**ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY**

# A FOURIER PSEUDO-SPECTRAL METHOD FOR THE ROSENAU-KORTEWEG-DE VRIES -REGULARIZED LONG WAVE EQUATION

**M.Sc. THESIS**

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**Department of Mathematical Engineering**

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**JUNE 2019**

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**<u>ISTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ</u>**

# ROSENAU-KORTEWEG-DE VRIES - REGULARIZED UZUN DALGA DENKLEMLERİ İÇİN FOURIER SPEKTRAL YÖNTEMİ

**YÜKSEK LİSANS TEZİ**

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**HAZİRAN 2019**

Ayşe Buse Uzuner, a M.Sc. student of İTU Graduate School of Science Engineering and Technology student ID 509151201, successfully defended the thesis/dissertation entitled "A FOURIER PSEUDO-SPECTRAL METHOD FOR THE ROSENAU-KORTEWEG-DE VRIES-REGULARIZED LONG WAVE EQUATION", which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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Özyeğin University

**Date of Submission : 03 May 2019**
**Date of Defense : 11 June 2019**

***To my family,***

## FOREWORD

I would like to express my sincere thanks to my advisor Assoc. Prof. Dr. Gülçin Mihriye Muslu who has always supported me from the very beginning. She shared her knowledge and experience with me without any limit.

I would like to thank to Res. Assis. Göksu Oruç for helping me.

I would like to express my endless appreciate to my father Süleyman Uzuner, my mother Aytül Uzuner and my brother Hamza Uzuner who always supported me in every area. I couldn't do without you.

June 2019 Ayşe Buse UZUNER

(Mathematical Engineer)

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## ABBREVIATIONS

**KdV** : Korteweg-de Vries

**RLW** : Regularized Long Wave

**RK4** : Runge Kutta

**ODE** : Ordinary Differential Equation

**FFT** : Fourier Transform

**IFFT** : Inverse Fourier Transform

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**A FOURIER PSEUDO-SPECTRAL METHOD FOR THE ROSENAU-KORTEWEG-DE VRIES-REGULARIZED LONG WAVE EQUATION**

**SUMMARY**

The dynamics of shallow water waves is governed by several models. Some of these well-known models are Korteweg-de Vries (KdV) equation, Boussinesq equation, Kawahara equation, Peregrine equation, Benjamin–Bona–Mahony equation and several others. These equations are studied exhaustively in the literature. In this thesis, we concern with the Rosenau-Korteweg-de Vries-Regularized Long Wave equation (Rosenau-KdV-RLW) given by

$$u_t + c_1 u_x + c_2 u_{xxx} - c_3 u_{xxt} + c_4 u_{xxxxt} + c_5 u^n u_x = 0$$

where $c_1, c_2, c_5$ are real constants and $c_3, c_4$ are positive constants. Here $u(x,t)$ is the real-valued function where $x$ and $t$ are the spatial and temporal variables, respectively. In this thesis study, we propose a Fourier pseudo-spectral method for the Rosenau-KdV-RLW equation.

In this thesis study, we propose a Fourier pseudo-spectral method for the Rosenau-KdV-RLW equation. The topics covered in the thesis study are as follows:

In Chapter 1, we introduce the Rosenau-KdV-RLW equation and its special cases. We present some information about the literature and conserved quantities.

In Chapter 2, we propose a Fourier pseudo-spectral method for the Rosenau-KdV-RLW equation. In order to obtain the numerical solutions of the Rosenau-KdV-RLW equation, we apply the discrete Fourier transform. Then, an ordinary differential equation is derived in terms of Fourier coefficients. This differential equation is solved by using the 4th order Runge-Kutta (RK4) method. As the last step, numerical solution is obtained by using inverse Fourier transforms. In this thesis, the Rosenau-KdV-RLW equation and its special cases, the Rosenau-KdV equation, Rosenau-RLW equation and Rosenau equation, are investigated by using the Fourier spectral method. For each equation, we compare our numerical results with the exact solution if it exists and numerical solutions given in literature. The numerical results show that the Fourier pseudo-spectral method produces more accurate results. We also show that our scheme conserves the discrete mass and energy very well.

In Chapter 3, the results obtained in this thesis are presented.

**ROSENAU-KORTEWEG-DE VRIES-DÜZENLENMİŞ UZUN DALGA DENKLEMLERİ İÇİN FOURIER SÖZDE-SPEKTRAL YÖNTEMİ**

## ÖZET

Su dalgalarının dinamiği literatürde birçok denklemle ifade edilmiştir. İlk olarak, Korteweg ve De-Vries zayıf doğrusal olmayan sığ su dalgalarını modellemek amacıyla Korteweg-de Vries (KdV) denklemini

$$u_t + u_{xxx} - 6uu_x = 0$$

türetmişlerdir. Ancak yoğun ayrık sistemlerin dinamiğinin analizinde ve dalga-duvar etkileşimlerinin açıklanmasında KdV denklemi yetersiz kalmıştır. KdV denklemindeki bu eksikliğin giderilmesi için Rosenau tarafından

$$u_t + u_x + (u^2)_x + u_{xxxxt} = 0$$

denklemi türetilmiştir. Daha sonra, Zou $u_{xxx}$ akışkan olmayan (viscous) terimini Rosenau denklemine ekleyerek, doğrusal olmayan dalgaları daha da iyi modelleyen Rosenau-KdV denklemini

$$u_t + u_x + u_{xxx} + u_{xxxxt} + uu_x = 0$$

önermiştir. Peregrine ise uzun dalga denklemlerini modelleyen düzenlenmiş (regularized) uzun dalga (RLW) denklemini

$$u_t + u_x + \delta uu_x - \mu u_{xxt} = 0$$

modellemiştir. Burada $\delta$ ve $\mu$ reel sayılardır. Sığ su dalgalarını modelleyen bir diğer denklem, Rosenau-Korteweg-de Vries-Düzenlenmiş uzun dalga denklemleri (Rosenau-KdV-RLW) en genel haliyle

$$u_t + c_1u_x + c_2u_{xxx} - c_3u_{xxt} + c_4u_{xxxxt} + c_5u^nu_x = 0$$

şeklinde ifade edilmektedir. $c_1, c_2, c_5$ terimleri reel sayılar olup, $c_3, c_4$ terimleri pozitif sayılardır.

Bu tezin amacı, Rosenau-Korteweg-de Vries-Düzenlenmiş uzun dalga denklemi için bir Fourier spektral yöntem yardımıyla sayısal çözüm bulmak ve literatürde varolan sayısal

çözümlerden ve analitik çözümden yararlanarak sayısal çözümünün doğruluğunu test etmektir.

Tez çalışmasında işlenen konular aşağıda belirtildiği şekildedir:,

Bölüm 1'de Rosenau-Korteweg-de Vries-Düzenlenmiş uzun dalga denklemi (Rosenau-KdV-RLW) hakkında genel bilgiler verilmiştir. Denklemin özel halleri ve Rosenau-KdV-RLW denkleminin korunan büyüklükleri sunulmuş ve literatürde yapılan çalışmalar hakkında kısaca bilgi verilmiştir.

Bölüm 2' de literatürde varolan Rosenau-KdV-RLW denkleminin sayısal çözümünün elde edilmesi amacıyla kullanılan çalışmalardan kısaca bahsedilmiştir. Rosenau-KdV-RLW denkleminin sayısal çözümünün incelenmesi için bu tezde Fourier sözde-spektral yöntemi önerilmiştir. Bu amaçla, ilk olarak denkleme ayrık Fourier dönüşümü uygulanarak bir adi diferansiyel denklem türetilmiştir. Oluşturulan bu diferansiyel denklem 4. mertebeden Runge-Kutta (RK4) yöntemi kullanılarak çözülmüştür. Son olarak ters Fourier dönüşümleri kullanılarak sayısal çözüm inşa edilmiştir.

