ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

EXACT SOLITON SOLUTIONS OF CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH THIRD ORDER DISPERSION

M.Sc. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Programme

SEPTEMBER 2019

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ÜÇÜNCÜ MERTEBEDEN DİSPERSİYON İÇEREN KÜBİK NONLİNEER SCHRÖDINGER DENKLEMİNİN SOLİTON TİPİ ÇÖZÜMLERİ

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Canan Simge TOKATLI, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509161203 successfully defended the thesis entitled "EXACT SOLITON SOLUTIONS OF CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH THIRD ORDER DISPERSION", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To all people who believe on my own way,

FOREWORD

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(Mathematical Engineer)

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ABBREVIATIONS

NLS	**:** Nonlinear Schrödinger
CNLS	**:** Cubic Nonlinear Schrödinger
3OD	**:** Third Order Dispersion
ODE	**:** Ordinary Differantial Equation
$\mathscr{P}\mathscr{T}$	**:** Parity - Time
SR	**:** Spectral Renormalization
Eq.	**:** Equation

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EXACT SOLITON SOLUTIONS OF CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH THIRD ORDER DISPERSION

SUMMARY

Positive sciences are related to each other in several ways. It can be seen that the relation between mathematics and the other scientific areas is mostly about differential equations. In recent years, there has been considerable interest in nonlinear wave problems. Nonlinear optical wave equations guide to understand nature in various disciplines such as quantum mechanics, nonlinear optics and biology; moreover, these equations are useful for improvement of innovative methods. Solitons are localized nonlinear waves which are used for understanding of complex nonlinear systems. The shape of wave is kept stable after collision. Solitons are related to several scientific fields such as quantum mechanics, nonlinear optics, plasma physics etc. Since soliton theory is an interdisciplinary topic, there has been noticeable studies about optical solitons in the last decades. Nonlinear Schrödinger (NLS) equation is used for modelling nonlinear propagations of optical pulses in one picosecond. In recent years, there has been considerable interest in the solutions of NLS equation. The analytical and numerical solutions of the cubic NLS equation were investigated in literature. Third-order dispersion term has considerable effects on inter-continent data transmission. Therefore, the investigations of these effects are significant for improvement of data transmission. External potentials affect stabilities and shapes of optical pulses. The potentials (lattices) which have parity-time symmetry ($\mathscr{PT}$-symmetry) are used in quantum mechanics and nonlinear optics problems frequently. In literature, there are various investigations about stability of NLS equations with $\mathscr{PT}$-symmetry. In this study, soliton solutions and stabilities of a NŞS equation which has cubic nonlinearity and third order dispersion is investigated in external potential with $\mathscr{PT}$-symmetry. The model is given below:

$$iu_z + au_{xx} + i\beta u_{xxx} + \alpha|u|^2u + V_{PT}u = 0. \tag{1}$$

Here, u refers to a differentiable complex-valued, slowly varying amplitude, u_{xx} refers to diffraction, z is a scaled propagation distance, β refers to coefficient of third order dispersion term and V_{PT} refers to external potential. In this study, external potential with $\mathscr{PT}$-symmetry is identified as:

$$\begin{aligned} V_{PT} &= V(x) + iW(x) \\ &= [V_0 + V_1\text{sech}(x) + V_2\text{sech}^2(x) + V_3\text{sech}^3(x)] \qquad (2) \\ &\quad + i[W_2\text{sech}^2(x)\tanh(x) + W_1\text{sech}(x)\tanh(x) + W_0\tanh(x)] \end{aligned}$$

Here, $V(x)$ corresponds to the real part which is an even function, $W(x)$ corresponds to the imaginary part which is a odd function.

