ON CONFORMAL CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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YARI SİMETRİK KONNEKSİYONLU RİEMANN UZAYLARININ KONFORM EĞRİLİK TENSÖRLERİ

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to my family...
FOREWORD
I want to present thanks to my advisor Prof. Dr. Fatma Özdemir who provides endless knowledge and source to me. Moreover I would like to express my gratitude to my family for their infinite support and love.

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Abbreviations

M : Manifold
\( g_{ij} \) : Metric tensor
\( \nabla_k \) : Covariant derivative
\( \{^i_{jk}\} \) : Connection coefficients of Levi-Civita connections
\( R^i_{jkl} \) : Curvature tensor of Riemannian manifold
\( C^i_{jkl} \) : Conformal curvature tensor of Riemannian manifold
\( \Gamma^i_{jk} \) : Connection coefficient of semi-symmetric Riemannian manifold
\( \tilde{R}^i_{jkl} \) : Curvature tensor of semi-symmetric Riemannian manifold
\( \tilde{C}^i_{jkl} \) : Conformal curvature tensor of semi-symmetric Riemannian manifold
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SUMMARY

In this thesis, we study on Riemannian manifolds with semi symmetric metric connection, which was introduced in [1-10].

In the first chapter, differentiable manifold, Riemannian manifold, affine connection, Riemannian connection, the covariant derivative of Riemannian manifold with respect to Riemannian connection are given. Then, the curvature tensors of Riemannian manifold are given and their properties of the curvature tensor are mentioned [2,4].

Afterwards, the conformal curvature tensor of Riemannian manifold is obtained and properties of the conformal curvature tensor are given.

In the second chapter, we study on Riemannian manifolds with semi symmetric metric connection. The connection coefficient of Riemannian manifold with semi-symmetric metric connection is given as

\[ \Gamma^l_{jk} = \{ l \}_{jk} + \delta^l_k \pi_j - \pi^l g_{jk}, \]

and the curvature tensor of Riemannian manifold with semi-symmetric metric connection is obtained such that

\[ R^h_{ijk} = R^h_{ijk} + \delta^h_j \pi^i_k - \delta^h_i \pi^k_j + g_{ij} g^{hr} \pi_{rk} - g_{ik} g^{hr} \pi_{rj}. \]

In the third chapter, conformal curvature tensor of Riemannian manifold with semi-symmetric metric connection is given as [10]

\[ C^h_{jkl} = \overline{R}^h_{jkl} + \frac{1}{(n-2)} (\delta^h_j \overline{R}_{i k} - \delta^h_i \overline{R}_{j k} - g_{i k} g^{hl} \overline{R}_{l j} - g_{i j} \overline{R}_{h k}) + \frac{R}{(n-1)(n-2)} (\delta^h_j \overline{R}_{i k} - \delta^h_i \overline{R}_{j k}). \]

After that, the properties of the tensor \( C_{mjk} \) are investigated. In the last part, properties of the conformal curvature tensor of Riemannian manifold with semi-symmetric metric connection are mentioned and we proved that the conformal curvature tensor satisfies the relation

\[ C_{mijk} + C_{mikl,j} + C_{miljk} = \frac{1}{n-3} (g_{ik} C_{mj} + g_{mj} C_{ikl} + g_{kl} C_{mjk} + g_{mk} C_{ilj} + g_{ij} C_{mik} + g_{ml} C_{ijk}). \]
YARI SİMETRİK KONNEKSİYONLU RIEMANN UZAYLARININ KONFORM EĞRİLİK TENSÖRLERİ

ÖZET

Bu tezimizde yarı simetrik metrik konneksiyonlu Riemann manifoldları incelenmiştir [1-10].

Tezin birinci bölümünde, diferansiyellenebilir manifold, Riemann manifoldu, affin konneksiyon, Riemann konneksiyonu, Riemann konneksiyonuna göre kovaryant türev tanmları detaylı bir şekilde verilmiştir. Daha sonra, local koordinatlarda Riemann manifoldunun eğrilik, kovaryant eğrili̇k, Ricci eğrili̇k tensöleri ve skaler eğrili̇k tanımları ve bu büyükliklere ait özellikler verilmiştir [2,4].

Birinci bölümünün ikinci kısmında Riemann manifoldlarına ait konform dönüşüm altında konneksiyon katsayısı ve eğrili̇k tensörü aşağıdaki şekilde elde edilmiştir.

\[ C_{lkji} = R_{lkji} - \frac{1}{n-2} (g_{kj}R_{li} - g_{ki}R_{lj} + g_{li}R_{kj} - g_{lj}R_{ki}) + \frac{R}{(n-1)(n-2)} (g_{kj}g_{li} - g_{ki}g_{lj}). \]

Sonrasında ise, bu eğrili̇k tensörü'nün özellikleri verilmiştir.

İkinci bölümde, literatürde verilen yarı simetrik metrik konneksiyonlu Riemann manifoldu ile ilgili çalışmalar üzerinde durulmuş ve çalışma boyunca bu manifold RS ile gösterilmiştir. Yarı simetrik metrik konneksiyonlu Riemann manifolduna ait konneksiyon katsayıları ve bu katsayılarnın tekliği ispat edilmiştir. Riemann manifoldu üzerinde bu konneksiyona göre üreten eğrili̇k tensörü'nün hesapları detaylı bir şekilde verilmiştir ve yarı simetrik metrik konneksiyonlu Riemann manifolduna ait konneksiyon katsayıları ve bu katsayıların tekliği ispat edilmiştir. Bu konneksiyon katsayılarnın eğrili̇k tensörü'nün genel ifadesine yerleştirilmesiyle yarı simetrik metrik konneksiyonlu Riemann manifolduna ait eğrili̇k tensörü aşağıdaki şekilde elde edilmiştir

\[ \bar{R}_{mi,jk} = R_{mi,jk} + g_{mk}\pi_{ij} - g_{jm}\pi_{ik} + g_{lj}\pi_{mk} - g_{ik}\pi_{mj}. \]

Devamında ise, bu eğrili̇k tensörü'nün özellikleri açıka verilmiştir.

Üçüncü bölümde, [10] dan yararlanarak, yarı simetrik metrik konneksiyonlu Riemann manifoldunun konform dönüşüm altında sahip olduğu konneksiyon katsayısı, aşağıdaki şekilde

\[ \Gamma_{jk}^i = \Gamma_{jk}^i - \delta_{ji}q_j + g_{jk}q_j \]
bulunmuştur.

Daha sonra aşağıdaki kısaltmalar kullanılarak,

\[ q_{ij} = \nabla_j q_j + q_i q_j - \frac{1}{2} g_{ij} q_i q_j, \]

ve

\[ W_{ij} = -q_{ij} + q_i q_j, \]

[10] da elde edilen yeni konneksiyon katsayısına göre eğrilik tensörü ,

\[ R^h_{ijk} = R^h_{ijk} + \delta^h_k W_{ij} - \delta^h_j W_{ik} + g_{ij} g^{hr} W_{rk} - g_{ik} g^{hr} W_{rj}, \]

olarak elde edilir.

Buradan yarı simetrik konneksiyonlu Riemann manifoldunun konform eğrilik tensörü,

\[ \bar{C}^{h}_{jkl} = \bar{R}^{h}_{jkl} + \frac{1}{(n-2)} (\delta^{h}_{j} \bar{R}_{ik} - \delta^{h}_{i} \bar{R}_{jk} - \delta^{h}_{k} \bar{R}_{ij}) + \frac{R}{(n-1)(n-2)} (\delta^{h}_{j} \bar{R}_{ik} - \delta^{h}_{i} \bar{R}_{jk}) \]

olarak bulunmuştur.

Sonrasında, tezimizde yarı simetrik metrik konneksiyonlu konform eğrilik tensörünün, Riemann manifoldunun konform eğrilik tensörüne eşit olduğu gösterilmiştir.

Yarı simetrik metrik konneksiyonlu konform eğrilik tensörünün, aşağıdaki özellikleri çıkarılmıştır

\begin{align*}
(1) & \quad \bar{C}_{mij} + \bar{C}_{imj} = 0, \\
(2) & \quad \bar{C}_{mij} + \bar{C}_{mik} = 0.
\end{align*}

Devamında ise, yarı simetrik metrik konneksiyonlu Riemann manifoldunun konform eğrilik tensörünün,

\[ R_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)} (g_{ik} R_{j} - g_{ij} R_{k}), \]

\[ R_{ikl} = R_{ik,l} - R_{il,k} + \frac{1}{2(n-1)} (g_{il} R_{k} - g_{ik} R_{l}), \]

\[ R_{ilj} = R_{il,j} - R_{ij,l} + \frac{1}{2(n-1)} (g_{ij} R_{l} - g_{il} R_{j}), \]

kısaltmaları altında, aşağıdaki eşitliğini

\[ \bar{C}_{mij,l} + \bar{C}_{mkl,j} + \bar{C}_{mil,j} = \frac{1}{n-3} (g_{ik} \bar{C}_{mj} + g_{mj} \bar{C}_{ikl} + g_{ml} \bar{C}_{ik} + g_{mk} \bar{C}_{il} + g_{ij} \bar{C}_{mik} + g_{ml} \bar{C}_{ijk}), \]

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sağladığı gösterilmiştir.
Daha sonra, yarı simetrik metrik konneksiyonlu Riemann manifoldu için konform rekürantlık koşulu, aşağıdaki şekilde verilmiştir

\[ \nabla_l C_{hijk} = \lambda_l C_{hijk}, \]