Bölüm 1 de en genel haliyle verilmiş olan Rosenau-KdV-RLW denklemi, bazı katsayılarının sıfır olması halinde farklı denklemlere indirgenmektedir. Bu tezde sunulan Fourier spektral yöntemi ile en genel Rosenau-KdV-RLW denklemi ve bu denklemin özel halleri olan Rosenau-KdV denklemi, Rosenau-RLW denklemi ve Rosenau denklemi incelenmiştir.

En genel durum olan Rosenau-KdV-RLW denklemi için, Fourier spektral yöntemi kullanılarak elde edilen sayısal sonuçların zamanda 4. mertebeden yakınsaklığa ve uzayda da eksponansiyel yakınsaklığa sahip olduğu gösterilmiştir. Zamandaki ayrıklaştırma için RK4 yöntemi kullanıldığından sayısal olarak elde edilen 4. mertebe yakınsaklık, uzayda da Fourier dönüşümü kullandığımız için eksponansiyel yakınsama beklentimiz ile uyumludur. Literatürde varolan araştırmalarda bildiğimiz kadarıyla, en iyi neticeler kompakt fark şeması yöntemi ile elde edilmiştir. Kompakt fark şeması yardımıyla elde edilen sayısal çözümler ile Fourier sözde-spektral şeması elde ettiğimiz çözümler karşılaştırılmıştır. Gerçek çözüm ile sayısal çözüm arasındaki hata hesaplanmış ve hatanın $10^{-12}$ mertebesinde olduğu gözlenmiştir. Bu da bize önerilen şemanın çözümleri yakalamakta oldukça başarılı olduğunu göstermektedir. Ayrıca, şemanın enerji ve kütleyi oldukça iyi koruduğu da yine bu bölümde gösterilmiştir.

Rosenau-KdV-RLW denkleminin özel durumları olan Rosenau-KdV ve Rosenau-RLW denklemleri için de literatürde varolan çalışmalardan faydalanılarak, gerçek çözüm ile Fourier sözde-spektral yöntemi kullanılarak elde sayısal çözüm arasındaki hata hesaplanmışve bu durumlar da da şemanın analitik çözüme hızla yakınsadığı gözlenmiştir. Ayrıca, enerji ve kütlenin korunduğu da gene bu bölümde gösterilmiştir. Son olarak, bir diğer özel hal olan Rosenau denklemi incelenmiştir. Rosenau denkleminin çözümlerinin yerel varlığı ve tekliği ile ilgili çalışmalar olmasına rağmen, analitik çözüm elde edilememiştir. Bu sebeple bu kısımda hata analizi gerçekleştirilememiştir. Şemamızın doğruluğunu test etmek için korunan büyüklükler incelenmiştir.

Bölüm 3’ te bu tezde elde edilen sonuçlar sunulmuştur. Bildiğimiz kadarıyla literatürde iyi bilinen, su dalga denklemleri olan KdV, Boussinesq, Kawahara, Peregrine, Benjamin–Bona–Mahony denklemleri ile ilgili birçok sayısal çalışma olmasına ragmen, su dalgalarını modelleyen bir diğer denklem olan Rosenau-KdV-RLW hakkında çok az sayısal çalışmaya rastlanmıştır.  Bu tez kapsamında elde edilen sonuçların, litaratürde varolan sonuçlara göre daha iyi olduğu gözlenmiştir.

# 1. THE ROSENAU KORTEWEG-DE VRIES-REGULARIZED LONG WAVE EQUATION

## 1.1 Introduction

In this chapter, we introduce the Rosenau Korteweg-de Vries-Regularized Long Wave equation and its special cases. In Section 1.2, we present some information about the literature and conserved quantities.

## 1.2 The Rosenau Korteweg-de Vries-Regularized Long Wave Equation

The Korteweg-de Vries (KdV) equation is the partial differential equation

$$u_t + u_{xxx} - 6uu_x = 0 \qquad (1.1)$$

derived by Korteweg-de Vries [1] to model weakly nonlinear shallow water waves. In the analysis of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described by the KdV equation. In order to overcome the deficiency of the KdV equation, Rosenau proposed the equation [2]

$$u_t + u_x + (u^2)_x + u_{xxxxt} = 0\,. \qquad (1.2)$$

For further consideration of the nonlinear wave, Zou [3] added the viscous term $u_{xxx}$ to the Rosenau equation and proposed the Rosenau–KdV equation

$$u_t + u_x + u_{xxx} + u_{xxxxt} + uu_x = 0\,. \qquad (1.3)$$

The regularized long wave (RLW) equation is

$$u_t + u_x + \delta uu_x - \mu u_{xxt} = 0 \qquad (1.4)$$

derived by Peregrine[4] to model the development of an undular bore. Than, Benjamin proposed the RLW equation as an alternative preferable to the more classical Korteweg–de Vries (KdV) equation, for modeling a larger class of physical phenomena. The incorporation of eqs. (1)–(4) yield the Rosenau-KdV-RLW equation

$$u_t + c_1 u_x + c_2 u_{xxx} - c_3 u_{xxt} + c_4 u_{xxxxt} + c_5 u^n u_x = 0 \qquad (1.5)$$

where $c_1, c_2, c_5$ are real constants and $c_3, c_4$ are positive constants [5]. Rosenau-KdV-RLW equation is used for describing the dynamics of shallow water waves that appear along lake shores.

The conserved quantities corresponding to mass and energy are respectively given as follows:

$$M(t) = \int_{\mathbb{R}} u(x,t)dx \tag{1.6}$$

$$E(t) = \int_{\mathbb{R}} u^2(x,t)dx + c_3 \int_{\mathbb{R}} u_x^2(x,t)dx + c_4 \int_{\mathbb{R}} u_{xx}^2(x,t)dx\,. \tag{1.7}$$

Solitary waves and shock waves of the Rosenau equation are obtained in [6] by using the ansatz method and the semi-inverse variational principle. By using the multipliers approach in Lie symmetry analysis, some additional conservation laws different from the mass and energy are derived in [7].

# 2. THE PSEUDO-SPECTRAL METHOD FOR THE ROSENAU-KDV-RLW EQUATION

## 2.1 Introduction

In this chapter, we propose a Fourier pseudo-spectral method for the solving Rosenau-KdV-RLW equation and its particular cases. In section 2.2, we introduce the numerical scheme. In section 2.3, we present some numerical experiments to test the accuracy of the proposed scheme.

## 2.2 The Numerical Scheme

We study the Rosenau-KdV-RLW equation with a Fourier pseudo-spectral method for the space variable, a fourth-order Runge Kutta scheme for time. The spatial period $x \in [x_1, x_r]$ is scaled to $X \in [0, 2\pi]$ by using the $X = 2\pi(x - x_r)/(x_r - x_1)$ transformation. In this case the Rosenau-KdV-RLW equation (1.5) becomes

$$u_t + c_1(\frac{2\pi}{x_r - x_1})u_X + c_2(\frac{2\pi}{x_r - x_1})^3 u_{XXX} - c_3(\frac{2\pi}{x_r - x_1})^2 u_{XXt} + c_4(\frac{2\pi}{x_r - x_1})^4 u_{XXXXt}$$

$$+(\frac{c_5}{n+1})(\frac{2\pi}{x_r - x_1})(u^{n+1})_X = 0 . \quad (2.1)$$

The interval $[0, 2\pi]$ is divided into N equal subintervals with $\Delta x = 2\pi / N$ .The spatial grid points are given by $X_j = 2\pi j / N \quad j = 0,1,2....,N$ . In this thesis, the solution $u(X_j, t)$ is denoted by $U_j(t)$.