In Chapter 1, definition of soliton and its relation to the other disciplines are briefly discussed. NLS equation is introduced and some developments about NLS equation are expressed. The importance of higher order dispersion terms in NLS equation is explained. NLS equation with $\mathscr{PT}$-symmetry and third order dispersion term is introduced. Purpose of this thesis, literature review and hypothesis are stated respectively.
In Chapter 2, Spectral Renormalization Method which is the numerical method for solving NLS equation with third order dispersion and $\mathscr{PT}$-symmetric potential is explained. Spectral Renormalization Method is essentially a Fourier iteration method. In this chapter, this method is modified to our problem. Numerical solutions are obtained for NLS equations with $\mathscr{PT}$-symmetric potential and third order dispersion term.
In Chapter 3, exact solutions of NLS equation with $\mathscr{PT}$-symmetric potential and third order dispersion term are investigated. The structure of $\mathscr{PT}$-symmetric potential is defined. In order to produce analytical solutions, the solution ansatz $u(x,z) = f(x)e^{i(\mu z+g(x))}$ is suggested. $f(x)$ and $g(x)$ show two real-valued functions. The structure of used $\mathscr{PT}$-symmetric potential is substituted in the equation with this solution ansatz and then the exact form of this potential is identified. It is verified that the real part of the potential is an even function and the imaginary part is an odd function. The exact solution and the numerical solution are plotted in order to compare the numerically obtained solution to the exact solution. It is seen from the related figure that SR method is an effective method for obtaining solutions for these type of equations. Moreover, it is also proved that the exact and the numerical solutions are in very good agreement.
In Chapter 4, stability analysis of NLS equation with $\mathscr{PT}$-symmetric potential and third order dispersion term is investigated. Firstly, Split-Step Fourier method has been modified to our equation to analyse the nonlinear stability properties of obtained solitons. Also, linear stability analysis is investigated by linear spectrum. Linear spectrums are compared for various values of β.
In Chapter 5, obtained results are summarized. Furthermore, possible future studies are briefly discussed.
In this thesis, MATLAB2018b computer programme is used and all of the results are produced by the use of this programme.

ÜÇÜNCÜ MERTEBEDEN DİSPERSİYON İÇEREN KÜBİK NONLİNEER SCHRÖDINGER DENKLEMİNİN SOLİTON TİPİ ÇÖZÜMLERİ

ÖZET

Pozitif bilimsel çeşitli yönlerle birbirleriyle ilişki içerisindedir. Matematiğin diğer bilimlerle olan ilişkisi incelendiğinde en çok paydanın diferansiyel denklemlerde olduğu görülmektedir. Son yıllarda nonlineer (doğrusal olmayan) dalga problemlerine olan ilgi oldukça artmıştır ve bu konu hakkında çeşitli bilimsel araştırmalar yürütülmektedir. Lineer olmayan dalga denklemleri kuantum, optik ve biyoloji gibi konularda doğayı anlamaya ve bu alanlarda inovatif yöntemler geliştirmeye yardımcı olur. Bu bilim dallarındaki doğrusal olmayan sistemlerin araştırılmasına öncelikli olarak kısmi türevli diferansiyel denklemin çözümünün elde edilmesiyle başlanır. Bu denklemlerin çözümlerinin bir kısmı soliton olarak adlandırılan doğrusal olmayan dalga tipindedir. Solitonlar quantum mekaniği, nonlineer optik, plazma fiziği gibi pek çok alanda elde edilen alan denklemlerinin çözümlerinde ortaya çıkmaktadır. Özellikle optik problemlerin çözümünde örneğin fiber optik kablolarda veri iletimi probleminde NLS denklemi ile bu denklemin bazı varyantları ve bunların soliton tipi çözümleri ortaya çıkmaktadır. NLS denklemi genellikle bir pikosaniyelik zaman ölçeğinde optik atımların doğrusal olmayan yayılımlarını modellemekte yaygın olarak kullanılmaktadır. NLS denklemi Erwin Schrödinger tarafından ortaya konmuştur ve standart formunda aşağıdaki gibi tanımlanmaktadır:

$$iu_z + u_{xx} + \alpha|u|^2u = 0. \tag{3}$$

Literatürde, NLS denkleminin analitik ve sayısal çözümleri araştırılmıştır; ancak bu araştırmalar genellikle (3) denklemindeki gibi ikinci mertebeden dispersiyon terimi içermektedir. Daha yüksek mertebeden (üçüncü ve dördüncü mertebe dispersiyon) terimlerinin probleme katkısını inceleyen çalışmalar daha kısıtlı sayıdadır. Literatürdeki bazı çalışmalarda üçüncü mertebeden dispersiyon teriminin, kıtalar arası veri iletimine etkilerinden bahsedilmiştir. Dolayısıyla yüksek mertebeden dispersiyon terimlerinin etkisini araştırmak, nonlineer optikte veri iletimi problemleri için oldukça önemlidir. Dış potansiyeller, optik atımların kararlılığına ve biçimlerine etki eden diğer bir faktördür. Parite-zaman ($\mathscr{PT}$) simetrisine sahip potansiyeller (kafes/latis), kuantum mekaniği ve nonlineer optik problemlerinde yaygın olarak kullanılmaktadır. Literatürde $\mathscr{PT}$-simetrisine sahip dıp potansiyeller içeren çeşitli tipte NLS denklemlerinin çözümleri ve çözümlerinin kararlılığıyla ilgili çok sayıda çalışma bulunmaktadır.

Bu çalışmada, kübik nonlineerite ve üçüncü mertebeden dispersiyon içeren doğrusal olmayan Schrödinger denkleminin bir $\mathscr{PT}$-simetrisine sahip dış potansiyel altında soliton tipi çözümleri ve çözümlerin kararlılığı incelenmiştir. Bu model aşağıda verilmiştir:

$$iu_z + au_{xx} + i\beta u_{xxx} + \alpha|u|^2u + V_{PT}u = 0. \tag{4}$$

Yukarıdaki denklemde u kompleks değerli elektrik alanın yavaş değişen genliğini, u_{xx} kırınım terimini, β üçüncü mertebeden dispersiyonu modelleyen terimin katsayısını, z yayılım mesafesini, V_{PT} ise dış potansiyeli temsil etmektedir. Bu çalışmadaki $\mathscr{PT}$-simetrisine sahip dış potansiyel aşağıdaki gibi belirlenmiştir:

$$\begin{aligned} V_{PT} &= V(x) + iW(x) \\ &= [V_0 + V_1 \operatorname{sech}(x) + V_2 \operatorname{sech}^2(x) + V_3 \operatorname{sech}^3(x)] \\ &+ i[W_2 \operatorname{sech}^2(x)\tanh(x) + W_1 \operatorname{sech}(x)\tanh(x) + W_0 \tanh(x)]. \end{aligned} \quad (5)$$

Yukarıdaki denklemde $V(x)$ çift fonksiyon özelliği gösteren reel kısmı, $W(x)$ ise tek fonksiyon özelliği gösteren sanal kısmı temsil etmektedir.

Bölüm 1'de, solitonların tanımından ve diğer bilim dalları ile ilişkisinden bahsedilmiştir. NLS denklemi tanıtılmış ve tarihsel gelişimine değinilmiştir. Yüksek mertebeden dispersiyon teriminin NLS denklemindeki önemine değinilmiştir. Bu çalışmada kullanılan $\mathscr{PT}$-simetrisine sahip bir dış potansiyel ve üçüncü mertebeden dispersiyon terimi içeren NLS denklemi tanıtılmıştır. Bu bölümde sırasıyla tezin amacı, literatür araştırması ve tezin hipotezine değinilmiştir.
Bölüm 2'de, tezdeki problemi modelleyen denklemin sayısal çözümünde kullanılan Spektral Renormalizasyon metodu anlatılmıştır. Temelde bir Fourier iterasyon yöntemi olan ve daha sonra Ablowitz ve Musslimani tarafından geliştirilen Spektral Renormalizasyon metodu açıklanarak model denkleme uygulanmıştır. Spektral Renormalizasyon yöntemi ile çözüm elde etmek için aşağıdaki Gaussian başlangıç koşulu kullanılmıştır :