Buna ek olarak, konform olarak düz olma koşulunun, yarı simetrik metrik konneksiyonlu Riemann manifoldunun konform eğrilik tensörünün sıfıra eşit olmasına sağlanacağı söylenmiştir. Ayrıca, RS için Ricci rekürantlık koşulu,

\[ \nabla_l R_{ij} = \lambda_l R_{ij}, \]

seklinde verilmiştir. Buna bağlı olarak açıkça görülür ki, yarı simetrik metrik konneksiyonlu Riemann manifoldunun skaler eğriliği şu koşulu sağlar,

\[ \nabla_l R = \lambda R. \]

Yukarıdaki üç koşul kullanılarak, yarı simetrik metrik konneksiyonlu Ricci rekürant bir Riemann manifoldu, konform rekürantlık ya da konform olarak düz olma koşulunu sağladığında, yarı simetrik metrik konneksiyonlu Riemann manifoldunun kendisinin de rekürantliği sağladığı gösterilmiştir.
Daha sonra, ([6], Lemma 2) de verilen yarı simetrik metrik konneksiyonlu Riemann manifoldu, eğer konform olarak düz olma koşulunu sağlarlsa,

\[ \nabla_l C_{kijh} + \nabla_j C_{lkih} + \nabla_k C_{ljih} = (1 - n)(p_l C_{kijh} + p_j C_{lkih} + p_k C_{ljih}) \]

ifadesini de sağladığı göstermek için gerekli olan aşağıdaki lemma ispatlanmıştır. [6] da gösterildiği gibi \( c_j, p_j \) and \( B_{hijk} \) sayıları aşağıdaki denklemleri sağlamaktadır,

\[ c_j B_{hijk} + p_l B_{lijk} + p_l B_{hlijk} + p_j B_{lilk} + p_k B_{hjl} = 0, \]

\[ B_{hijk} + B_{hjki} + B_{hlkj} = 0, \]

ve

\[ B_{hijk} = B_{jkhi} = -B_{hikj}. \]

Yukarıdaki denklemler sonucu ya her \( B_{hijk} = 0 \), ya da her \( b_j = c_j + 2p_j = 0 \) dır.
1 INTRODUCTION

Riemannian spaces endowed with some special connections are being used defining physical spaces, especially in the theories of gravity context, in addition to its wide geometrical richness. This thesis is devoted to study manifolds with a special connection, semi-symmetric metric connection. We present basic definitions and fundamental theorems of curvatures in Riemannian manifolds in the differential geometric frame, and then extend our knowledge to semi-symmetric metric connection case. In literature, calculations of curvature tensor of Riemannian and semi symmetric spaces were studied by many authors [1-10].

In the first chapter, we study on differentiable manifolds. We give the definitions of affine connection, Riemannian metric, Riemannian connection explicitly. Then, we examine these concepts in local coordinates. In the second section of first chapter we work on conformal curvature of Riemannian manifold and how changes the curvature tensor under a conformal mapping.

In the second chapter, we study on Riemannian manifold with semi symmetric metric connection denoted by RS. The uniquennes of the connection coefficient of RS is proved and the curvature tensor with respect to the connection coefficients of RS is obtained [6]. Then, the properties of this curvature tensor are examined in detail.

In the third section, we study conformal curvatures on Riemannian space with semi symmetric metric connection. In [10], the calculations of conformal curvature tensor are given in detail. In this thesis, after a brief calculations of conformal curvature, we give the covariant derivative of conformal curvature tensor of Riemannian manifold with semi symmetric metric connection rigorously. We hope that the relations and results about covariant derivative of conformal curvature will be helpfull for readers.

1.1 Riemannian Manifold

Let \( M \) be an \( n \) dimensional Haussdorff space and \( p \) be any point of \( M \). If there exists an open neighbourhood \( U \) of \( p \), and \( U \) is a homeomorphism to an \( n \) dimensional Euclidean space \( \mathbb{R}^n \)
, then $M$ is called $n$ dimensional manifold. If there exists a differentiable homeomorphism between $U$ and $\mathbb{R}^n$ then, the manifold is called differentiable manifold.

It is given definition of a manifold more explicitly in following.

Let $M$ be a Hausdorff space and $\Lambda$ be the set indices and for open sets $U_i$, $\phi_i(U_i)$ be an open subset of $\mathbb{R}^n$ under $\phi$ homeomorphism. If a family of collection $\{(U_i, \phi_i)\}_{i \in \Lambda}$ on $M$ satisfies the following properties

1. $M = \bigcup_{i \in \Lambda} U_i$, 
2. for any pair $i$ and $j \in \Lambda$, the differentiable mapping $\phi_j \circ \phi_i^{-1}$ is such that 
   $$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$
3. the collection $(U_i, \phi_i)_{i \in \Lambda}$ is maximal with respect to the condition (1) and (2).

then the collection $\{(U_i, \phi_i)\}_{i \in \Lambda}$ defines an $n$-dimensional differentiable structure.

If a function $\phi$ is a one-to-one, into and continuous function and has continuous inverse, then it is called as a homeomorphism. If a homeomorphism $\phi$ is differentiable then, it is called differentiable homeomorphism i.e, diffeomorphism.

Let $U$ be an open subset of $M$ and $\phi : U \to \mathbb{R}^n$ be a homeomorphism. If $\phi$ has all partial derivatives of order $k$ and these derivatives are continuous, then the function $\phi$ is differentiable class of $C^k$.

If it can be defined a differentiable structure on $M$ class of $C^k$, then $M$ is called a differentiable manifold class of $C^k$ [5].

**Definition 1.1:** Let $T(M)$ be a set of fields of tangent vectors class of $C^\infty$ and $X$, $Y$ and $Z \in T(M)$ and $f, g \in C^\infty(M, \mathbb{R})$. For any map $\nabla$ defined as

$$\nabla : T(M) \times T(M) \to T(M),$$

$$(X, Y) \to \nabla_X Y, \quad (X, Y \in T(M))$$

if the following properties are satisfied

1. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z,$
2. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$
3. $\nabla_{gX} Y = g \nabla_X Y,$
4. $\nabla_{X}(gY) = (Xg)Y + g \nabla_X Y,$

then $\nabla$ is called an affine connection on $M$.

**Definition 1.2:** If there exist a bilinear, symmetric and positive definite and class of $C^\infty$ function
denoted by \( <,> \) on \( M \), then \( M \) is called Riemannian manifold and also the function \( <,> \) is called Riemannian metric.

Let \( T(M) \) be a tangent space at \( p \in M \) and any vector fields \( X, Y \) and \( Z \in T(M) \). If the function \( <,> \) defined as

\[
<,>: T_p(M) \times T_p(M) \rightarrow R
\]

\[
<,>(X,Y) \rightarrow <X,Y>
\]

satisfied the followings

(1). \( <X + Y, Z> = <X, Z> + <Y, Z> \),

\( <X, Y + Z> = <X, Y> + <X, Z> \),

(2). \( <X, Y> = <Y, X> \),

(3). \( <X, X> \geq 0 \quad X \neq 0 \),

(4). If \( X \) and \( Y \) is class of \( C^\infty \) in a neighbourhood \( U \) of \( p \), then, the inner product \( <X, Y>_p = <X_p, Y_p> \) is class of \( C^\infty \) in a neighbourhood \( U \).

Also, the Riemannian metric \( <X, Y> \) can be expressed \( g(X, Y) \).

Furthermore, if the affine connection \( \nabla \), also satisfies the following two property

(1). \( \nabla_X Y - \nabla_Y X = [X,Y] \),

and

(2). \( Z <X,Y> = <\nabla_Z X, Y> + <X, \nabla_Z Y> \),

then \( \nabla \) is called Riemannian connection.

We now give the definition of metric tensor in local coordinates since many tensor calculations in our thesis is based on local coordinates.

In 1854, Riemann defined the infinitesimal distance \( ds \) between infinitesimally close two points \( x^i \) and \( x^i + dx^i \) in \( n \)-dimensional spaces as

\[
ds^2 = g_{ij} dx^i dx^j, \quad (i, j = 1, 2, \cdots, n), \tag{1.1}
\]

where \( x^i \) are coordinate functions and \( g_{ij} \) is a non-singular symmetric tensor of rank 2 with type (0,2). \( g_{ij} \) is function of \( x^i \)'s and called Riemannian metric tensor. Such a space and a geometry are characterized by Riemannian metric tensor, also they are called Riemannian space and Riemannian geometry.

3
Let $C$ be a smooth curve with parameterized by $x^i = x^i(t)$ in $M$ and $x^i$ and, $x^i + dx^i$ be two very close points on the curve $C$. Any vector $U$ is parallel transported along the curve $C$ by

$$dU^i = -\Gamma^i_{jk} U^j dx^k; \quad (1.2)$$

here $\Gamma^i_{jk}$ denotes the connection coefficients of the parallel transport.

We now give the statement of fundamental theorem for Riemannian manifolds.

**Theorem 1.1.** For a Riemannian manifold $M$ with its metric tensor $g_{ij}$ defined (1.1) there exists unique connection that is compatible with Riemannian metric tensor $g_{ij}$ of $M$.