The discrete Fourier transform of the sequence $\{U_j\}$

$$U_k = \mathbb{F}_k[U_j] = \frac{1}{N}\sum_{j=0}^{N-1} U_j e^{-ikX_j}, -N/2 \le k \le N/2 - 1 \quad (2.2)$$

gives the Fourier coefficients. As the same way, the inversion formula is given,

$$U_j = \mathbb{F}_j^{-1}[U_k] = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} U_k e^{ikX_j}, \; j = 0,1,2,....,N-1 . \quad (2.3)$$

For the calculating, the discrete Fourier transform and its inverse for a function $f(x)$, we use MATLAB functions “fft” and “ifft”. Using the discrete Fourier transform to (2.1) we have

$$(U_k)_t + c_1(\frac{2\pi}{x_r - x_1}) ik(U_k) + c_2(\frac{2\pi}{x_r - x_1})^3 (ik)^3 (U_k)$$
$$-c_3(\frac{2\pi}{x_r - x_1})^2 (ik)^2 (U_k)_t + c_4(\frac{2\pi}{x_r - x_1})^4 (ik)^4 (U_k)_t + (\frac{c_5}{n+1})(\frac{2\pi}{x_r - x_1})(ik)(U_j^{n+1})_k = 0 \quad . \tag{2.4}$$

This equation can be written in the following form,

$$(U_k)_t = S[-c_1 U_k + c_2(\frac{2\pi k}{x_r - x_1})^2 U_k - \frac{c_5}{n+1}(U_j^{n+1})_k], \tag{2.5}$$

where

$$S = \frac{2\pi ik}{x_r - x_1}[\frac{1}{1 + c_3(\frac{2\pi k}{x_r - x_1})^2 + c_4(\frac{2\pi k}{x_r - x_1})^4}] \ . \tag{2.6}$$

We use the fourth order Runge-Kutta (RK4) method for solving ODE system (2.4) in time. The time interval $[0,T]$ is divided $M$ equal subintervals with $\Delta t = \frac{T}{M}$. Here temporal grid points are $t_m = \frac{mT}{M}$ . The value of the Fourier components at $t_m$ is then denoted by $U_k^m$. RK4 method for the solution of the ODE (2.5) is given

$$U_k^{m+1} = U_k^m + \frac{\Delta t}{6}(l_{1,k}^m + 2l_{2,k}^m + 2l_{3,k}^m + l_{4,k}^m) \ , \tag{2.7}$$

where

$$
\begin{aligned}
l_{1,k}^{m} &= S[-c_1 U_k^m + c_2 (\frac{2\pi k}{x_r - x_1})^2 U_k^m - \frac{c_5}{n+1} (U_j^{n+1})_k^m] \\
l_{2,k}^{m} &= S\left\{ \frac{c_5}{n+1} \mathbb{F}_k [U_j^m + \mathbb{F}_j^{-1} (\frac{l_{1,k}^m}{2})^{n+1}] + (-c_1 U_k^m + c_2 (\frac{2\pi k}{x_r - x_1})^2 U_k^m + \frac{l_{1,k}^m}{2} \right\} \\
l_{3,k}^{m} &= S\left\{ \frac{c_5}{n+1} \mathbb{F}_k [U_j^m + \mathbb{F}_j^{-1} (\frac{l_{2,k}^m}{2})^{n+1}] + (-c_1 U_k^m + c_2 (\frac{2\pi k}{x_r - x_1})^2 U_k^m + \frac{l_{2,k}^m}{2} \right\} \\
l_{4,k}^{m} &= S\left\{ \frac{c_5}{n+1} \mathbb{F}_k [U_j^m + \mathbb{F}_j^{-1} (l_{3,k}^m)^{n+1}] + (-c_1 \widetilde{U_k^m + c_2} (\frac{2\pi k}{x_r - x_1})^2 \widetilde{U_k^m + l_{3,k}^m} \right\}
\end{aligned}
\tag{2.8}
$$

Approximate solution is obtained by using inverse Fourier transformation.

### 2.3 Numerical Experiments

In this subsection we want to show that our proposed Fourier pseudo-spectral scheme is more accurate and the scheme illustrates the fourth-order convergence in time and spectral accuracy in space. $L_\infty$ - *error* norm is defined as

$$L_\infty \text{-} error = \max_i |u_i - U_i| \ , \tag{2.9}$$

where $u_i$ defines the exact solution at $u(X_i, t)$. The convergence order in spatial direction is defined as,

$$p \approx \frac{\ln(\frac{L_\infty \text{-} error_{N_2}}{L_\infty \text{-} error_{N_1}})}{\ln(\frac{N_1}{N_2})} \tag{2.10}$$

The convergence order in temporal direction is defined as,

$$p \approx \frac{\ln(\frac{L_\infty \text{-} error_{M_2}}{L_\infty \text{-} error_{M_1}})}{\ln(\frac{M_1}{M_2})} \tag{2.11}$$

As mentioned in previous sections, the fast Fourier transform (FFT) used in MATLAB (fft & ifft) to calculate Fourier transform and the inverse Fourier transform. We compare our numerical results with the results in the literature.

#### **2.**3.1 Rosenau-KdV-RLW **equation**

We study the approximate solution of the Rosenau-KdV-RLW equation (1.5) for $c_1 = c_2 = c_3 = c_4 = c_5 = n = 1$.The exact solitary wave solution is given in [5],

$$u(x,t)=\frac{5}{456}(-25+13\sqrt{457})\operatorname{sech}^4[\frac{1}{\sqrt{288}}\sqrt{-13+\sqrt{457}}(x-(\frac{241+13\sqrt{457}}{266})t)]. \quad (2.12)$$

This solution shows a solitary wave initially at $x_0=0$ moving to the right with the speed $c\approx 1.95$. The initial conditions (2.12) is given,

$$u(x,0)=\frac{5}{456}(-25+13\sqrt{457})\operatorname{sech}^4[(\frac{1}{\sqrt{288}}\sqrt{-13+\sqrt{457}})x] . \quad (2.13)$$

The problem is solved on the space interval $-100\le x\le 100$ for times up to $T=20$.

To test the Fourier pseudo-spectral method, we demonstrate some numerical experiments for various values of M and N. M is the number of grid points in time, N is the number of grid points in space.

In the first numerical experiment, we get N as a number of the grid points in space fixed and we change M which is the number of the grid points in time(M). We take N=8000 to ensure that the error due to the spatial discretization is negligible. $L_\infty$-*errors* for the time $T=20$ are listed in Table 2.1.

**Table 2.1** : The convergence rates in time with N=8000.

| M | $L_\infty$-*error* | Order |
|---|---|---|
| 100 | 1.510925 x $10^{-4}$ | - |
| 500 | 2.096000 x $10^{-7}$ | 4.08866 |
| 1000 | 1.289006 x $10^{-8}$ | 4.02330 |
| 4000 | 4.971778 x $10^{-11}$ | 4.00914 |
| 6000 | 9.803269 x $10^{-12}$ | 4.00440 |

The computed orders show that the Fourier pseudo-spectral method has the fourth order convergence in time.

In the second numerical experiment, we get the number of the grid points in time M fixed and we change the number of the grid points in space N. We take M=8000 to minimize temporal error. $L_\infty$-*errors* for the time $T=20$ are listed in Table 2.2.

**Table 2.2** : The convergence rates in space with M=8000.

| N | $L_\infty$-error | Order |
|---|---|---|
| 10 | 1.11553 | - |
| 50 | 0.23006 | 0.980928 |
| 100 | 0.00110 | 7.705789 |
| 200 | 2.40759 x $10^{-9}$ | 18.804057 |

These results show that the numerical solutions (2.12) obtained using the Fourier pseudo-spectral scheme converges rapidly to the accurate solution in space, which indicates exponential converges.