$$w_0 = e^{-x^2} \quad (6)$$

Burada 10^{-12} mertebesinde yakınsama elde edilmiştir. Spektral Renormalizasyon yöntemi ile $\mathscr{PT}$-simetrisine sahip bir dış potansiyel ve üçüncü mertebeden dispersiyon terimi içeren NLS denklemi için soliton tipi sayısal çözüm elde edilmiştir.
Bölüm 3'te, $\mathscr{PT}$-simetrisine sahip bir dış potansiyel ve üçüncü mertebeden dispersiyon terimi içeren NLS denkleminin analitik çözümü araştırılmıştır. $\mathscr{PT}$-simetrik potansiyelin yapısı bu bölümde ayrıntılı olarak türetilmiştir. Bu potansiyelde dayalı analitik çözüm üretebilmek için aşağıdaki çözüm önerisi uygulanmıştır:

$$u(x,z) = f(x)e^{i(\mu z + g(x))} \quad (7)$$

Burada $f(x)$ ve $g(x)$ yapısı henüz belli olmayan reel değerli fonksiyonları ifade etmektedir. Bu bölümde, yukarıdaki çözüm önerisi kullanılarak alan denkleminin kesin çözümleri ve $\mathscr{PT}$-simetrik potansiyelin yapısı elde edilmiştir. Kompleks potansiyelin reel kısmı

$$V(x) = V_0 + V_1 \operatorname{sech}(x) + V_2 \operatorname{sech}^2(x) + V_3 \operatorname{sech}^3(x) \quad (8)$$

olarak hesaplanmıştır. Bu fonksiyonun çift fonksiyon özelliği taşıdığı gösterilmiştir. Kompleks potansiyelin sanal kısmı ise,

$$W(x) = W_0 \tanh(x) + W_1 \operatorname{sech}(x)\tanh(x) + W_2 \operatorname{sech}^2(x)\tanh(x). \quad (9)$$

olarak hesaplanmış olup bu fonksiyonun tek fonksiyon özelliği taşıdığı gösterilmiştir. Bölüm 2'de elde edilen sayısal çözüm ile bu bölümde elde edilen analitik çözüm üst

üste çizdirilerek çözümlerin üst üste düştüğü gözlenmiştir. Dolayısıyla hem Bölüm 2'de kullanılan SR yönteminin bu tip denklemlerin çözümüne uygunluğu gösterilmiş, hem de bulunan kesin çözümün sayısal çözüm ile uyumlu olduğu ispatlanmıştır.
Bölüm 4'te, tezin model denklemi olan $\mathscr{P}\mathscr{T}$-simetrik bir potansiyel ve üçüncü mertebe dispersiyon terimi içeren NSL denkleminin kararlılık (stabilite) analizi incelenmiştir. Öncelikle elde edilmiş olan solitonların lineer olmayan (nonlineer) stabilite özelliklerini incelemek için kullanılacak olan Ayrık-Adımlı Fourier metodu anlatılmıştır. Daha sonra bu yöntem $\mathscr{P}\mathscr{T}$-simetrik bir potansiyel ve üçüncü mertebe dispersiyon içeren model NLS denklemine ana hatları aşağıda gösterildiği biçimde uyarlanmıştır. Öncelikle u_z terimi çözülürse model denklem aşağıdaki formda ifade edilebilir:

$$u_z = iau_{xx} - (\beta u_{xxx}) + i\alpha|u|^2 u + iV_{\mathscr{P}\mathscr{T}} u \tag{10}$$

Buarada aşağıdaki operatörler tanımlanmıştır:

$$\begin{aligned} M &= i(a\partial_{xx} + i\beta\partial_{xxx}) \\ N &= i(\alpha|u|^2 + V_{\mathscr{P}\mathscr{T}}) \end{aligned} \tag{11}$$