Let $x$ and $\bar{x}$ be coordinate functions on $M$ and, also, $\Gamma^i_{jk}$ and $\bar{\Gamma}^i_{jk}$ connection coefficients for corresponding these coordinates, respectively. Under general coordinate transformations, $\Gamma^i_{jk}$ transforms as

$$\bar{\Gamma}^i_{ml} \frac{\partial x^i}{\partial x^h} = \frac{\partial^2 x^i}{\partial x^m \partial x^l} + \Gamma^i_{jk} \frac{\partial x^j}{\partial x^m} \frac{\partial x^k}{\partial x^l}, \quad (1.3)$$

therefore, $\Gamma^i_{ml}$ is not a tensor. The coefficients of the connection $\Gamma^i_{jk}$ of the Riemannian manifold is sometimes called Levi-Civita.

It is observed that the followings are

$$g^{ih} g_{jh} = \delta^i_j, \quad \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad (1.4)$$

The coefficients of Levi-Civita connection are expressed by $\{^i_{jk}\}$ explicitly, and it is given by [2]

$$\{^i_{jk}\} = g^{ij} [jk, h], \quad [jk, h] = \frac{1}{2} \left( \frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right), \quad (1.5)$$

where $[jk, h]$ is the first kind of Christoffel symbol, and $\{^i_{jk}\}$ is the second kind of Christoffel symbol.

It is obvious that these symbols are symmetric with respect to the indices $j$ and $k$.

In general, on a manifold $M$ the connection coefficients $\Gamma^i_{jk}$ can be decomposed of symmetric and anti-symmetric part as

$$\Gamma^i_{jk} = \Gamma^i_{(jk)} + \Gamma^i_{[jk]}, \quad (1.6)$$

Here, the symmetric part is defined by

$$\Gamma^i_{(jk)} = \frac{1}{2} \left( \Gamma^i_{jk} + \Gamma^i_{kj} \right), \quad (1.7)$$
and the anti-symmetric part is defined by
\[
\Gamma_{[jk]}^i = \frac{1}{2} (\Gamma_{jk}^i - \Gamma_{kj}^i). \tag{1.8}
\]
In spite of $\Gamma_{jk}^i$ is not a tensor, their anti-symmetric part $\Gamma_{[jk]}^i$ is a tensor and it is called torsion tensor of the connection.

It is known that the Riemannian connection $\Gamma_{jk}^i$ is a symmetric and torsion free connection.

In a Riemannian manifold, components of a contravariant vector field $u^i$, and any covariant vector field $u_i$, and any tensor field $U_{hji}$ of rank 3, we have the followings [1]
\[
\nabla_j u^i = \frac{\partial u^i}{\partial x^j} + u^h \Gamma_{hji}^i, \tag{1.9}
\]
\[
\nabla_j u_i = \frac{\partial u_i}{\partial x^j} - u_k \Gamma_{kij}^i, \tag{1.10}
\]
\[
\nabla_k U_{hji} = \frac{\partial U_{hji}}{\partial x^k} + U_{kai} \Gamma_{hji}^a - U_{hja} \Gamma_{kij}^a - U_{hja} \Gamma_{kji}^a. \tag{1.11}
\]

From (1.11), it can be seen that the covariant derivative of the Riemannian metric tensor $g_{ij}$ is
\[
\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \{ a_k \} g_{ai} - \{ a_i \} g_{aj} = 0. \tag{1.12}
\]
Also, using (1.4) and (1.12), it is seen
\[
\nabla_k g^{ij} = 0. \tag{1.13}
\]

In general, the curvature tensor of any manifold $M$ is given by [4]
\[
R_{hji}^k = \partial_k \Gamma_{hji}^k - \partial_j \Gamma_{hki}^i + \Gamma_{hji}^a \Gamma_{ki}^a - \Gamma_{hjl}^a \Gamma_{lki}^a. \tag{1.14}
\]
where $\partial_k$ denotes the derivative with respect to coordinate functions of manifold $M$, i.e., $\partial_k = \frac{\partial}{\partial x^k}$.

If we use the Levi-Civita connection $\{ i_{jk} \}$ in (1.14), instead of $\Gamma_{jk}^i$, it is obtained that the curvature tensor of Riemannian manifold $M$ is
\[
R_{hji}^k = \partial_k \{ h_{ji} \} - \partial_j \{ h_{ki} \} + \{ h_{kl} \} \{ l_{ji} \} - \{ h_{jl} \} \{ l_{ki} \}, \tag{1.15}
\]
and $R_{hji}^k$ is called the Riemannian curvature tensor.

Multiplying (1.15) by the metric tensor $g_{ah}$ we get
\[
R_{hkji} = R_{kji}^a g_{ah}, \tag{1.16}
\]
and $R_{jkji}$ is called the covariant curvature tensor of Riemannian manifold. The following properties are valid for the Riemannian covariant curvature tensor $R_{jkji}$

\[(1). \quad R_{kjih} = -R_{jkih}, \quad (1.17)\]
\[(2). \quad R_{kjih} = -R_{kjhi}, \quad (1.18)\]
\[(3). \quad R_{kjih} = R_{ihkj}, \quad (1.19)\]

and from (1.18) and (1.19), it is concluded that

\[(4). \quad R_{khih} = R_{kjhh} = 0 . \quad (1.20)\]

In the equation (1.15), if the indices are changed cyclicly and adding obtained equations, we have

\[
R^h_{ikj} + R^h_{jik} + R^h_{kji} = 0 , \quad (1.21)
\]
that is called the first Bianchi identity for the Riemannian curvature tensor of $M$.

Multiplying the equation (1.21) by $g_{hm}$, the another representation of the first Bianchi identity is obtained as

\[
R_{mikj} + R_{mjik} + R_{mkji} = 0 , \quad (1.22)
\]

Another identity holds for the covariant derivative of the Riemannian curvature tensor

\[
\nabla_i R^h_{kji} + \nabla_j R^h_{kij} + \nabla_j R^h_{kli} = 0 , \quad (1.23)
\]
that is called the second Bianchi identity. Multiplying (1.23) by metric tensor $g_{hm}$, the second Bianchi identity can be written in the covariant form as

\[
R_{mkji,i} + R_{mklij} + R_{mkil,j} = 0 .
\]

Contracting the indices $h$ and $i$ in $R^h_{kji}$ in (1.15) we have

\[
R_{kj} = R^i_{kji} , \quad (1.24)
\]
and the $R_{kj}$ is called the Ricci curvature tensor. Also, we have

\[
R_{kj} = R_{akj}^a = g^{ab} R_{akj}b = g^{ba} R_{jba} = g^{ba} R_{bja} = R_{jk}, \quad (1.25)
\]
which shows that the Ricci curvature tensor is symmetric.

Multiplying $R_{kj}$ by $g^{jk}$, we get

\[
R = g^{ij} R_{kj} , \quad (1.26)
\]
which is called the scalar curvature $R$ of the Riemannian manifold.
### 1.2 The Conformal Curvature Of Riemannian Manifold

In this section, first we review the conformal curvature tensor of Riemannian manifold with respect to Levi-Civita connection which is studied in [6], then we examine properties of this tensor in detail.

Let \( M \) and \( N \) be two \( n \) dimensional Riemannian manifolds, \( p \) be a positive differentiable function on \( M \), \( g \) and \( g^* \) be two metric tensors defined on \( M \) and \( N \), respectively.

Now, we assume that the metric tensor \( g^* \) is obtained by using a one to one differentiable mapping \( p \) from \( M \) into \( N \) such that

\[
g^*_{ij} = p^2 g_{ij}. \tag{1.27}
\]

Especially, this transformation is called conformal mapping.

Also, from (1.27), we have

\[
g^{*ij} = p^{-2} g_{ij}^*. \tag{1.28}
\]

Under conformal mapping for the metric tensor \( g^*_{ij} \), connection coefficients are

\[
\{ h^*_{ji} \} = \frac{1}{2} g^{*th}(\frac{\partial g^*_{it}}{\partial x^j} + \frac{\partial g^*_{jt}}{\partial x^i} - \frac{\partial g^*_{ji}}{\partial x^t}). \tag{1.29}
\]

Substituting (1.27) and (1.28) in (1.29), the connection coefficients of Riemannian manifold transform into

\[
\{ h^*_{ji} \} = \{ h_{ji} \} + (\delta_i^h p_j + \delta_j^h p_i - g_{ji} p^h), \tag{1.30}
\]

where \( p_i = \partial p/\partial x^i \) and \( p^h = g^{hm} p_m \).

The relation (1.30) shows that how connection coefficients change under conformal transformations.