We show the difference between the exact solution and the numerical solution of the Rosenau-KdV-RLW for the final time $T = 20$ in Figure 2.1. We take space interval $-100 \le x \le 100$.

$$Error = |u_{num} - u_{exact}| \tag{2.14}$$

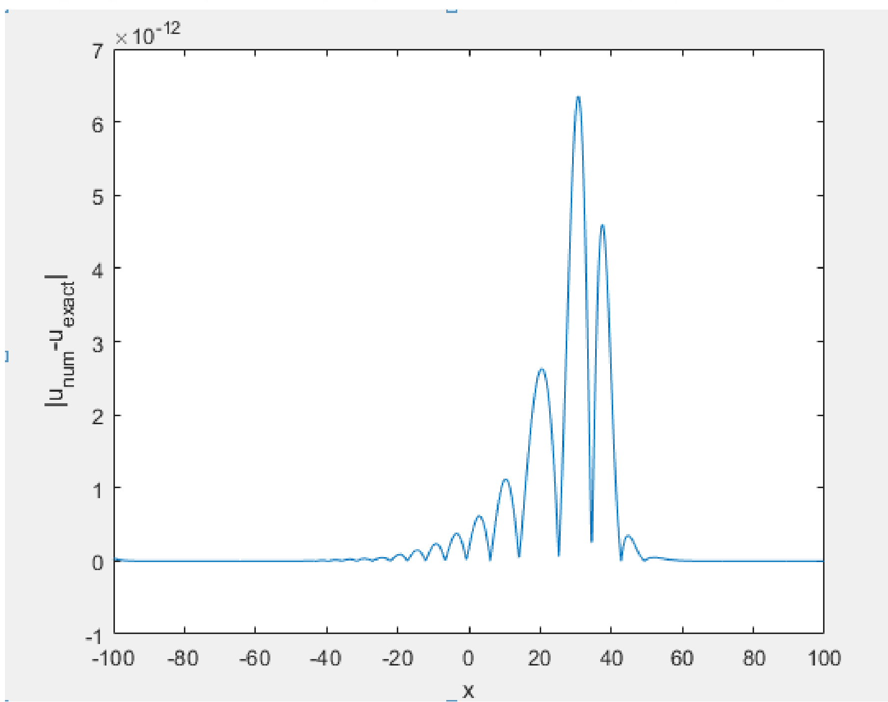


**Figure 2.1 :** The difference between exact and numerical solution of the Rosenau-KdV-RLW.

The $Error$ is about $10^{-12}$. The numerical solution is in good agreement with the exact solution.

We illustrate evolution of solitary waves of Rosenau KdV-RLW equation in Figure 2.2.

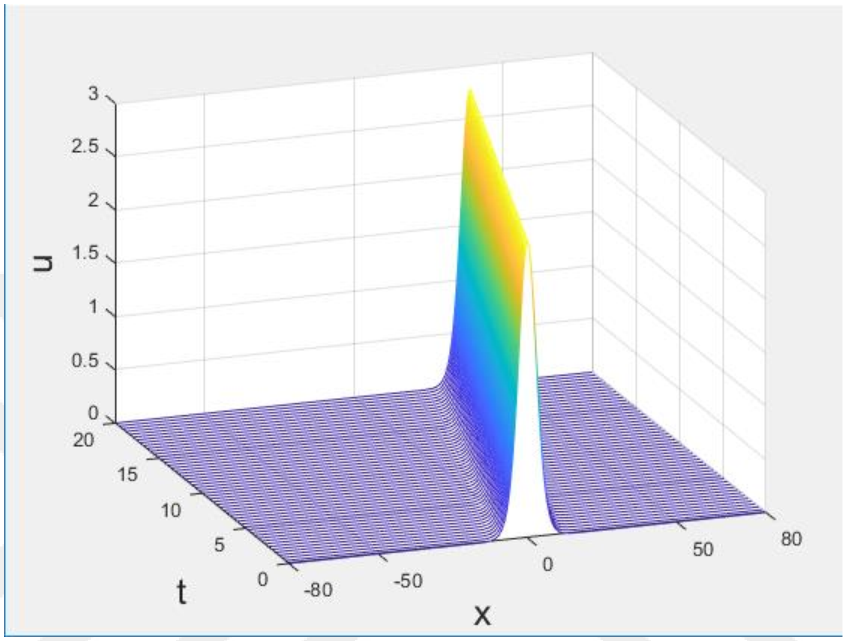


**Figure 2.2:** The Solitary wave profile of the Rosenau KdV-RLW equation.

As a numerical check of the proposed Fourier pseudo-spectral scheme, we present the evaluation of the change in conserved quantities mass and energy. In Figure 2.3, we show the mass' change evolution which is conserved.

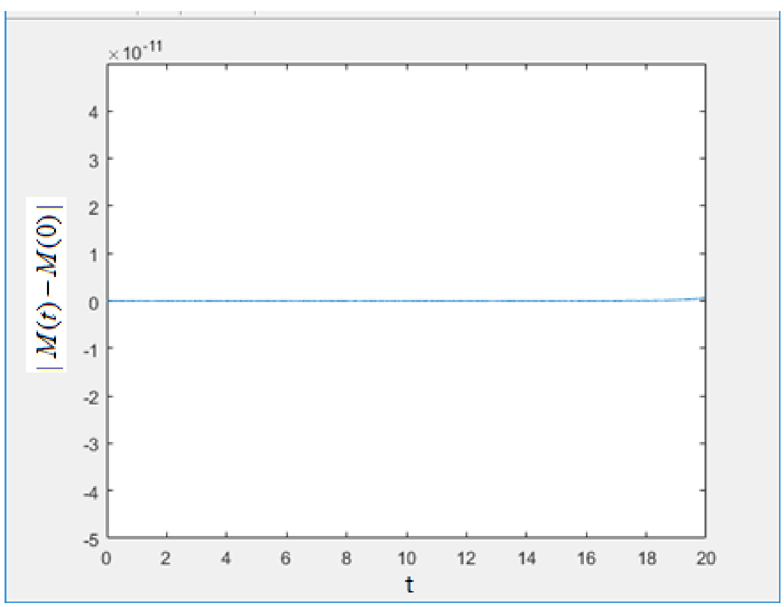


**Figure 2.3 :** Evolution of the change in mass $|M(t)-M(0)|$.

As we see our scheme conserves mass with $10^{-11}$ error. In Figure 2.4, we show the energy' change evolution which is conserved.

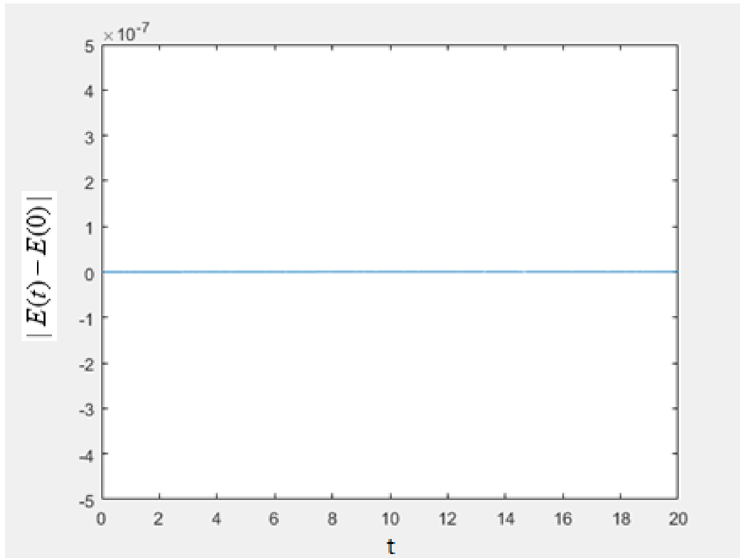


**Figure 2.4 :** Evolution of the change in energy $|E(t)-E(0)|$.

As we see our scheme conserves energy with $10^{-7}$ error.