Önce $u_z = Mu$ kısmı Fourier dönüşümü kullanılarak çözülmüş daha sonra $u_z = Nu$ parçası çözülmüş ve model NLS denkleminin çözümü Ayrık-Adımlı Fourier yöntemi kullanılarak elde edilmiştir. Kısaca özetlenen bu yöntem kullanılarak elde edilmiş olan soliton tipi çözümlerin çeşitli potansiyel derinliklerinde kararlılık analizleri araştırılmış ve üçüncü mertebeden dispersiyon teriminin katsayısı olan β'nın değişen değerleri için stabil ve stabil olmayan bölgeler belirlenmiştir. Daha sonra lineer spektrum kullanılarak elde edilen solitonların lineer stabilite özellikleri de incelenmiş ve β katsayısının farklı değerleri için lineer spektrumlar karşılaştırılmıştır.
Bölüm 5'te, önceki bölümlerde elde edilen çözümler ve bu çözümlerin stabilite özellikleri özetlenmiş ve ileride yapılabilecek çalışmalar tartışılmıştır. β katsayısının farklı değerleri için incelenen nonlineer stabilite analizi yapılmış ve lineer spektrumlar MATLAB bilgisayar programı ile çizilen grafikler ile yorumlanmıştır.
Bu tezde MATLABR2018b bilgisayar programı kullanılmış ve bütün çözümler bu program ile elde edilmiştir.

1. INTRODUCTION

Solitary waves (commonly referred to as solitons) have been the subject of intense theoretical and experimental studies in many different fields; especially, quantum mechanics, nonlinear optics and biology [1]. Mathematically, a soliton is a solitary wave that asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves [2]. In optics, a soliton refers as any optical field which does not change its shape during propagation due to the balance between linear and nonlinear effects in the medium [3]. Solitons can be used for speeding up the complicated experiments since they give the advantages for understanding sophisticated problems. In the context of nonlinear optics, solitons are classified as being either temporal or spatial depending on whether the confinement of light occurs in time or space during wave propagation. Temporal solitons represent optical pulses that maintain their shape, whereas spatial solitons represent self-guided beams that remain confined in the transfers directions orthogonal to the direction of propagation. In both cases, the pulse or the beam propagates through a medium without change in its shape is said to be self-localized [3]. Nonlinear Schrödinger Equation is an important nonlinear evolution equations which is used in brought areas in physics and applied mathematics. Nonlinear Schrödinger equation (NLS) is established by Erwin Schrödinger; moreover, it basically arises nonlinear systems [4]. Its pulses is 1 picosecond time-scale and it is usually determined as

$$iu_z + u_{xx} + \alpha |u|^2 u = 0. \tag{1.1}$$

where u refers to the differentiable complex-valued, gradually altering amplitude of the electric field; u_{xx} refers to diffraction; z is a scaled propagation distance; the coefficients α represents the sign of cubic nonlinearity of the medium. This NLS equation is referred to as being (1+1)-dimensional and constitutes the simplest form of the NLS equation. The bright and dark spatial solitons correspond to the choice of α as $+1$ and -1 respectively. In this thesis, we only consider the bright soliton type of solution.

It is known that NLS equation does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state lasers, where pulses are short as 10 femtoseconds are generated, the approximation breaks down. Therefore, one need to consider the third order dispersion for performance enhancement along trans-continental distances.

In this thesis, we investigate the cubic nonlinear Schrödinger equation with a $\mathscr{P}\mathscr{T}$-symmetric optical potential and third order dispersion which is given below:

$$iu_z + au_{xx} + i\beta u_{xxx} + \alpha|u|^2 u + V_{PT}u = 0. \tag{1.2}$$

Here, β is a third-order diffraction coupling constant taken as either negative or positive constant value and V_{PT} is a $\mathscr{P}\mathscr{T}$-symmetric external potential which the exact structure is defined in Chapter 3.

We will consider $\mathscr{P}\mathscr{T}$-symmetric potential as

$$V_{PT} = V(x) + iW(x) \tag{1.3}$$

The purpose of this thesis is to explore the exact and the numerical solutions of the equation (1.2) and discover the effect of the third order dispersion term $i\beta u_{xxx}$ on the soliton solutions and their stabilities.