Now, we examine how the curvature tensor of Riemannian manifold is changed by using \( g^*_{ij} \) and \( \{ h^*_{ji} \} \). If we use (1.30) in (1.15), the curvature tensor of Riemannian manifold is obtained as

\[
R^*_{kji} = \partial_k \{ h_{ji} \}^* - \partial_j \{ h_{ki} \}^* + \{ h_{kl} \}^* \{ l_{ji} \}^* - \{ h_{jl} \}^* \{ l_{ki} \}^*. \tag{1.31}
\]

Substituting (1.30) in (1.31), we get
Arranging terms in (1.32), we have

\[ R^h_{kji} = \partial_k \left[ \left\{ \frac{h}{j} \right\} + (\delta_j^h p_l + \delta_j^l p_j - g_{ji} p_k^h) \right] - \partial_j \left[ \left\{ \frac{h}{ki} \right\} + (\delta_k^h p_l + \delta_k^l p_k - g_{ki} p^h) \right] \\
+ \left\{ \frac{h}{kl} \right\} + (\delta_k^h p_l + \delta_k^l p_k - g_{kl} p^h) \right] \left[ \left\{ \frac{l}{ji} \right\} + (\delta_j^l p_l + \delta_j^l p_j - g_{ji} p^l) \right] \\
- \left\{ \frac{h}{jl} \right\} + (\delta_j^h p_l + \delta_j^l p_j - g_{ji} p^l) \right] \left[ \left\{ \frac{l}{ki} \right\} + (\delta_k^l p_l + \delta_k^l p_k - g_{ki} p^l) \right]. \] (1.32)

Arranging terms in (1.32), we have

\[ R^h_{kji} = R^h_{kji} + \delta_j^h \left( \partial_k p_l - \left\{ \frac{1}{ki} \right\} p_l - p_l p_k + \frac{1}{2} p_l p^l g_{ki} \right) - \delta_k^h \left( \partial_j p_l - \left\{ \frac{1}{ji} \right\} p_l - p_l p_j + \frac{1}{2} p_l p^l g_{ji} \right) \\
- g_{ji} \left( \partial_k p^h + \left\{ \frac{h}{ki} \right\} p^l + \frac{1}{2} \delta_k^h p^l p^l - p_k p^h \right) + g_{ki} \left( \partial_j p_l - \left\{ \frac{h}{jl} \right\} p^l + \frac{1}{2} \delta_j^h p^l p^l - p_j p^h \right). \]

Then, we reach

\[ R^h_{kji} = R^h_{kji} + \delta_j^h p^k - \delta_j^h p^k - g_{ji} p^h + g_{ki} p^l, \] (1.33)

and for the simplicity in calculations we define

\[ p^k_j = \nabla_j p_i - p_j p_i + \frac{1}{2} p_l p^l g_{ji}, \] (1.34)

and

\[ p^h_k = g_{kl} g^{jk}. \] (1.35)

Equating the indices \( h \) and \( i \), it is obtained

\[ R^k = R_{kj} + (n - 2) p_{kj} + p^h g_{kj}. \] (1.36)

After some calculations, we reach

\[ R^h_{kji} = R^h_{kji} - \delta_j^h (L^*_{ji} - L^*_{ij}) + \delta_j^h (L^*_{ki} - L^*_{ik}) - p^h_{ji} g_{ji} + p^h_{j} g_{ki}, \] (1.37)

where

\[ L_{ji} = \frac{1}{n-2} R_{ji} - \frac{R_{g_{ji}}}{2(n-1)(n-2)}. \] (1.38)

Thus, we arrive

\[ R^h_{kji} + \delta_j^h L^*_{ji} - \delta_j^h L^*_{ki} + L^h_{ki} g^h_{ji} - L^*_{j} g^h_{ki} = R^h_{kji} + \delta_j^h L^*_{ji} - \delta_j^h L_{ki} + L_{ki} g^h_{ji} - L^*_{ji} g^h_{ji}. \] (1.39)
Arranging (1.39), then we find that the invariant part under the conformal mapping of the Riemannian curvature tensor is

\[
C^h_{kj} = R^h_{kj} + \delta^h_i L^i_{ji} - \delta^h_j L^i_{ki} + L^i_{kl} g_{ji}^h g_{ki} - L^i_{ji} g_{ki}^h g_{ki}
\] (1.40)

and

\[
C^h_{kj} = R^h_{kj} + \delta^h_i L^i_{ji} - \delta^h_j L^i_{ki} + L^i_{kl} g_{ji}^h g_{ki} - L^i_{ji} g_{ki}^h g_{ki}
\] (1.41)

Consequently, it is obtained

\[
C^h_{kj} = C^h_{kj}
\] (1.42)

The tensor \( C^h_{kj} \) is called as the conformal curvature tensor.

Multiplying (1.41) by the metric tensor \( g_{hl} \), we have

\[
C^l_{ki} = R^l_{ki} - \frac{1}{n-2} (g_{kj} R_{li} - g_{ki} R_{lj} + g_{il} R_{kj} - g_{lj} R_{ki}) + \frac{R}{(n-1)(n-2)} (g_{kj} g_{li} - g_{ki} g_{lj})
\] (1.43)

that is called the conformal covariant curvature tensor of Riemannian manifold.

For conformal curvature tensor \( C^l_{ki} \), the following properties hold

\[
(1). \quad C^l_{kj} + C^l_{ki} = 0
\] (1.44)

\[
(2). \quad C^l_{kj} + C^l_{kij} = 0
\] (1.45)

and

\[
(3). \quad C^k_{kj} = 0
\] (1.46)

Also, it is seen that the conformal curvature tensor satisfies the following property similar to the first Bianchi identity of the Riemannian curvature tensor

\[
C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0
\] (1.47)
2 Riemannian Manifolds with a Semi-Symmetric Metric Connection

Riemannian manifolds are enriched by introducing several connections on them. In this chapter, we examine a Riemannian manifold with a special connection, semi-symmetric metric connection, and study its curvature tensors and their properties.

2.1 Semi-Symmetric Metric Connection

In 1924, Friedmann and Schouten, introduced the idea of the semi-symmetric linear connection for differentiable manifolds without the condition of the metricity. Later, in 1932 Hayden defined a metric connection with torsion on Riemannian manifold. In literature, many author K. Yano, T. Imai, Z. Nakao, K. Amur, S.S. Pujar studied on semi-symmetric metric spaces [7].

Let $M$ be a Riemannian manifold with metric tensor $g$, $\nabla$ be any linear connection, and $\nabla$ be Levi-Civita connection on $M$. The torsion tensor for any non-symmetric linear connection is defined as

$$T^h_{ij} = \Gamma^h_{ij} - \Gamma^h_{ji}. \tag{2.1}$$

In this thesis, we consider torsion tensor, in local coordinates, for $\nabla$

$$T^h_{ij} = \delta^h_j \pi_i - \delta^h_i \pi_j, \tag{2.2}$$

where $\pi$ is a one form.

Then, the connection $\nabla$ is called semi-symmetric connection on $M$. Furthermore, if, $\nabla$ satisfies the property $\nabla g = 0$, then $\nabla$ is also defined by semi symmetric metric connection.

Here, a Riemannian manifold $M$ with endowed connection $\nabla$ is called Riemannian manifold with semi-symmetric metric connection and it will be denoted by RS in this text.

Also, Yano studied spaces with semi symmetric metric connection and some properties of semi symmetric metric connection [8].

Let $v^h$ be a contravariant vector field, $w_i$ be a covariant vector field, $f$ be a scalar and $S_{ji}^h$ be a general tensor field on any manifold $M$ with torsion, then the following Ricci’s identities are satisfied.

$$\nabla_k \nabla_j v^h - \nabla_j \nabla_k v^h = R_{kji}^h v^j - 2T_{kj}^i \nabla_i v^h, \tag{2.3}$$
\[
\n\nabla_k \nabla_j w_i - \nabla_j \nabla_k w_i = -R_{k j}^h w_h - 2T_{k j}^i \nabla_i w_i, \tag{2.4}
\]

and

\[
\nabla_j \nabla_i f - \nabla_i \nabla_j f = -2T_{i j}^h \nabla_h f, \tag{2.5}
\]

and

\[
\nabla_i \nabla_i S_{j i}^h - \nabla_k \nabla_i S_{j i}^h = R_{ikj}^i S_{j i}^h - R_{lkj}^i S_{j i}^h - R_{lki}^i S_{j i}^h - 2T_{ik}^i \nabla_i S_{j i}^h, \tag{2.6}
\]

where \( R_{k j}^h \) is the Riemannian curvature tensor, and \( T_{i j}^h \) is the torsion tensor of the manifold \( M \) [2,3]. In Riemannian manifold it should be noted that for the above given Ricci identities, (2.3-2.6), the torsion tensor \( T_{i j}^h \) is identical zero.

There are many application areas of semi symmetric metric connection in gravitational, and mathematical physics and engineering problems.

Now, we examine the existence and uniqueness of the semi-symmetric metric connection \( \nabla \) in detail.

In the following proof, we use Yano and Imai’s notation and proof techniques for the uniqueness of the connection of RS.

**Theorem 2.1.1:** In a Riemannian manifold with semi-symmetric metric connection the connection coefficients of a semi-symmetric metric connection are determined uniquely as

\[
\Gamma_{h k j}^i = \left\{ h_{k j} \right\} + \delta_{h j}^i \pi_k - \pi_i g_{k j}, \tag{2.7}
\]

where \( \left\{ h_{jk} \right\} \) denotes the Riemannian connection coefficient, and \( \pi_j \) is a 1-form defined in (2.2).