Wongsaijai and Poochinapan [8] proposed a three-level weighted average implicit finite difference scheme for solving Rosenau-KdV-RLW equation. After them, Pan offered a new C–N pseudo-compact conservative numerical scheme for solving the generalized Rosenau-KdV-RLW equation [9]. But these schemes are second-order convergent in time and space variables. Recently, Ghiloufi et. Al [5] have proposed fourth order three level linearized compact difference scheme for solving Rosenau-KdV-RLW equation in 2018. In Table 2.3, we present the comparison of the solutions obtained by the Fourier pseudo-spectral method and the linearized compact difference method. As far as we know, the numerical results obtained by the the linearized compact difference scheme are the best results in literature. Since they have used to fourth order method in time. For the numerical experiment, spatial domain is chosen as the same interval $-100 \le x \le 100$ given in [5]. We choose the number of the grid points in space N=1000.

**Table 2.3 :** Comparison of $L_\infty$-*error* for N=1000.

| t | M | $L_\infty$-*error* LCD in [5] | $L_\infty$-*error* FPS (present study) |
|---|---|---|---|
| 0.8 | 80 | 3.887581 x $10^{-5}$ | 5.09772 x $10^{-11}$ |
| 1 | 100 | 4.480250 x $10^{-5}$ | 6.35609 x $10^{-11}$ |
| 4 | 400 | 1.482582 x $10^{-4}$ | 2.44781 x $10^{-10}$ |
| 8 | 800 | 2.772720 x $10^{-4}$ | 4.04208 x $10^{-10}$ |

As it is seen from the Table 2.3, our proposed scheme produces more accurate results.

### 2.3.2 Rosenau-KdV equation

In this subsection, we consider Rosenau-KdV equation. With $c_1 = c_2 = c_4 = c_5 = n = 1$ and $c_3 = 0$ coefficients, Rosenau-KdV-RLW equation reduces to Rosenau-KdV equation [5]. The exact solitary wave solution has the following form [5],

$$u(x,t) = (-\frac{35}{24} + \frac{35}{312}\sqrt{313})\,\text{sech}^4[\frac{1}{24}\sqrt{-26+2\sqrt{313}}\,(x-(\frac{1}{2}+\frac{1}{26}\sqrt{313})t)]\,. \qquad (2.15)$$

This solution represents a solitary wave initially at $x_0 = 0$ moving to the right with the speed $c \approx 0.68$. The initial condition corresponding to the solution (2.15) is given

$$u(x,0) = (-\frac{35}{24} + \frac{35}{312}\sqrt{313})\,\text{sech}^4[(\frac{1}{24}\sqrt{-26+2\sqrt{313}})x]\,. \qquad (2.16)$$

The problem is solved on the space interval $-40 \le x \le 60$ for times up to $T = 30$. Similar to Rosenau-KdW-RLW equation, we compare the Fourier pseudo-spectral solution and the linearized compact difference schemes solution [5] in Table 2.4. For the numerical experiment, spatial domain is chosen the same interval $-40 \le x \le 60$ given in [5]. We show the comparison of $L_\infty$-*error* by taking the number of the grid points in space N=1000.

**Table 2.4 :** Comparison of $L_\infty$-*error* for N=1000, M=300.

| t | $L_\infty$-*error* **LCD in [5]** | $L_\infty$-*error* **FPS (present study)** |
|---|---|---|
| 10 | $1.557832 \times 10^{-7}$ | $9.354536 \times 10^{-9}$ |
| 20 | $1.362388 \times 10^{-6}$ | $7.388451 \times 10^{-8}$ |
| 30 | $3.909021 \times 10^{-5}$ | $2.955806 \times 10^{-5}$ |

As it is seen from the Table 2.4, our proposed scheme produces more accurate results.

Figures 2.5 represents the evolution of the wave using N=1000, M=300. We see that the wave moves from left to right direction without changing its shape.

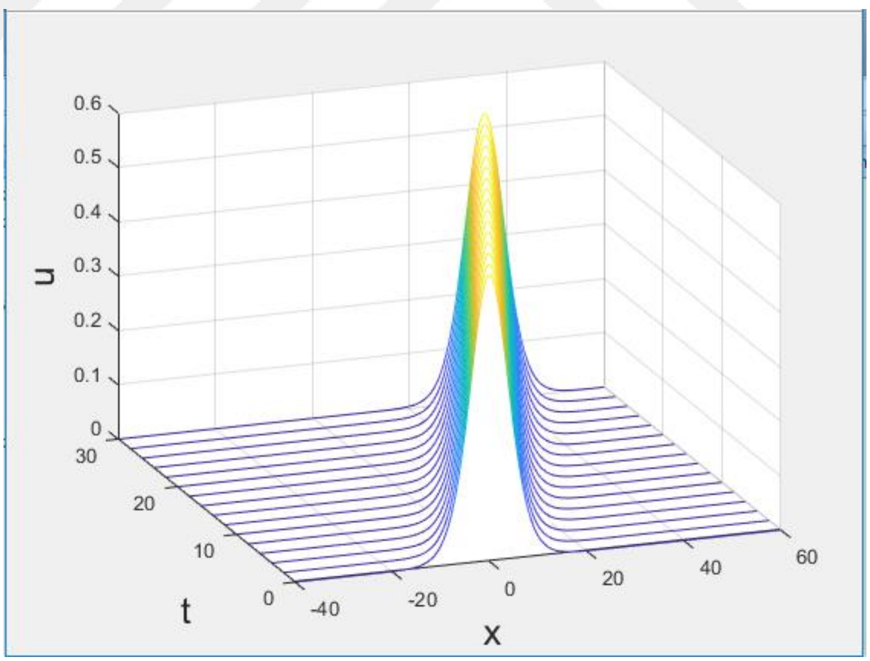


**Figure 2.5 :** Wave profile with $-40 \le x \le 60$ and $T = 30$.

We present the evolution of the change in the conserved quantity mass in Figure 2.5.

In Figure 2.6, we show the mass' change evolution which is conserved.

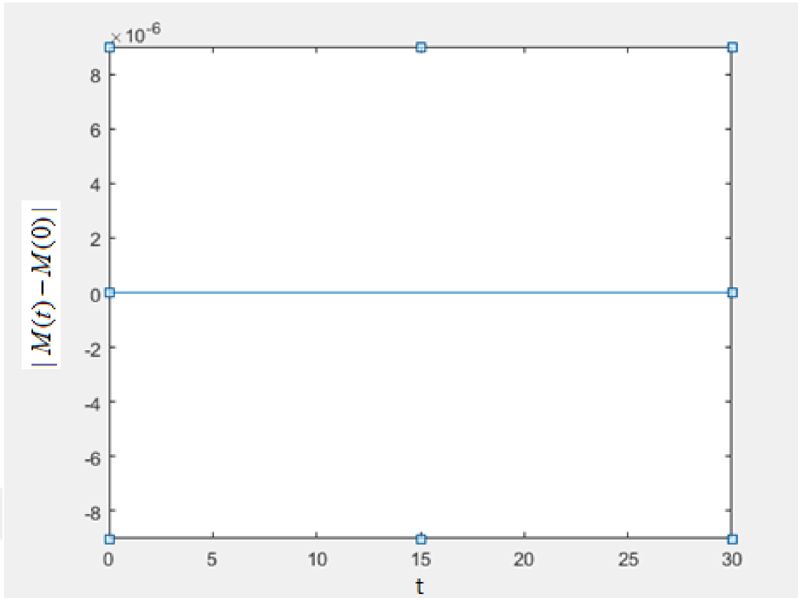


**Figure 2.6 :** Evolution of the change in mass $|M(t)-M(0)|$.

As we see our scheme conserves mass with $10^{-6}$ error. To eliminate the error due to the boundary effect we increase the spatial domain from $-40 \le x \le 60$ to $-100 \le x \le 100$. In Figure 2.7, we show the energy' change evolution which is conserved.