In order to solve Eq.(1.2) Spectral Renormalization method is applied. Spectral Renormalization method is essentially a Fourier iteration method and it was introduced by Petviashvili in 1975 [5]. It was first use to find localized solutions in the two-dimensional Korteweg-deVries equation (usually referred to as the Kadomtsev-Petviashvili equation or in short KP equation) [6]. Then, this method is strengthened by Ablowitz and his co-worker Musslimani in early 2000's [7]. The idea behind the method is to transform the underlying equation governing the soliton such as a NLS type equation into Fourier space and determine a nonlocal integral equation coupled to an algebraic equation. The coupling prevents the numerical scheme from diverging.

1.1 Purpose of Thesis

In this thesis, we investigate the effect of the external potential and third order dispersion term on the soliton solutions of the cubic nonlinear Schrödinger equation with $\mathscr{PT}$-symmetric potential and third order dispersion term. We aim to find an exact soliton type solution to this model equation and explore linear and nonlinear stability properties of the obtained solitons.

1.2 Literature Review

Nonlinear wave problems arise in various mathematical and physical fields such as nonlinear optics, plasma physics and quantum mechanics [8–10]. Solutions and stabilities of NLS equations are largely discussed by scientists for many years [2, 3, 11, 12]. Solitons appear as the solutions of common class of weakly nonlinear dispersive ordinary differential equations describing physical systems [13]. The recent study of Kartashov, Vysloukh and Torner is about lattice solitons that they adress variety type of optical lattices and potential stabilization of these structures [14]. Stability analysis of optical solitons in a periodic $\mathscr{PT}$-symmetric potential is explored by Musslimani and co-workers [15]. Optical soliton solutions of NLS equations with $\mathscr{PT}$-symmetric optical lattices are investigated by various scientists [16–19]. The existence and stability of lattice solitons were reported in parity-time symmetric mixed linear-nonlinear optical lattices. It is also revealed that the parameters of the linear lattice periodic potential have considerable role in controlling regions of stability domains [20]. Göksel et al. investigated the existence and stability properties of solitons of the (1+1)D cubic-quintic NLS equations with $\mathscr{PT}$-symmetric external potential. They obtained the solutions by means of Spectral Renormalization method for varying potential depths. Stability and instability regions of solitons were investigated by linear spectrum analysis [21]. The numerical existence of fundamental solitons in saturable media on crystal and certain type of quasicrystal lattices were investigated and the nonlinear stability of the fundamental solitons were studied by using numerical methods such as finite difference method and fourth-order Runge-Kutta method. The effects of the potential depth and applied external electrical field on the gap width were also studied in [22]. The existence and stability of solitons in $\mathscr{PT}$-symmetric optical lattices with spatially periodic modulation of the local strength of the nonlinear media were investigated. Additionally, the effects of spatial modulation of the nonlinearity on stability of solitons in $\mathscr{PT}$–symmetric optical lattices were revealed [23]. The effects of additional higher order dispersive term in cubic NLS equations were studied by Karlsson [24]. Third order dispersive term was not included in their study. They only focused on effects of fourth order dispersive term in the related work. Wazwaz and Kaur investigated exact analytical solutions for NLS equations with normal dispersive regimes by using variational iteration method [25]. Soliton type solutions of cubic NLS equations are produced [26, 27]. The exact solutions of the nonlinear Schrödinger equation with cubic and quintic space were explored by using canonical transformations in the presence of time-dependent and inhomogeneous external potentials. The importance of $\mathscr{P}\mathscr{T}$-symmetry was searched to guarantee the conservation of the average energy of the system [28]. Stability properties and bad-gap structures of higher order NLS equations with periodic lattice were investigated in [29]. The recent studies are mostly about soliton dynamics of higher order multi-dimensional NLS equations [30]. Yan and Chen investigated the stability of bright solitons in the generalized NLS equations with several types of $\mathscr{P}\mathscr{T}$-symmetric potentials and they showed that their stability is verified by the linear stability spectrum. They also explored the interactions of two solitons [31].