**Proof:** Let \( \nabla \) be derivative with respect to the Riemannian connection, and \( \overline{\nabla} \) be covariant derivative with semi-symmetric metric connection then, we have the covariant derivative of the metric tensor with respect to \( \overline{\nabla} \)

\[
\overline{\nabla}_k g_{i j} = \partial_k g_{i j} - \Gamma_{k i}^h g_{h j} - g_{i h} \Gamma_{k j}^h = 0, \tag{2.8}
\]

and the covariant derivative of the Riemannian metric \( g_{ij} \) with respect to Levi-Civita connection

\[
\nabla_k g_{i j} = \partial_k g_{i j} - \left\{ h_{k i} \right\} g_{h j} - g_{i h} \left\{ h_{j k} \right\} = 0. \tag{2.9}
\]

Subtracting (2.9) from (2.8), we obtain

\[
g_{h j} \left[ \Gamma_{k i}^h - \left\{ h_{k i} \right\} \right] + g_{i h} \left[ \Gamma_{k j}^h - \left\{ h_{j k} \right\} \right] = 0, \tag{2.10}
\]
If we define $U_{ki}^h$ such that

$$U_{ki}^h = \Gamma_{ki}^h - \left\{ \begin{array}{c} h \\ ki \end{array} \right\}, \quad (2.11)$$

and substituting (2.11) into (2.10) then, we find

$$g_{hj}U_{ki}^h + g_{ih}U_{kj}^h = 0. \quad (2.12)$$

From the equation (2.1), we have

$$T_{ij}^h = U_{ij}^h - U_{ji}^h, \quad (2.13)$$

and

$$U_{ij}^h = T_{ij}^h + U_{ji}^h. \quad (2.14)$$

Substituting (2.14) into (2.12), the equation (2.12) becomes

$$(T_{ki}^h + U_{ik}^h)g_{hj} + (T_{kj}^h + U_{jk}^h)g_{ih} = 0. \quad (2.15)$$

If we change the indices of (2.12) cyclicly, we get

$$U_{ki}^h g_{hj} + U_{kj}^h g_{ih} = 0, \quad (2.16)$$

and

$$U_{jk}^h g_{hi} + U_{ji}^h g_{kh} = 0, \quad (2.17)$$

and

$$U_{ij}^h g_{hk} + U_{ik}^h g_{jh} = 0, \quad (2.18)$$

summing up (2.16) and (2.18) and subtracting (2.17), we get

$$(U_{ki}^h + U_{ik}^h)g_{jh} + (U_{kj}^h - U_{jk}^h)g_{ih} + (U_{ij}^h - U_{ji}^h)g_{kh} = 0.$$

Using (2.14), we have

$$(T_{ki}^h + 2U_{ik}^h)g_{jh} + T_{kj}^h g_{ih} + T_{ij}^h g_{kh} = 0. \quad (2.19)$$

Also, using (2.1), we have

$$T_{ji}^h = \delta_j^h \pi_i - \delta_i^h \pi_j, \quad (2.20)$$

and

$$T_{ji}^h = \delta_j^h \pi_j - \pi^j g_{ij}. \quad (2.21)$$
Multiplying (2.19) by $g^{jl}$, and using (2.21) and (2.22) the (2.19) becomes

$$T_{ki}^l - T_{ki}^l - T_{ik}^l + 2U_{ik}^l = 0,$$ \hspace{1cm} (2.22)

and

$$2U_{ik}^l = T_{ki}^l + T_{ik}^l + T_{ki}^l.$$ \hspace{1cm} (2.23)

Then, from (2.23), we obtain

$$U_{ik}^l = \frac{1}{2}(T_{ki}^l + T_{ik}^l + T_{ki}^l),$$ \hspace{1cm} (2.24)

Using (2.24), (2.20) and (2.21) in (2.11), then we uniquely determine the connection coefficient of the semi-symmetric metric connection as

$$\Gamma^h_{jk} = \left\{ \begin{array}{c} h \\ jk \end{array} \right\} + \delta^h_k \pi_j - \pi^h g_{jk}.$$
2.2 The Curvature Of Riemannian Manifold with Semi-Symmetric Metric Connection

In [6] it is examined the curvature tensor of RS and its properties. In this section, we review these concepts and give the calculation of the curvatures of this manifold.

Lemma 2.2.1: Let \( \nabla_k \) denote the covariant derivative with respect to the Riemannian connection and \( \overline{\nabla}_k \) be the covariant derivative of the connection for the RS. Then, we have
\[
\overline{\nabla}_k g_{ij} = 0.
\]

Proof: Using (2.7) and (2.8), we have
\[
\overline{\nabla}_k g_{ij} = \partial_k g_{ij} - g_{hj} \Gamma^h_{ik} - g_{ih} \Gamma^h_{jk} - g_{ik} \pi_j + g_{ij} g^{hr} \pi_r.
\]

Next, we derive the curvature tensor of RS, and steps of calculations.

Theorem 2.2.1: The curvature tensor of Riemannian manifold with semi-symmetric connection is given
\[
\overline{R}^h_{ijk} = R^h_{ijk} + \delta_{i}^{h} \pi_{jk} - \delta_{j}^{h} \pi_{ik} + g_{ij} g^{hr} \pi_r - g_{ik} g^{hr} \pi_r,
\]
where
\[
\pi_{ij} = \nabla_j \pi_i - \pi_i \pi_j + \frac{1}{2} g_{ij} \pi^s \pi_s.
\]
and \( \nabla \) denotes the Levi-Civita connection, and \( R^h_{ijk} \) is the Riemannian curvature tensor [2].

Proof: If we substitute (2.7) in general definition of the curvature tensor stated in (1.14), then we obtain
\[
\overline{R}^h_{ijk} = \partial_j \left( \left\{ \frac{h}{ik} \right\} + \delta_{i}^{h} \pi_{jk} - \delta_{j}^{h} \pi_{ik} + g_{ij} g^{hr} \pi_r - g_{ik} g^{hr} \pi_r \right) - \partial_k \left( \left\{ \frac{h}{ij} \right\} + \delta_{j}^{h} \pi_{ik} - g_{ij} \pi^h \right)
\]
\[
+ \left( \left\{ \frac{h}{sj} \right\} + \delta_{j}^{h} \pi_{s} - g_{sj} \pi^h \right) \left( \left\{ \frac{s}{ik} \right\} + \delta_{i}^{h} \pi_{s} - g_{is} \pi^h \right) \]
After some computations, curvature tensor of RS becomes

\[
R^h_{ijk} = R_{ijk}^h + \delta_k^h (\partial_j \pi_i) - (\partial_j g_{ik}) \pi^h - g_{ik} (\partial_j \pi^h) - \delta_i^h (\partial_k \pi_j) - (\partial_k g_{ij}) \pi^h - g_{ij} (\partial_k \pi^h)
\]

\[
+ \left\{ \frac{h}{s} \right\} (\delta_k^s \pi_i - g_{ik} \pi^s) - \left\{ \frac{h}{s} \right\} (\delta_j^s \pi_i - g_{ij} \pi^s)
\]

\[
+ \delta_j^h \pi_s \left( \left\{ \frac{s}{ik} \right\} + \delta_k^s \pi_i - g_{ik} \pi^s \right)
\]

\[
- g_{sj} \pi^h \left( \left\{ \frac{s}{ik} \right\} + \delta_k^s \pi_i - g_{ik} \pi^s \right)
\]

\[
- \delta_k^h \pi_s \left( \left\{ \frac{s}{ij} \right\} + \delta_j^s \pi_i - g_{ij} \pi^s \right)
\]

\[
+ g_{sk} \pi^h \left( \left\{ \frac{s}{ij} \right\} + \delta_j^s \pi_i - g_{ij} \pi^s \right).
\]

(2.28)

Arranging (2.29), we reach

\[
\bar{R}^h_{ijk} = R^h_{ijk} + \delta_k^h (\nabla_j \pi_i - \pi_j \pi_i + \frac{1}{2} g_{ij} \pi^s \pi_s) - \delta_i^h (\nabla_k \pi_j - \pi_k \pi_j + \frac{1}{2} g_{ik} \pi^s \pi_s) - g_{kj} g^{hr} (\nabla_j \pi_r - \pi_j \pi_r + \frac{1}{2} g_{kr} \pi^s \pi_s)
\]

\[
+ g_{ij} g^{hr} (\nabla_k \pi_r - \pi_k \pi_r + \frac{1}{2} g_{kr} \pi^s \pi_s),
\]

(2.29)

also, using (2.27), we obtain the curvature tensor of RS

\[
\overline{R}^h_{ijk} = \bar{R}_{ijk}^h + \delta_k^h \pi_{ij} - \delta_i^h \pi_{jk} + g_{ij} g^{hr} \pi_{rk} + g_{ik} g^{hr} \pi_{jr},
\]

(2.30)

where \(\bar{R}_{ijk}^h\) denotes the curvature tensor of Riemannian manifold.

Since \(\bar{R}_{mijk} = g_{mh} \bar{R}_{ijk}^h\), if we multiply both sides of the equation (2.26) by \(g_{mh}\) we get the covariant curvature tensor of RS

\[
\overline{R}_{mijk} = \bar{R}_{mijk} + g_{mk} \pi_{ij} - g_{jm} \pi_{ik} + g_{ij} \pi_{mk} - g_{ik} \pi_{mj}.
\]

(2.31)

Contracting the indices \(h\) and \(k\) in (2.26), it is obtained that the Ricci curvature tensor \(\overline{R}_{ij}\) in the form

\[
\overline{R}_{ij} = R_{ij} + \delta_i^h \pi_{ij} - \delta_j^h \pi_{ih} + g_{ij} g^{rh} \pi_{rh} - g_{ih} g^{hr} \pi_{rj},
\]
or
\[ R_{ij} = R_{ij} + (n - 2) \pi_{ij} + g_{ij} \pi \]  
(2.32)

where \( \pi = g^{mk} \pi_{mk} \) and \( R_{ij} \) represents the Ricci tensor of Riemannian manifold.

