aligned}\overline{C}_{mijk} &= R_{mijk} + g_{mk}\pi_{ij} - g_{jm}\pi_{ik} + g_{ij}\pi_{mk} - g_{ik}\pi_{mj}\\ &+\frac{1}{n-2}\left[g_{jm}(R_{ik} + (n-2)\pi_{ik} + g_{ik}\pi) - g_{km}((R_{ij} + (n-2)\pi_{ij}) + g_{ij}\pi)\right]\\ &+\frac{1}{n-2}\left[g_{ik}(R_{mj} + (n-2)\pi_{mj} + g_{mj}\pi) - g_{ij}(R_{mk} + (n-2)\pi_{mk} + g_{mk}\pi\right]\\ &+\frac{R + 2(n-1)\pi}{(n-1)(n-2)}(g_{ij}g_{mk} - g_{ik}g_{jm}).\end{aligned} \tag{3.7}$$

If we arrange the above terms, we have

$$\overline{C}_{mijk} = R_{mijk} + \frac{1}{n-2}(g_{jm}R_{ik} - g_{km}R_{ij} + g_{ik}R_{mj} - g_{ij}R_{mk}) + \frac{R}{(n-1)(n-2)}(g_{ij}g_{mk} - g_{ik}g_{jm}), \tag{3.8}$$

from which we conclude that (3.8) is equal to the conformal curvature of Riemannian manifold in (1.43).

Theorem 3.1: The conformal curvature tensor of Riemannian manifold with semi-symmetric metric connection has properties

$$(\mathbf{1}). \qquad \overline{C}_{mijk}+\overline{C}_{imjk}=0, \tag{3.9}$$

and

$$(\mathbf{2}). \qquad \overline{C}_{mijk}+\overline{C}_{mikj}=0. \tag{3.10}$$

Proof:

(1). Interchanging the indices i and m on (3.6), we get

$$\overline{C}_{imjk}=\overline{R}_{imjk}+\frac{1}{n-2}\big(g_{ij}\,\overline{R}_{mk}-g_{ik}\,\overline{R}_{mj}+g_{mk}\,\overline{R}_{ij}-g_{mj}\,\overline{R}_{ik}\big)+\frac{\overline{R}}{(n-1)(n-2)}\big(g_{mj}g_{ik}-g_{mk}g_{ij}\big). \tag{3.11}$$

Summing up (3.6) and (3.11) and, using (2.39), we have
$\overline{C}_{mijk}+\overline{C}_{imjk}=\ \overline{R}_{mijk}+\overline{R}_{imjk}=0.$

(2). Interchanging the indices k and j on (3.6), we get

$$\overline{C}_{mikj}=\overline{R}_{mik}+\frac{1}{n-2}\big(g_{mk}\,\overline{R}_{ij}-g_{mj}\,\overline{R}_{ik}+g_{ij}\,\overline{R}_{mk}-g_{ik}\,\overline{R}_{mj}\big)+\frac{\overline{R}}{(n-1)(n-2)}\big(g_{ik}g_{mj}-g_{ij}g_{mk}\big). \tag{3.12}$$

From (3.6), (3.12), and using (2.38)
$\overline{C}_{mijk}+\overline{C}_{mikj}=\overline{R}_{mijk}+\overline{R}_{mikj}=0$.

We now examine the covariant derivative of the conformal curvature tensor of Riemannian manifold.

Theorem 3.2: The conformal curvature tensor of Riemannian manifold satisfies the relation

$$C_{mijk,l}+C_{mikl,j}+C_{milj,k}=\frac{1}{n-3}\left[g_{ik}\,C_{mjl}+g_{mj}\,C_{ikl}+g_{il}\,C_{mkj}+g_{mk}\,C_{ilj}+g_{ij}\,C_{mlk}+g_{ml}\,C_{ijk}\right], \tag{3.13}$$

where

$$C_{ijk}=C^{r}{}_{ijk,r}. \tag{3.14}$$

Proof: By taking covariant derivative of the equation (3.5) , we have

$$C^{h}_{\ ijk,l} = R^{h}_{\ ijk,l} + \frac{1}{(n-2)}\left(\delta^{h}_{j} R_{ik,l} - \delta^{h}_{k} R_{ij,l} + g_{ik} R^{h}_{\ j,l} - g_{ij} R^{h}_{\ k,l}\right) + \frac{R_{,l}}{(n-1)(n-2)}(\delta^{h}_{k} g_{ij} - \delta^{h}_{j} g_{ik}), \tag{3.15}$$

chancing the indices cyclicy l , j, and k in (3.15)