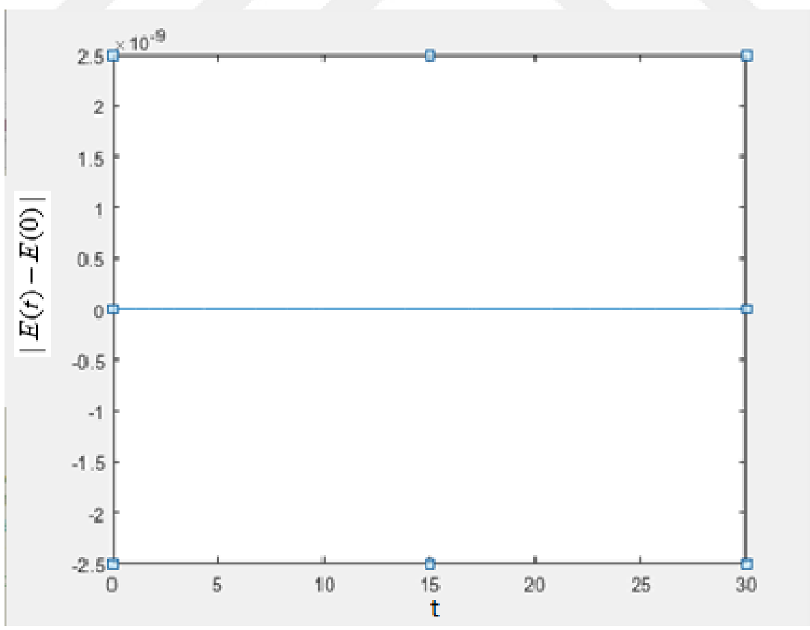


**Figure 2.7 :** Evolution of the change in energy .$|E(t)-E(0)|$.

As we see our scheme conserves energy with $10^{-9}$ error. We show the difference between exact solution and numerical solution of the Rosenau-KdV for final $T=30$ in Figure 2.8. We take space interval $-100 \le x \le 100$.

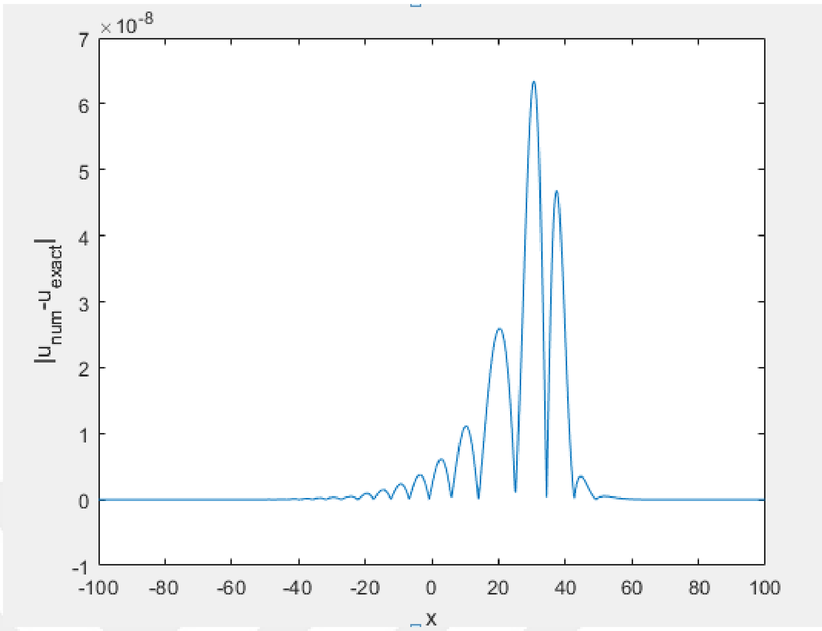


**Figure 2.8 :** The absolute difference between exact and numerical solution of the Rosenau-KdV.

### 2.3.3 Rosenau-RLW equation

In this subsection, we will consider Rosenau-RLW equation. With $c_1 = c_3 = c_4 = c_5 = n = 1$ and $c_2 = 0$ coefficients, Rosenau-KdV-RLW equation reduces to Rosenau-RLW equation. The exact solitary wave solution has the following form [5],

$$u(x,0) = \frac{15}{19}\operatorname{sech}^4[\frac{\sqrt{13}}{26}(x - \frac{169}{133}t)]. \tag{2.17}$$

This solution represents a solitary wave initially at $x_0 = 0$ moving to the right with the speed $c \approx 0.17$. The initial condition corresponding to solution (2.17) is given

$$u(x,0) = \frac{15}{19}\operatorname{sech}^4[\frac{\sqrt{13}}{26}x]. \tag{2.18}$$

The problem is solved on the space interval $-60 \le x \le 60$ for times up to $T = 20$.

Similar to Rosenau-KdW-RLW equation, we compare the Fourier pseudo-spectral solution and the linearized compact difference schemes solution [5] in Table 2.5. For the numerical experiment, spatial domain is chosen the same interval $-60 \le x \le 60$ given in [5]. We show the comparison of $L_\infty$*-errors* by taking the number of the grid points in space N=600.

**Table 2.5 :** Comparison of $L_\infty$*-errors* for N=600.

| t | M | $L_\infty$-error **LCD** | $L_\infty$-error **FPS** |
|---|---|---|---|
| 2 | 200 | 2.0929 x $10^{-5}$ | 2.4389 x $10^{-12}$ |
| 8 | 800 | 7.1461 x $10^{-5}$ | 1.2480 x $10^{-11}$ |
| 10 | 1000 | 7.7335x $10^{-5}$ | 5.1100x $10^{-11}$ |
| 20 | 2000 | 1.2323 x $10^{-4}$ | 5.8795 x $10^{-8}$ |

As it is seen from the Table 2.5, our proposed scheme produces more accurate results.

Figures 2.9 represents the evolution of the wave using N=600, M=2000. We see that the wave moves from left to right direction without changing its shape.

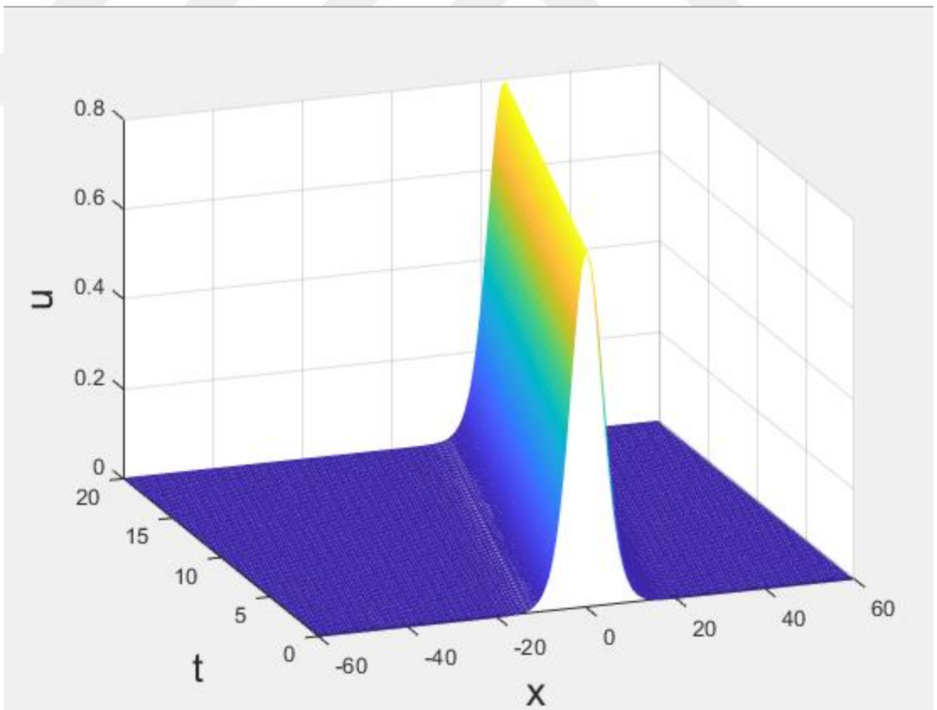


**Figure 2.9 :** Wave profile with $-60 \le x \le 60$ and $T = 20$.

We present the evolution of the change in the conserved quantity mass in Figure 2.9.In Figure 2.10, we show the mass' change evolution which is conserved.