The symmetric part of the Ricci tensor \( R_{ij} \) is
\[ \bar{R}_{(ij)} = \frac{1}{2} (R_{ij} + R_{ji}), \]  
(2.33)
and the anti-symmetric part of the Ricci tensor \( R_{ij} \) is
\[ \bar{R}_{[ij]} = \frac{1}{2} (R_{ij} - R_{ji}). \]  
(2.34)

On the other hand, for the anti-symmetric part of any tensor \( \pi_i \), it is used the notation
\[ 2\nabla_j [\pi_i] = \nabla_j \pi_i - \nabla_i \pi_j, \]  
(2.35)

From (2.27), it can be seen that
\[ 2\nabla_j [\pi_i] = \pi_{ij} - \pi_{ji}. \]  
(2.36)

Using (2.32) in (2.33), we get
\[ \bar{R}_{(ij)} = R_{ij} + \frac{1}{2} (n - 2)(\nabla_j \pi_i + \nabla_i \pi_j) + g_{ij} \pi \]
and
\[ \bar{R}_{[ij]} = R_{[ij]} + (n - 2)\nabla_j [\pi_i] \]

Multiplying the (2.32) by \( g^{ij} \), we have the scalar curvature for RS as the form
\[ \bar{R} = R + 2(n - 1)\pi \]  
(2.37)

where \( R \) denotes the scalar curvature of Riemannian manifold.

Next, we examine the properties of the curvature tensor of RS and the first Bianchi identity.

**Theorem 2.2.2:** The curvature tensor of semi-symmetric metric Riemannian manifold has the following properties

1. \( R_{mik} + R_{mjk} = 0 \),  
(2.38)
2. \( R_{mijk} + R_{imjk} = 0 \),  
(2.39)
3. \( R_{ijk} + R_{jik} + R_{ki} = 2(\delta^h_i \nabla_k [\pi_j] + \delta^h_j \nabla_i [\pi_k] + \delta^h_k \nabla_j [\pi_i]) \).  
(2.40)
which is called the first Bianchi identity.

Proof:

(1). Interchancing the indices $j$ and $k$ on the equation (2.31), we have

$$\mathcal{R}_{mikj} = R_{mikj} + g_{mj}\pi_{ik} - g_{mk}\pi_{ij} + g_{ik}\pi_{mj} - g_{ij}\pi_{mk},$$

(2.41)

and summing up (2.31) and (2.41), and from (1.18), we obtain

$$\mathcal{R}_{mijk} + \mathcal{R}_{mikj} = R_{mijk} + R_{mikj} = 0.$$

(2.42)

(2). Interchancing the indices $i$ and $m$ in the equation (2.31), we get

$$\mathcal{R}_{imjkh} = R_{imjkh} + g_{ik}\pi_{mjh} - g_{mjh}\pi_{ik} + g_{mj}\pi_{ih} - g_{ih}\pi_{mj},$$

(2.43)

and using (2.31), (2.42), and (1.17), it is seen that covariant curvature tensor $\mathcal{R}_{mijk}$ is anti symmetric with respect to the first two indices, i.e,

$$\mathcal{R}_{mijk} + \mathcal{R}_{mikj} = R_{mijk} + R_{mikj} = 0.$$

(2.44)

(3). Using (2.26), interchancing the indices $i, j$ and $k$ cyclicly, we get

$$\mathcal{R}^{h}_{ijk} = R^{h}_{ijk} + \delta^{h}_{k}\pi_{ij} - \delta^{h}_{j}\pi_{ik} + g_{ij}g^{rh}\pi_{rj} - g_{ik}g^{rh}\pi_{jr},$$

(2.45)

and

$$\mathcal{R}^{h}_{jki} = R^{h}_{jki} + \delta^{h}_{l}\pi_{jk} - \delta^{h}_{k}\pi_{lj} + g_{jk}g^{rh}\pi_{rl} - g_{lj}g^{rh}\pi_{rk},$$

(2.46)

and

$$\mathcal{R}^{h}_{kij} = R^{h}_{kij} + \delta^{h}_{j}\pi_{ki} - \delta^{h}_{i}\pi_{kj} + g_{ki}g^{rh}\pi_{rj} - g_{kj}g^{rh}\pi_{ri}.$$  

From (2.44), (2.45), and (2.46), we reach the first Bianchi identity for RS

$$\mathcal{R}^{h}_{ijk} + \mathcal{R}^{h}_{jki} + \mathcal{R}^{h}_{kij} = R^{h}_{ijk} + R^{h}_{jki} + R^{h}_{kij} + \delta^{h}_{i}(\pi_{jk} - \pi_{kj}) + \delta^{h}_{j}(\pi_{ki} - \pi_{ik}) + \delta^{h}_{k}(\pi_{ij} - \pi_{ji}).$$

(2.47)

In the right hand side of the equation (2.47), using (2.36) and the first Bianchi identity for Riemannian manifold given in (1.21), we have the following relation.

$$\mathcal{R}^{h}_{ijk} + \mathcal{R}^{h}_{jki} + \mathcal{R}^{h}_{kij} = 2(\delta^{h}_{i}\nabla_{[i}\delta_{j]} + \pi^{h}_{i}\nabla_{[i}\pi_{j]} + \pi^{h}_{j}\nabla_{[j}\delta_{i]}).$$
3 THE CONFORMAL CURVATURE OF RIEMANNIAN MANIFOLD WITH SEMI-SYMMETRIC METRIC CONNECTIN

In this chapter, we examine the conformal curvature tensor with respect to the semi-symmetric metric connection and its some properties are presented.

Let \( M \) and \( \overline{M} \) be Riemannian manifolds with semi symmetric metric connections and \( \Gamma^h_{ij} \) and \( \overline{\Gamma}^h_{ij} \) be connection coefficients and \( R^h_{ijk} \) be Riemannian curvatute tensor and \( \overline{R}^h_{ijk} \) be the curvature of Riemannian manifold with semi symmetric metric connection of \( M \) and \( \overline{M} \), respectively. We consider a conformal mapping \( \sigma \) from \( M \) to \( \overline{M} \) and examine how the connection coefficients and curvatures of these connections are under conformal mapping \( \sigma \) and we review their relationship between them.

In [10], It is shown that connection coefficients for semi-symmetric metric connection given (2.7), transforms into

\[
\Gamma^i'_{jk} = \Gamma^i_{jk} - \delta^i_k q_j + g_{ik} q^j, \tag{3.1}
\]

under conformal mapping \( \sigma \) and then the curvature tensor of Riemannian manifold with semi-symmetric metric connection becomes

\[
\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k W_{ij} - \delta^h_j W_{ik} + g_{ij} g^{hr} W_{rk} - g_{ik} g^{hr} W_{rj}, \tag{3.2}
\]

where \( \overline{R}^h_{ijk} \) denotes the transformed curvature tensor and \( \overline{R}^h_{ijk} \) denotes the curvature of the semi symmetric metric connection.

Here

\[
q_{ij} = \nabla_j q_i + q_i q_j - \frac{1}{2} g_{ij} g^{rs} q_r q_s, \tag{3.3}
\]

and

\[
W_{ij} = -q_{ij} + q_i q_j. \tag{3.4}
\]

In [10], under the conformal mapping, the invariant part of the curvature tensor of Riemannian manifold with semi symmetric metric connection is given as
\[
\overline{C}_{ijkl} = \overline{R}_{ijkl} + \frac{1}{n-2} (\delta^j_k R_{ik} - \delta^k_j R_{ij} + g_{ik} g_{jl} R_{lk} - g_{ij} g_{lk} R_{ik}) + \frac{\overline{R}}{(n-1)(n-2)} (\delta^h_i g_{jk} - \delta^h_j g_{ik}),
\]

which is called conformal curvature tensor of these manifolds.

Multiplying (3.5) by \( g_{hm} \) it is obtained the covariant conformal curvature tensor as

\[
\overline{C}_{mijk} = \overline{R}_{mijk} + \frac{1}{n-2} (g_{jm} \overline{R}_{ik} - g_{km} \overline{R}_{ij} + g_{ik} \overline{R}_{mj} - g_{ij} \overline{R}_{mk}) + \frac{\overline{R}}{(n-1)(n-2)} (g_{km} g_{ij} - g_{jm} g_{ik}).
\]

By using an alternative method, we study the conformal curvature tensor of RS and restate its definition [6], and we give the calculation of the following lemma in explicitly to be helpfull characterizing the spaces with respect to their curvature invariants.

**Lemma 3.1:** The conformal curvature tensor with respect to Riemannian connection is equal to the conformal curvature tensor with respect to semi symmetric metric connection.

**Proof:** Using

\[
\overline{C}_{mijk} = R_{mijk} + \frac{1}{n-2} (g_{jm} R_{ik} - g_{km} R_{ij} + g_{ik} R_{mj} - g_{ij} R_{mk}) + \frac{\overline{R}}{(n-1)(n-2)} (g_{km} g_{ij} - g_{jm} g_{ik})
\]

and substituting (2.31), (2.32) and (2.37) in (3.6), we get

\[
\overline{C}_{mijk} = R_{mijk} + g_{mk} \pi_{ij} - g_{jm} \pi_{ik} + g_{ij} \pi_{mk} - g_{ik} \pi_{mj}
\]

\[
+ \frac{1}{n-2} \left[ g_{jm} (R_{ik} + (n-2) \pi_{ik} + g_{ik} \pi) - g_{km} ((R_{ij} + (n-2) \pi_{ij}) + g_{ij} \pi) \right]
\]

\[
+ \frac{1}{n-2} \left[ g_{ik} (R_{mj} + (n-2) \pi_{mj} + g_{mj} \pi) - g_{ij} (R_{mk} + (n-2) \pi_{mk} + g_{mk} \pi) \right]
\]

\[
+ \frac{\overline{R} + 2(n-1)\pi}{(n-1)(n-2)} (g_{ij} g_{mk} - g_{ik} g_{jm}).
\]

If we arrange the above terms, we have

\[
\overline{C}_{mijk} = R_{mijk} + \frac{1}{n-2} (g_{jm} R_{ik} - g_{km} R_{ij} + g_{ik} R_{mj} - g_{ij} R_{mk}) + \frac{\overline{R}}{(n-1)(n-2)} (g_{ij} g_{mk} - g_{ik} g_{jm}),
\]
from which we conclude that (3.8) is equal to the conformal curvature of Riemannian manifold in (1.43).