It is known that Hamiltons which are defined in quantum mechanics must be Hamiltonian for real spectrum. Recently, Bender and his co-workers showed that Hermitian property is not an obligation for real spectrum in the $\mathscr{P}\mathscr{T}$-symmetry [32, 33].

1.3 Hypothesis

Existence and stability properties of the solutions of cubic NLS equation with a $\mathscr{PT}$-symmetric potential are highly related to an additional third order dispersion term. The existence of a positive third order dispersion (3OD) term has a positive effect on the nonlinear stability of the solitons of Eq.(1.2). Additionally, the existence of 3OD affects the linear stability of the obtained solitons. Eq.(1.2) becomes more stable with 3OD term; moreover, the existence and stability of the solitons affected by the potential depths of the $\mathscr{PT}$-symmetric potential defined in the thesis.

2. NUMERICAL METHODS

Well-known Spectral Renormalization method will be modified to find numerical solution of Eq.(1.2).

2.1 Spectral Renormalization Method

There are variety of numerical methods in order to obtain soliton type solutions. Spectral Renormalization method is one of these techniques which is essentially Fourier iteration proposed by Petviashvili, in order to find localized solutions in the two-dimensional Korteweg-deVries (KP) equation. Later, Ablowitz and Musslimani extended this method [2] with usage of nonlinear wave guides to compute self-localized states. This method can be used to computed self-localized states of nonlinear wave guides that is flexible and can be applied to many nonlinear systems involve nonlinearities with different homogeneities such as cubic-quintic or as saturable nonlinearity.

This method can be applied to the cubis NLS equation with a $\mathscr{P}\mathscr{T}$ symmetric potential and 3OD as follows:

$$iu_z + au_{xx} + i\beta u_{xxx} + \alpha|u|^2 u + V_{PT} u = 0. \tag{2.1}$$

Using the ansatz $u(x,z) = f(x)e^{i\mu z}$ where $f(x)$ is a complex-valued function and μ is the eigenvalue, we have following set of equations,

$$\begin{aligned} u_z &= i\mu f e^{i\mu z} \\ u_x &= f_x e^{i\mu z} \\ u_{xx} &= f_{xx} e^{i\mu z} \\ u_{xxx} &= f_{xxx} e^{i\mu z} \\ |u|^2 &= |f|^2 \end{aligned} \tag{2.2}$$

Substituting the equations Eq. (2.2) into Eq. (2.1), the following nonlinear equation for f is obtained

$$-\mu f e^{i\mu z} + a f_{xx} e^{i\mu z} + i\beta f_{xxx} e^{i\mu z} + \alpha |f|^2 f e^{i\mu z} + V_{PT} f e^{i\mu z} = 0. \tag{2.3}$$

After cancelling the exponential term we have

$$-\mu f + a f_{xx} + i\beta f_{xxx} + \alpha |f|^2 f + V_{PT} f = 0. \tag{2.4}$$

After applying Fourier transformation to Eq. (2.4) we get

$$\mathscr{F}\{-\mu f\} + \mathscr{F}\{a f_{xx}\} + \mathscr{F}\{i\beta f_{xxx}\} + \mathscr{F}\{\alpha |f|^2 f\} + \mathscr{F}\{V_{PT} f\} = \mathscr{F}\{0\}. \tag{2.5}$$

where $\mathscr{F}$ indicates Fourier transformation. Due to the properties of this transformation, we get Eq. (2.6)

$$-\mu \hat{f} - a(k_x)^2 \hat{f} - \beta {k_x}^3 \hat{f} + \alpha \mathscr{F}\{|f|^2 f\} + \mathscr{F}\{(V + iW) f\} = 0 \tag{2.6}$$

where $\mathscr{F}(f) = \hat{f}$ and k_x are Fourier variables. Solving Eq. (2.6) for $\hat{f}$ yields

$$\hat{f} = \frac{\alpha \mathscr{F}\{|f|^2 f\} + \mathscr{F}\{(V + iW) f\}}{[\mu + a{k_x}^2 + \beta {k_x}^3]} \tag{2.7}$$