$$C^{h}_{\ ikl,j} = R^{h}_{\ ikl,j} + \frac{1}{(n-2)}\left(\delta^{h}_{k} R_{il,j} - \delta^{h}_{l} R_{ik,j} + g_{il} R^{h}_{\ k,j} - g_{ik} R^{h}_{\ l,j}\right) + \frac{\overline{R}_{,j}}{(n-1)(n-2)}(\delta^{h}_{l} g_{ik} - \delta^{h}_{k} g_{il}), \tag{3.16}$$

and

$$C^{h}_{\ ilj,k} = R^{h}_{\ ilj,k} + \frac{1}{(n-2)}\left(\delta^{h}_{l} R_{ij,k} - \delta^{h}_{j} R_{il,k} + g_{ij} R^{h}_{\ l,k} - g_{il} R^{h}_{\ j,k}\right) + \frac{R}{(n-1)(n-2)}(\delta^{h}_{j} g_{il} - \delta^{h}_{l} g_{ij}). \tag{3.17}$$

For the simplicity in our calculations, the following abbreviations are used

$$R_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik} R_{,j} - g_{ij} R_{,k}) \ , \tag{3.18}$$

and

$$R_{ikl} = R_{ik,l} - R_{il,k} + \frac{1}{2(n-1)}(g_{il} R_{,k} - g_{ik} R_{,l}) \ , \tag{3.19}$$

and

$$R_{ilj} = R_{il,j} - R_{ij,l} + \frac{1}{2(n-1)}(g_{ij} R_{,l} - g_{il} R_{,j}) \ . \tag{3.20}$$

By summing up (3.15) , (3.16) and (3.17) , we obtain

$$\begin{aligned}
C^{h}_{\ ijk,l} + C^{h}_{\ ikl,j} + C^{h}_{\ ilj,k} &= R^{h}_{\ ijk,l} + R^{h}_{\ ikl,j} + R^{h}_{\ ilj,k} \\
&+ \frac{1}{(n-2)}\left[\delta^{h}_{j}\left(R_{ik,l} - R_{il,k} + \frac{1}{2(n-1)}(g_{il} R_{,k} - g_{ik} R_{,l})\right)\right] \\
&+ \frac{1}{(n-2)}\left[\delta^{h}_{k}\left(R_{il,j} - R_{ij,l} + \frac{1}{2(n-1)}(g_{ij} R_{,l} - g_{il} R_{,j})\right)\right] \\
&+ \frac{1}{(n-2)}\left[\delta^{h}_{l}\left(R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik} R_{,j} - g_{ij} R_{,k})\right)\right] \\
&+ \frac{1}{(n-2)}\left[g_{ik} R^{h}_{j,l} - g_{ik} R^{h}_{l,j} + \frac{1}{2(n-1)}(g^{hm} g_{ml} g_{ik} R_{,j} - g^{hm} g_{mj} g_{ik} R_{,l})\right] \\
&+ \frac{1}{(n-2)}\left[g_{ij} R^{h}_{l,k} - g_{ij} R^{h}_{k,l} + \frac{1}{2(n-1)}(g^{hm} g_{mk} g_{ij} R_{,l} - g^{hm} g_{ml} g_{ij} R_{,k})\right]
\end{aligned}$$

$$+\frac{1}{(n-2)}\left[g_{il}\,R^h_{k,j}-g_{il}\,R^h_{j,k}+\frac{1}{2(n-1)}(g^{hm}g_{mj}g_{il}\,R_{,k}-g^{hm}g_{mk}g_{il}\,R_{,j})\right]. \tag{3.21}$$

Now, if we arrange and simplicity the terms of the equation (3.21), then we have

$$\begin{aligned}
C^h{}_{ijk,l}+C^h{}_{ikl,j}+C^h{}_{ilj,k} &= R^h{}_{ijk,l}+R^h{}_{ikl,j}+R^h{}_{ilj,k} \\
&+\frac{1}{(n-2)}\left[\delta^h_j R_{ikl}+\delta^h_l R_{ijk}+\delta^h_k R_{ilj}\right] \\
&+\frac{1}{(n-2)}g_{ik}g^{hm}\left[\left(R_{mj,l}-R_{ml,j}+\frac{1}{2(n-1)}(g_{ml}R_{,j}-g_{mj}R_{,l})\right)\right] \\
&+\frac{1}{(n-2)}g_{ij}g^{hm}\left[\left(R_{ml,k}-R_{mk,l}+\frac{1}{2(n-1)}(g_{mk}R_{,l}-g_{ml}R_{,k})\right)\right] \\
&+\frac{1}{(n-2)}g_{il}g^{hm}\left[\left(R_{mk,j}-R_{mj,k}+\frac{1}{2(n-1)}(g_{mj}R_{,k}-g_{mk}R_{,j})\right)\right].
\end{aligned} \tag{3.22}$$