**Theorem 3.1:** The conformal curvature tensor of Riemannian manifold with semi-symmetric metric connection has properties

(1). \( C_{mijk} + C_{imj} = 0, \)  \hspace{1cm} (3.9)

and

(2). \( C_{mijk} + C_{mikj} = 0. \)  \hspace{1cm} (3.10)

**Proof:**

(1). Interchanging the indices \( i \) and \( m \) on (3.6), we get

\[
C_{imjk} = R_{imjk} + \frac{1}{n-2} \left( g_{ij} R_{mk} - g_{ik} R_{mj} + g_{mk} R_{ij} - g_{mj} R_{ik} \right) + \frac{R}{(n-1)(n-2)} (g_{mj} g_{ik} - g_{mk} g_{ij}).
\]

(3.11)

Summing up (3.6) and (3.11) and, using (2.39), we have

\[
C_{mijk} + C_{imjk} = R_{mijk} + \bar{R}_{mijk} = 0.
\]

(2). Interchanging the indices \( k \) and \( j \) on (3.6), we get

\[
C_{mikj} = R_{mik} + \frac{1}{n-2} \left( g_{mk} R_{ij} - g_{mj} R_{ik} + g_{ij} R_{mk} - g_{ik} R_{mj} \right) + \frac{R}{(n-1)(n-2)} (g_{ik} g_{mj} - g_{ij} g_{mk}).
\]

(3.12)

From (3.6), (3.12), and using (2.38)

\[
C_{mijk} + C_{mikj} = R_{mijk} + \bar{R}_{mikj} = 0.
\]

We now examine the covariant derivative of the conformal curvature tensor of Riemannian manifold.

**Theorem 3.2:** The conformal curvature tensor of Riemannian manifold satisfies the relation

\[
C_{ijk,l} + C_{mik,j} + C_{mij,k} = \frac{1}{n-3} \left[ g_{ik} C_{mjl} + g_{mj} C_{ikl} + g_{il} C_{mkj} + g_{mk} C_{ilj} + g_{ij} C_{mlk} + g_{ml} C_{ijk} \right],
\]

(3.13)

where

\[
C_{ijk} = C'_{ijk,r}.
\]

(3.14)
Proof: By taking covariant derivative of the equation (3.5), we have
\[
C^h_{ijk,l} = R^h_{ijk,l} + \frac{1}{(n-2)} \left( \delta^h_j R_{ik,l} - \delta^h_l R_{ij,k} + g_{ik} R^h_{j,l} - g_{ij} R^h_{k,l} \right) + \frac{R_j}{(n-1)(n-2)} \left( \delta^h_k g_{ij} - \delta^h_j g_{ik} \right),
\]
(3.15)

changing the indices cyclicly \( l, j, \) and \( k \) in (3.15)
\[
C^h_{ikl,j} = R^h_{ikl,j} + \frac{1}{(n-2)} \left( \delta^h_k R_{il,j} - \delta^h_l R_{ik,j} + g_{il} R^h_{j,k} - g_{ik} R^h_{j,j} \right) + \frac{R_j}{(n-1)(n-2)} \left( \delta^h_j g_{il} - \delta^h_l g_{ij} \right),
\]
(3.16)

and
\[
C^h_{ilj,k} = R^h_{ilj,k} + \frac{1}{(n-2)} \left( \delta^h_i R_{lj,k} - \delta^h_l R_{il,k} + g_{lj} R^h_{k,i} - g_{il} R^h_{k,j} \right) + \frac{R_j}{(n-1)(n-2)} \left( \delta^h_j g_{li} - \delta^h_i g_{lj} \right).
\]
(3.17)

For the simplicity in our calculations, the following abbreviations are used
\[
R_{ijk} = R_{ijk} - R_{ik,j} + \frac{1}{2(n-1)} (g_{ik} R_{i,j} - g_{ij} R_{i,k}),
\]
(3.18)

and
\[
R_{ikl} = R_{ikl} - R_{il,k} + \frac{1}{2(n-1)} (g_{il} R_{i,k} - g_{ik} R_{i,l}),
\]
(3.19)

and
\[
R_{ilj} = R_{ilj} - R_{ij,l} + \frac{1}{2(n-1)} (g_{ij} R_{i,l} - g_{il} R_{i,j}).
\]
(3.20)

By summing up (3.15), (3.16) and (3.17), we obtain
\[
C^h_{ijk,l} + C^h_{ikl,j} + C^h_{ilj,k} = R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k}
\]
\[
+ \frac{1}{(n-2)} \left[ \delta^h_j (R_{ik,l} - R_{il,k} + \frac{1}{2(n-1)} (g_{il} R_{ik} - g_{ik} R_{i,l})) \right]
\]
\[
+ \frac{1}{(n-2)} \left[ \delta^h_k (R_{il,j} - R_{ij,l} + \frac{1}{2(n-1)} (g_{ij} R_{il} - g_{il} R_{i,j})) \right]
\]
\[
+ \frac{1}{(n-2)} \left[ \delta^h_l (R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)} (g_{ik} R_{ij} - g_{ij} R_{i,k})) \right]
\]
\[
+ \frac{1}{(n-2)} \left[ g_{ik} R^h_{j,l} - g_{ik} R^h_{l,j} + \frac{1}{2(n-1)} (g^{hm} g_{ml} g_{ik} R_{j} - g^{hm} g_{mj} g_{ik} R_{l}) \right]
\]
\[
+ \frac{1}{(n-2)} \left[ g_{ij} R^h_{l,k} - g_{ij} R^h_{k,l} + \frac{1}{2(n-1)} (g^{hm} g_{mk} g_{ij} R_{l} - g^{hm} g_{ml} g_{ij} R_{k}) \right]
\]
\( + \frac{1}{(n-2)} [g_{il} R_{k,j}^h - g_{il} R_{j,k}^h + \frac{1}{2(n-1)} (g^{hm} g_{mj} g_{il} R_{k} - g^{hm} g_{mk} g_{il} R_{j})] \). \quad (3.21) \)

Now, if we arrange and simplicity the terms of the equation (3.21), then we have

\[
C_{ijk,l} + C_{ikl,j} + C_{ilj,k} = R_{ijk,l} + R_{ikl,j} + R_{ilj,k}
+ \frac{1}{(n-2)} \left[ \delta^h_{j} R_{ikl} + \delta^h_{i} R_{jik} + \delta^h_{l} R_{ijl} \right]
+ \frac{1}{(n-2)} g_{ik} g^{hm} \left[ (R_{mj,l} - R_{ml,j} + \frac{1}{2(n-1)} (g_{ml} R_{j} - g_{mj} R_{l})) \right]
+ \frac{1}{(n-2)} g_{ij} g^{hm} \left[ (R_{ml,k} - R_{mk,l} + \frac{1}{2(n-1)} (g_{mk} R_{l} - g_{ml} R_{k})) \right]
+ \frac{1}{(n-2)} g_{il} g^{hm} \left[ (R_{mk,j} - R_{mj,k} + \frac{1}{2(n-1)} (g_{mj} R_{k} - g_{mk} R_{j})) \right].
\]

Also if we multiply the equation (3.18) by \( g^{ih} \), we obtain

\[
g^{ih} R_{ijk} = g^{ih} (R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)} (g_{ik} R_{j} - g_{ij} R_{k})). \quad (3.23) \)

If we arrange the terms in (3.23), then the equation becomes

\[
R^h_{jk} = (R^h_{j,k} - R^h_{k,j} + \frac{1}{2(n-1)} (\delta^h_{j} R_{j,k} - \delta^h_{j} R_{k,j})). \quad (3.24) \)

In the equation (3.24), if we equal the indices \( h \) and \( j \), then

\[
R^j_{jk} = (R^j_{j,k} - R^j_{k,j} + \frac{1}{2(n-1)} (\delta^j_{k} R_{j,k} - \delta^j_{k} R_{k,j})),
R^j_{jk} = R_{k} - \frac{1}{2} R_{k} - \frac{1}{2} R_{k}.
\]

Consequently, we reach

\[
R^j_{jk} = 0. \quad (3.26) \)

In addition, by interchanging the indices \( j \) and \( k \) in (3.18), we have

\[
R^{ik}_{jk} = R^{ik}_{k,j} - R^{ik}_{j,j} + \frac{1}{2(n-1)} (g_{ij} R_{k} - g_{ik} R_{j}).
\]

By summing up (3.18) and (3.27), we find

\[
R_{ijk} + R^{ik}_{jk} = 0
\]
By substituting (3.18), (3.19) and (3.20) in (3.22), and using (3.24) and (3.26), then we get

\[ C^h_{ijk,l} + C^h_{ikl,j} + C^h_{ilj,k} = \frac{1}{n-2} \left[ \delta^h_j R_{ikl} + \delta^h_l R_{ijk} + \delta^h_k R_{ilj} + g_{ij} R^h_{lk} + g_{il} R^h_{jk} + g_{ik} R^h_{lj} \right]. \]  

(3.29)