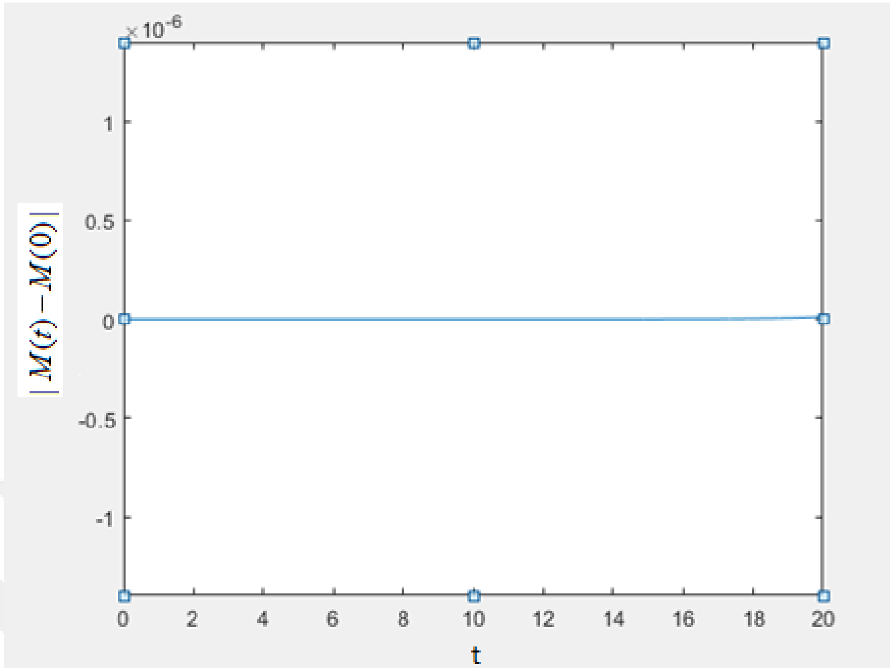


**Figure 2.10 :** Evolution of the change in mass $|M(t)-M(0)|$.

As we see our scheme conserves mass with $10^{-6}$ error.

In Figure 2.11, we show the energy' change evolution which is conserved.

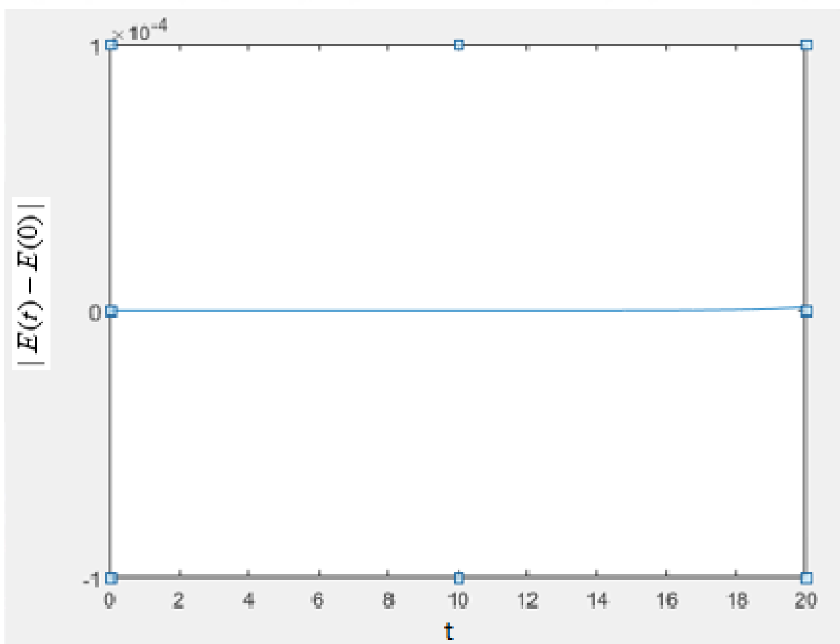


**Figure 2.11 :** Evolution of the change in energy $|E(t)-E(0)|$.

As we see our scheme conserves energy with $10^{-4}$ error. We show the difference between exact solution and numerical solution of the Rosenau- RLW for the final time $T=20$ equation in Figure 2.12. We take space interval $-60 \le x \le 60$.

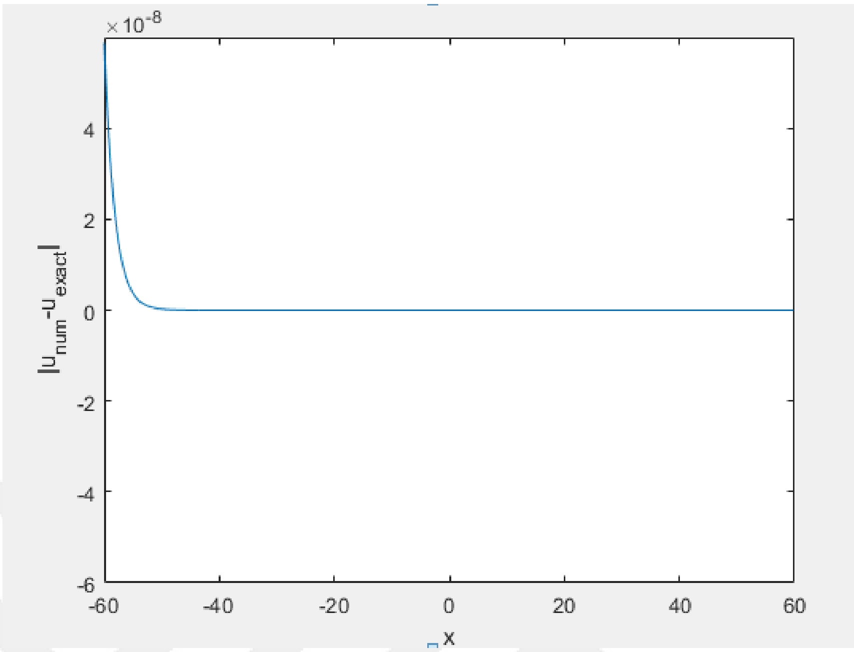


**Figure 2.12 :** The absolute difference between exact and numerical solution of the Rosenau-RLW.

The *Error* is about $10^{-8}$. This numerical experiments illustrated that our method is more accurate.

### 2.3.4 Rosenau equation

With $c_1 = c_4 = c_5 = n = 1$ and $c_2 = c_3 = 0$ coefficients, Rosenau-KdV-RLW equation reduces to Rosenau [10].We consider the initial condition as

$$u(x,0) = x^4(1-x)^4 \ . \tag{2.19}$$

The problem is solved on the space interval $0 \le x \le 1$.Figure 2.13 represents time evolution of the solution N=100, M=100. We see that the wave moves from left to right direction without changing its shape.