Also if we multiply the equation (3.18) by g^{ih} , we obtain

$$g^{ih}R_{ijk}=g^{ih}\left(R_{ij,k}-R_{ik,j}+\frac{1}{2(n-1)}(g_{ik}\,R_{,j}-g_{ij}\,R_{,k})\right). \tag{3.23}$$

If we arrange the terms in (3.23), then the equation becomes

$$R^h{}_{jk}=(R^h{}_{j,k}-R^h{}_{k,j}+\frac{1}{2(n-1)}(\delta^h_k R_{,j}-\delta^h_j R_{,k}). \tag{3.24}$$

In the equation (3.24), if we equal the indices h and j, then

$$\begin{aligned}
R^j{}_{jk} &= (R^j{}_{j,k}-R^j{}_{k,j})+\frac{1}{2(n-1)}(\delta^j_k R_{,j}-\delta^j_j R_{,k}), \\
R^j{}_{jk} &= R_{,k}-\frac{1}{2}R_{,k}-\frac{1}{2}R_{,k}.
\end{aligned} \tag{3.25}$$

Consequently, we reach

$$R^j{}_{jk}=0. \tag{3.26}$$

In addition, by interchanging the indices j and k in (3.18), we have

$$R_{ikj}=R_{ik,j}-R_{ij,k}+\frac{1}{2(n-1)}(g_{ij}\,R_{,k}-g_{ik}\,R_{,j}). \tag{3.27}$$

By summing up (3.18) and (3.27), we find

$$R_{ijk}+R_{ikj}=0 \tag{3.28}$$

By substituting (3.18), (3.19) and (3.20) in (3.22), and using (3.24) and (3.26), then we get

$$C^h{}_{ijk,l}+C^h{}_{ikl,j}+C^h{}_{ilj,k}=\frac{1}{n-2}[\delta^h_j R_{ikl}+\delta^h_l R_{ijk}+\delta^h_k R_{ilj}+g_{ij}R^h{}_{lk}+g_{il}R^h{}_{kj}+g_{ik}R^h{}_{jl}]. \quad (3.29)$$

If we equal the indices h and k in (3.29), we get

$$C^h{}_{ijh,l}+C^h{}_{ihl,j}+C^h{}_{ilj,h}=\frac{1}{n-2}[\delta^h_j R_{ihl}+\delta^h_l R_{ijl}+\delta^h_h R_{ilj}+g_{ij}R^h{}_{lh}+g_{il}R^h{}_{hj}+g_{ih}R^h{}_{jl}]. \quad (3.30)$$

By using (3.28) and (3.26) in (3.30), then we have

$$C^h{}_{ilj,h}=\frac{n-3}{n-2}R_{ilj}. \quad (3.31)$$

Substituting (3.31) into (3.29) leads to

$$C^h{}_{ijk,l}+C^h{}_{ikl,j}+C^h{}_{ilj,k}=\frac{1}{n-3}[\delta^h_j C_{ikl}+\delta^h_l C_{ijk}+\delta^h_k C_{ilj}+g_{ij}C^h{}_{lk}+g_{il}C^h{}_{kj}+g_{ik}C^h{}_{jl}]. \quad (3.32)$$

Multiplying (3.32) by g_{hm}

$$C_{mijk,l}+C_{mikl,j}+C_{milj,k}=\frac{1}{n-3}[g_{jm}C_{ikl}+g_{lm}C_{ijk}+g_{km}C_{ilj}+g_{ij}C_{mlk}+g_{il}C_{mkj}+g_{ik}C_{mjl}].$$

Then, from (lemma 3.1) and (theorem 3.2) we can state the conformal curvature tensor with respect to semi symmetric connection satisfies the relation

$$\overline{C}_{mijk,l}+\overline{C}_{mikl,j}+\overline{C}_{milj,k}=\frac{1}{n-3}[g_{ik}\overline{C}_{mjl}+g_{mj}\overline{C}_{ikl}+g_{il}\overline{C}_{mkj}+g_{mk}\overline{C}_{ilj}+g_{ij}\overline{C}_{mlk}+g_{ml}\overline{C}_{ijk}], \quad (3.33)$$

where

$$\overline{C}_{ijk}=\overline{C}^r{}_{ijk,r}. \quad (3.34)$$

Now, we define conformally recurrent manifold and conformally flat manifold for the Riemannian manifold with semi-symmetric metric connection.