If we equal the indices \( h \) and \( k \) in (3.29), we get

\[ C^h_{ijh,l} + C^h_{ihl,j} + C^h_{ilj,h} = \frac{1}{n-2} \left[ \delta^h_j C^h_{ikl} + \delta^h_l C^h_{ijk} + \delta^h_k C^h_{ilj} + g_{ij} C^h_{mk} + g_{il} C^h_{mkj} + g_{ik} C^h_{mlj} \right]. \]  

(3.30)

By using (3.28) and (3.26) in (3.30), then we have

\[ C^h_{ilj,h} = \frac{n-3}{n-2} R_{lj}. \]  

(3.31)

Substituting (3.31) into (3.29) leads to

\[ C^h_{ijk,l} + C^h_{ikl,j} + C^h_{ilj,k} = \frac{1}{n-3} \left[ \delta^h_j C^h_{ikl} + \delta^h_l C^h_{ijk} + \delta^h_k C^h_{ilj} + g_{ij} C^h_{mlk} + g_{il} C^h_{mkj} + g_{ik} C^h_{mlj} \right]. \]  

(3.32)

Multiplying (3.32) by \( g_{lm} \)

\[ C_{mijk,l} + C_{mikl,j} + C_{milj,k} = \frac{1}{n-3} \left[ g_{jm} C_{ikl} + g_{lm} C_{ijk} + g_{km} C_{ilj} + g_{ij} C_{mkj} + g_{il} C_{mkj} + g_{ik} C_{mlj} \right]. \]

Then, from (lemma 3.1) and (theorem 3.2) we can state the conformal curvature tensor with respect to semi symmetric connection satisfies the relation

\[ \overline{C}_{mijk,l} + \overline{C}_{mikl,j} + \overline{C}_{milj,k} = \frac{1}{n-3} \left[ g_{ik} \overline{C}_{mjl} + g_{mj} \overline{C}_{ikl} + g_{il} \overline{C}_{mkj} + g_{mk} \overline{C}_{itj} + g_{ij} \overline{C}_{mlk} + g_{ml} \overline{C}_{ijk} \right], \]

(3.33)

where

\[ \overline{C}_{ijk} = \overline{C}^{r}_{ijk}. \]  

(3.34)

Now, we define conformally recurrent manifold and conformally flat manifold for the Riemannian manifold with semi-symmetric metric connection.

**Definition 3.1:** If conformal curvature tensor of a Riemannian manifold with semi-symmetric metric connection satisfies the condition

\[ \nabla_{ij} \overline{C}_{hijk} = \lambda_{i} \overline{C}_{hijk}, \]

(3.35)

where \( \lambda_{i} \) is a recurrent vector field, then the manifold is called conformally recurrent manifold.
**Definition 3.2:** If conformal curvature tensor of a Riemannian manifold with semi-symmetric connection is equal to zero, then the manifold is called conformally flat manifold.

**Definition 3.3:** If the Ricci tensor of RS satisfies the relation

\[ \nabla_l \bar{R}_{ij} = \lambda_l \bar{R}_{ij}, \]  
(3.36)

where \( \lambda_l \) is a recurrent vector field, then the manifold is called Ricci recurrent manifold.

It is easily seen that from (3.36) if a Riemannian manifold with semi-symmetric metric connection is Ricci recurrent, then the manifold satisfies the relation

\[ \nabla_l \bar{R} = \lambda_l \bar{R}. \]  
(3.37)

**Theorem 3.3:** If a Ricci-recurrent Riemannian manifold with semi-symmetric metric connection satisfies properties of conformally recurrent or conformally flat, then it is recurrent.

**Proof:** Riemannian manifold with semi-symmetric metric connection holds the relations in (3.35),(3.36) and (3.37)

If we calculate the covariant derivative of \( \bar{C}_{hijk} \), then we have the following equations

\[ \nabla_l \bar{C}_{hijk} = \nabla_l \bar{R}_{hijk} - \frac{1}{n-2} (g_{ij} \nabla_l \bar{R}_{hk} - g_{ik} \nabla_l \bar{R}_{hj} + g_{hk} \nabla_l \bar{R}_{ij} - g_{j} \nabla_l \bar{R}_{ik}) \]

\[ + \frac{\nabla_l \bar{R}}{(n-1)(n-2)} (g_{ij} g_{hk} - g_{ik} g_{hj}). \]  
(3.38)

Using (3.36), the equation (3.38) turns

\[ \nabla_l \bar{C}_{hijk} = \nabla_l \bar{R}_{hijk} - \frac{1}{n-2} (g_{ij} \lambda_l \bar{R}_{hk} - g_{ik} \lambda_l \bar{R}_{hj} + g_{hk} \lambda_l \bar{R}_{ij} - g_{j} \lambda_l \bar{R}_{ik}) \]

\[ + \frac{\lambda_l \bar{R}}{(n-1)(n-2)} (g_{ij} g_{hk} - g_{ik} g_{hj}). \]  
(3.39)

By arranging the equation (3.39) , we have

\[ \nabla_l \bar{C}_{hijk} = \nabla_l \bar{R}_{hijk} + \lambda_l \left( - \frac{1}{n-2} (g_{ij} \bar{R}_{hk} - g_{ik} \bar{R}_{hj} + g_{hk} \bar{R}_{ij} - g_{j} \bar{R}_{ik}) + \right. \]

\[ \frac{\bar{R}}{(n-1)(n-2)} (g_{ij} g_{hk} - g_{ik} g_{hj}) \right), \]  
(3.40)

From the (3.40) it is concluded the following equation

\[ \nabla_l \bar{C}_{hijk} = \nabla_l \bar{R}_{hijk} + \lambda_l (\bar{C}_{hijk} - \bar{R}_{hijk}). \]  
(3.41)
If Ricci recurrent Riemannian manifold with semi-symmetric metric connection is conformally recurrent, then we have

$$\nabla_l R_{hijk} = \lambda_l R_{hijk}$$  \hspace{1cm} (3.42)

If also, Ricci recurrent Riemannian manifold with semi-symmetric metric connection is conformally flat, then we have again

$$\nabla_l R_{hijk} = \lambda_l R_{hijk}.$$  

For the proof of the following proposition, we need (Lemma 3.2) in [6], therefore, we give the proof of the (lemma 3.2) explicitly. If $M$ is a manifold with a semi symmetric metric connection satisfying

$$c_l B_{hijk} + p_h B_{lijk} + p_i B_{hljk} + p_j B_{hilk} + p_k B_{hijl} = 0,$$  \hspace{1cm} (3.45)

then each $B_{hijk}$ is equal to zero or each $b_j = c_j + 2p_j$ is equal to zero.

**Proof:** Let non-zero $b_q$ be a one of the b’s. If we get equal $l, h, k$ and $q$ in (3.45), then we have

$$c_q B_{qijq} + p_q B_{qijq} + p_i B_{qqjq} + p_j B_{qiqq} + p_k B_{qiik} = 0,$$  \hspace{1cm} (3.48)

by arranging (3.48),

$$B_{qijq}(c_q + 2p_q) + p_i B_{qqjq} + p_j B_{qiqq} = 0$$  \hspace{1cm} (3.49)

from (3.46) and (3.47) we have

$$B_{qqjq} = 0$$  \hspace{1cm} (3.50)

so,

$$B_{qijq}(c_q + 2p_q) = B_{qijq} b_q = 0.$$  \hspace{1cm} (3.51)
Since $b_q$ is non-zero, we have

$$B_{qijq} = 0$$  \hspace{1cm} (3.52)

Now, we equal the indices $k, h$ and $q$ in (3.45) and using (3.52), we get

$$p_q(B_{lijq} + B_{qijl}) = 0$$  \hspace{1cm} (3.53)

If we get $p_q = 0$, since $b_q = c_q + 2p_q$ we have $c_q = b_q \neq 0$.

Now, we get equal the indices $l, h$ and $q$ in (3.45), then we have

$$c_qB_{qijk} + p_qB_{qijk} + p_lB_{qijkl} + p_jB_{qijk} + p_kB_{qijl} = 0$$  \hspace{1cm} (3.54)

from (3.45), (3.48), (3.52), we have

$$B_{qijk} = 0$$  \hspace{1cm} (3.55)

for all $i, j$ and $k$ since $c_q \neq 0$.

Now, we get equal the indices $l$ and $q$ in (3.45), then we have

$$c_qB_{hijk} + p_hB_{qijk} + p_lB_{hijkl} + p_jB_{hiqk} + p_kB_{hijq} = 0.$$  \hspace{1cm} (3.56)

so, we have

$$c_qB_{hijk} = 0,$$  \hspace{1cm} (3.57)

since $c_q \neq 0$. $B_{hijk} = 0$ for all $i, j, k$ and $h$.

If we get $p_q \neq 0$ in (3.53), then we have

$$B_{lijq} + B_{qijl} = 0,$$  \hspace{1cm} (3.58)

thus, it is seen from (3.58)

$$B_{lijq} = -B_{qijl},$$  \hspace{1cm} (3.59)

and from (3.47), we have

$$B_{lijq} = -B_{qjli}$$  \hspace{1cm} (3.60)

If we get equal the indices $h$ and $q$ in (3.46), then

$$B_{qijk} + B_{qjki} + B_{qkij} = 0.$$  \hspace{1cm} (3.61)

The equation (3.61) can be written as

$$B_{qjki} + B_{qjki} + B_{qjki} = 0.$$  \hspace{1cm} (3.62)
Consequently we have

$$B_{hijk} = 0.$$  

We will examine the covariant derivative of the conformal curvature tensor of RS.
REFERENCES


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