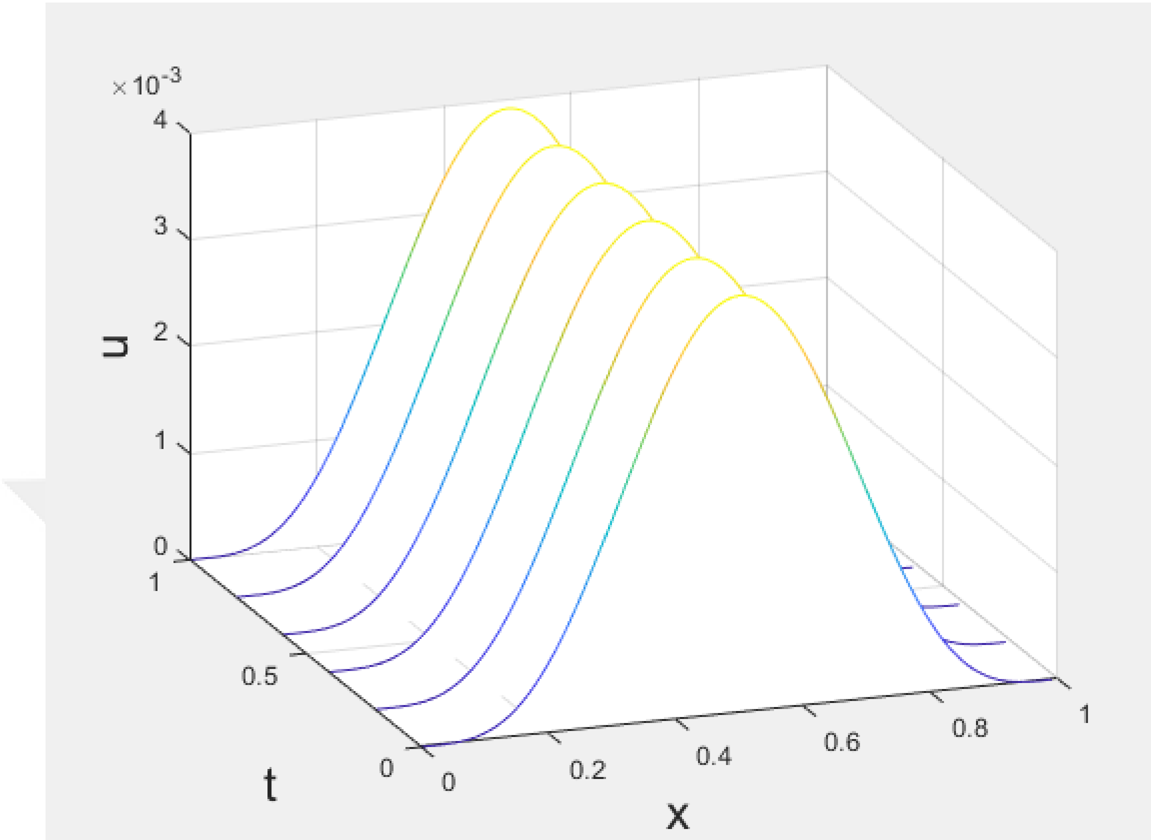


**Figure 2.13 :** Time evolution of the solution with $0 \leq x \leq 1$ and $T = 1$.

The existence and uniqueness of the solution for the Rosenau equation were proved by Park (1992). However, it is difficult to find the analytical solution. Besides the theoretical analysis, much work has been done on the numerical methods for the Rosenau equation, as seen in Manickam et al. (1998) [11], Chung and Pani (2001), Omrani et al. (2008) [12]. That' s why we cannot present error.

We present the evolution of the change in the conserved quantity mass and energy with using Fourier pseudo spectral scheme.

In Figure 2.14, we show the mass' change evolution which is conserved and in Figure 2.15, we show the energy' change evolution which is conserved.

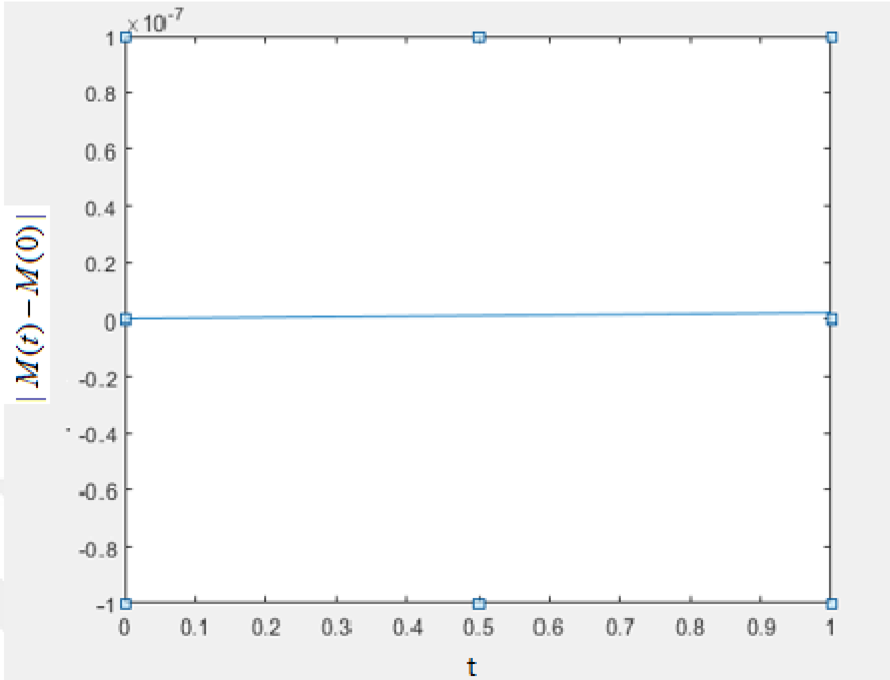


**Figure 2.14 :** Evolution of the change in mass $|M(t)-M(0)|$.

As we see our scheme conserves mass with $10^{-7}$ error.

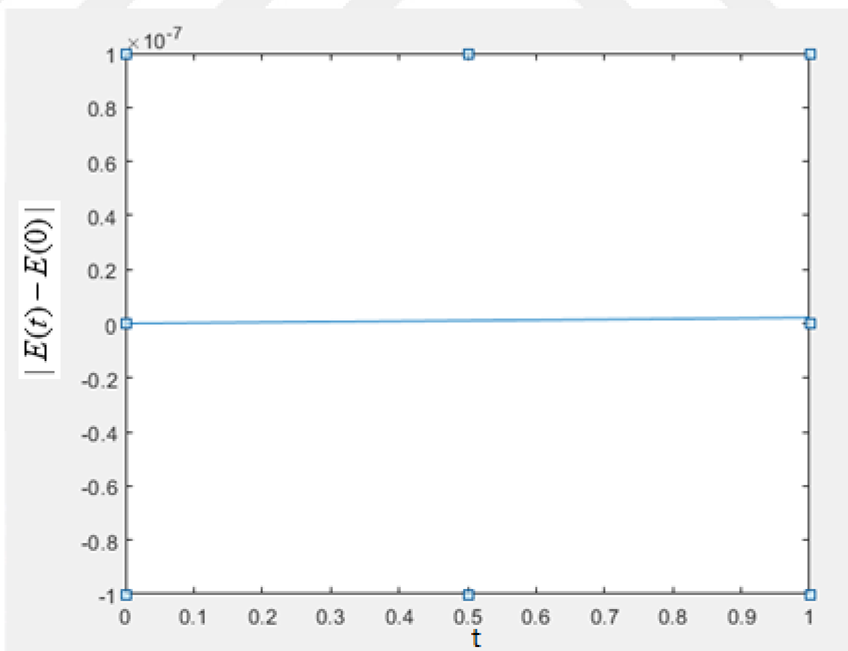


**Figure 2.15 :** Evolution of the change in energy $|E(t)-E(0)|$.

As we see our scheme conserves energy with $10^{-7}$ error.

## 3. CONCLUSIONS

As a conslusion, we proposed a Fourier pseudo-spectral method for the Rosenau-KdV-RLW equation and its particular cases; the Rosenau-KdV equation, the Rosenau-RLW equation and the Rosenau equation. For each equation, we compare our numerical results with the exact solution if it exists and numerical solutions given in literature. The numerical results show that the Fourier pseudo-spectral method produces more accurate results.  We also show that our scheme conserves the mass and energy very well.

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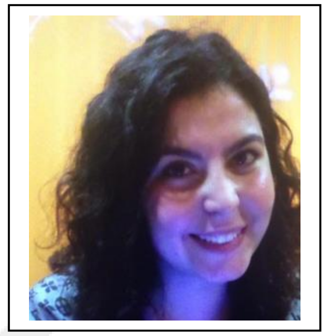
**CURRICULUM VITAE**



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