Definition 3.1: If conformal curvature tensor of a Riemannian manifold with semi-symmetric metric connection satisfies the condition

$$\overline{\nabla}_l\overline{C}_{hijk}=\lambda_l\overline{C}_{hijk}, \quad (3.35)$$

where λ_l is a recurrent vector field, then the manifold is called conformally recurrent manifold.

Definition 3.2: If conformal curvature tensor of a Riemannian manifold with semi-symmetric connection is equal to zero, then the manifold is called conformally flat manifold.

Definition 3.3: If the Ricci tensor of RS satisfies the relation

$$\overline{\nabla}_l \overline{R}_{ij} = \lambda_l \overline{R}_{ij}, \tag{3.36}$$

where λ_l is a recurrent vector field, then the manifold is called Ricci recurrent manifold.

It is easily seen that from (3.36) if a Riemannian manifold with semi-symmetric metric connection is Ricci recurrent, then the manifold satisfies the relation

$$\overline{\nabla}_l R = \lambda_l R. \tag{3.37}$$

Theorem 3.3: If a Ricci-recurrent Riemannian manifold with semi-symmetric metric connection satisfies properties of conformally recurrant or conformally flat, then it is recurrent.

Proof: Riemannian manifold with semi-symmetric metric connection holds the relations in (3.35),(3.36) and (3.37)
If we calculate the covariant derivative of $\overline{C}_{hijk}$, then we have the following equations

$$\begin{aligned}\overline{\nabla}_l \overline{C}_{hijk} = \nabla_l \overline{R}_{hijk} - \frac{1}{n-2}(g_{ij}\overline{\nabla}_l \overline{R}_{hk} - g_{ik}\overline{\nabla}_l \overline{R}_{hj} + g_{hk}\overline{\nabla}_l \overline{R}_{ij} - g_{hj}\overline{\nabla}_l \overline{R}_{ik}) \\ + \frac{\overline{\nabla}_l \overline{R}}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}).\end{aligned} \tag{3.38}$$

Using (3.36), the equation (3.38) turns

$$\begin{aligned}\overline{\nabla}_l \overline{C}_{hijk} = \overline{\nabla}_l \overline{R}_{hijk} - \frac{1}{n-2}(g_{ij}\lambda_l \overline{R}_{hk} - g_{ik}\lambda_l \overline{R}_{hj} + g_{hk}\lambda_l \overline{R}_{ij} - g_{hj}\lambda_l \overline{R}_{ik}) \\ + \frac{\lambda_l \overline{R}}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}).\end{aligned} \tag{3.39}$$

By arranging the equation (3.39) , we have

$$\begin{aligned}\overline{\nabla}_l \overline{C}_{hijk} = \overline{\nabla}_l \overline{R}_{hijk} + \lambda_l \big(- \frac{1}{n-2}(g_{ij}\overline{R}_{hk} - g_{ik}\overline{R}_{hj} + g_{hk}\overline{R}_{ij} - g_{hj}\overline{R}_{ik}) + \\ \frac{\overline{R}}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})\big).\end{aligned} \tag{3.40}$$

From the (3.40) it is concluded the following equation

$$\overline{\nabla}_l \overline{C}_{hijk} = \overline{\nabla}_l \overline{R}_{hijk} + \lambda_l (\overline{C}_{hijk} - \overline{R}_{hijk}). \tag{3.41}$$

If Ricci recurrent Riemannian manifold with semi-symmetric metric connection is conformally recurrent, then we have

$$\overline{\nabla}_l \overline{R}_{hijk} = \lambda_l \overline{R}_{hijk} \tag{3.42}$$

If also, Ricci recurrent Riemannian manifold with semi-symmetric metric connection is conformally flat, then we have again

$$\overline{\nabla}_l \overline{R}_{hijk} = \lambda_l \overline{R}_{hijk}.$$

For the proof of the following proposition, we need (Lemma 3.2) in [6], therefore, we give the proof of the (lemma 3.2) explicitly. If M is a manifold with a semi symmetric metric connection satisfying

$$\overline{\nabla}_l \overline{C}_{kjih} = 0. \tag{3.43}$$

Then, we have

$$\nabla_l C_{kjih} + \nabla_j C_{lkih} + \nabla_k C_{jlih} = (1-n)(p_l C_{kjih} + p_j C_{lkih} + p_k C_{jlih}). \tag{3.44}$$

Lemma 3.2: ([6], Lemma 2). Let c_j, p_j and B_{hijk} be numbers satisfying the following relations

$$c_l B_{hijk} + p_h B_{lijk} + p_i B_{hljk} + p_j B_{hilk} + p_k B_{hijl} = 0, \tag{3.45}$$

$$B_{hijk} + B_{hjki} + B_{hkij} = 0, \tag{3.46}$$

and

$$B_{hijk} = B_{jkhi} = -B_{hikj}, \tag{3.47}$$

then each B_{hijk} is equal to zero or each $b_j = c_j + 2p_j$ is equal to zero.

Proof: Let non-zero b_q be a one of the b's. If we get equal l, h, k and q in (3.45), then we have

$$c_q B_{qijq} + p_q B_{qijq} + p_i B_{qqjq} + p_j B_{qiqq} + p_q B_{qijk} = 0, \tag{3.48}$$

by arranging (3.48),

$$B_{qijq}(c_q + 2p_q) + p_i B_{qqjq} + p_j B_{qiqq} = 0 \tag{3.49}$$

from (3.46) and (3.47) we have

$$B_{qqjq} = 0 \tag{3.50}$$

so,

$$B_{qijq}(c_q + 2p_q) = B_{qijq} b_q = 0. \tag{3.51}$$

Since b_q is non-zero, we have

$$B_{qijq} = 0 \tag{3.52}$$

Now, we equal the indices k, h and q in (3.45) and using (3.52), we get

$$p_q(B_{lijq} + B_{qijl}) = 0 \tag{3.53}$$

If we get $p_q = 0$, since $b_q = c_q + 2p_q$ we have $c_q = b_q \neq 0$.

Now, we get equal the indices l, h and q in (3.45), then we have

$$c_q B_{qijk} + p_q B_{qijk} + p_i B_{qqjk} + p_j B_{qiqk} + p_k B_{qijq} = 0 \tag{3.54}$$

from (3.45), (3.48), (3.52) , we have

$$B_{qijk} = 0 \tag{3.55}$$

for all i, j and k since $c_q \neq 0$.

Now, we get equal the indices l and q in (3.45), then we have

$$c_q B_{hijk} + p_h B_{qijk} + p_i B_{hqjk} + p_j B_{hiqk} + p_k B_{hijq} = 0. \tag{3.56}$$

so, we have

$$c_q B_{hijk} = 0, \tag{3.57}$$

since $c_q \neq 0$, $\quad B_{hijk} = 0$ for all i, j, k and h.

If we get $p_q \neq 0$ in (3.53), then we have

$$B_{lijq} + B_{qijl} = 0, \tag{3.58}$$

thus, it is seen from (3.58)

$$B_{lijq} = -B_{qijl}, \tag{3.59}$$

and from (3.47), we have

$$B_{lijq} = -B_{qjli} \tag{3.60}$$

If we get equal the indices h and q in (3.46), then

$$B_{qijk} + B_{qjki} + B_{qkij} = 0. \tag{3.61}$$

The equation (3.61) can be written as

$$B_{qjki} + B_{qjki} +_{qijk} = 0, \tag{3.62}$$

Consequently we have

$$B_{hijk} = 0.$$

We will examine the covariant derivative of the conformal curvature tensor of RS.

REFERENCES

[1] **Weatherburn, C. E.,** 1966. An Introduction to Riemannian Geometry and Tensor Calculus. Cambridge University Press, Cambridge.

[2] **Eisenhart, L. P.,** 1927. Non-Riemannian Geometry. American Mathematical Society, Colloquim Publications, **Volume VIII.**

[3] **Shouten, J. A.,** 1954. Der Ricci-Kalkul. Springer, Berlin.

[4] **Kreyszig, E.,** 1968. Introduction to Differential Geometry and Riemannian Geometry. University of Toronto Press.

[5] **Carmo, M. P.do,** 1992. Riemannian Geometry. Mathematics: Theory and Applications. Boston, Mass.

[6] **Krawczyk J.,** 1988. On Some Riemannian Manifolds Admitting a Metric Semi-Symmetric Connection, Colloquium Mathematicum.

[7] **Yano K, Imai T,.** 1975.On Semi-Symmetric Metric F-connection Tensor N.S.,**Vol.29**

[8] **Yano K,** 1970. On Semi-Symmetric Metric Connection. Revue Romanie de Math. Pures et Appliqueees, **1579-1586**.

[9] **Eisenhart L. P.,** 1949. Riemannian Geometry, Princteon University Press.

[10] **Ercan Şelale,.** 2011. Yarı Simetrik Rekürant Metrik Konneksiyonlu Riemann Manifoldu, Marmara Üniversitesi.

APPENDICES

CURRICULUM VITAE

Name Surname: Sena Kaçar

Place and Date of Birth: Fatih, 27.03.1989

E-mail: uralse@itu.edu.tr

B. S.c.: Istanbul Technical University

M. S.c.: Istanbul Technical University