Since the scheme diverges, the equation (2.7) cannot be applied to find $f(x)$. New field variable $f(x) = \lambda w(x)$ with $\lambda \in R^+$ where λ is a parameter can be determined. After the arrangement of the equation (2.7) with the new field variable $f(x)$, we get

$$\lambda \hat{w} = \frac{\alpha \mathscr{F}\{|w|^2 |\lambda|^2 w \lambda\} + \mathscr{F}\{(V + iW) \lambda w\}}{\mu + a{k_x}^2 + \beta {k_x}^3} \tag{2.8}$$

simplifying this equation, we have

$$\hat{w} = \frac{\alpha \mathscr{F}\{|w|^2 |\lambda|^2 w\} + \mathscr{F}\{(V + iW) w\}}{\mu + a{k_x}^2 + \beta {k_x}^3} \tag{2.9}$$

Eq. (2.9) can be applied in an iterative method to investigate w; morever, the following iteration approach can be utilized for investigation of $\hat{w}$:

$$\hat{w}_{n+1} = \frac{\alpha |\lambda|^2 \mathscr{F}\{|w_n|^2 w_n\} + \mathscr{F}\{(V + iW) w_n\}}{\mu + a{k_x}^2 + \beta {k_x}^3}, \quad n \in N \tag{2.10}$$

with the initial condition taken as a Gaussian type function

$$w_0 = e^{-x^2} \tag{2.11}$$

where $|w_{n+1} - w_n| < 10^{-12}$ is the convergence criterion. After the multiplication of both sides of Eq. (2.9) by $(\mu + a{k_x}^2 + \beta {k_x}^3)$ and we get

$$(\mu + a{k_x}^2 + \beta {k_x}^3)\hat{w} = |\lambda|^2 \alpha \mathscr{F}\{|w|^2 w\} + \mathscr{F}\{(V + iW)w\}. \tag{2.12}$$

If we take all terms of Eq. (2.12) to the left side, we get the following equation

$$(\mu + a{k_x}^2 + \beta {k_x}^3)\hat{w} - |\lambda|^2 \alpha \mathscr{F}\{|w|^2 w\} - \mathscr{F}\{(V + iW)w\} = 0. \tag{2.13}$$

After the multiplication of Eq. (2.13) by the conjugate of $\hat{w}$, i.e. by $\hat{w}^*$ yields

$$(\mu + a{k_x}^2 + \beta {k_x}^3)|w|^2 - |\lambda|^2 \alpha \mathscr{F}\{|w|^2 w\}\hat{w}^* - \mathscr{F}\{(V + iW)w\}\hat{w}^* = 0. \tag{2.14}$$

Integrating Eq. (2.14) leads to

$$\begin{aligned} -\int_{-\infty}^{\infty} (\mu + a{k_x}^2 + \beta {k_x}^3)|w|^2 dk + |\lambda|^2 \int_{-\infty}^{\infty} \alpha \mathscr{F}\{|w|^2 w\}\hat{w}^* dk \\ + \int_{-\infty}^{\infty} \mathscr{F}\{(V + iW)w\}\hat{w}^* dk = 0 \end{aligned} \tag{2.15}$$

or in a more compact form

$$\begin{aligned} -\int_{-\infty}^{\infty} \left[-\mathscr{F}\{(V + iW)w\}\hat{w}^* + (\mu + a{k_x}^2 + \beta {k_x}^3)|w|^2\right] dk \\ + |\lambda|^2 \int_{-\infty}^{\infty} \alpha \mathscr{F}\{|w|^2 w\}\hat{w}^* dk = 0. \end{aligned} \tag{2.16}$$

Eq. (2.16) is a second order polynomial of λ in the form $P(\lambda) = a\lambda^2 + b$ then λ can be calculated exactly by the usage of the following formula:

$$\lambda_{1;2} = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{2.17}$$

where

$$a = \alpha \int_{-\infty}^{\infty} \mathscr{F}\{|w|^2 w\}\hat{w}^* dk \tag{2.18}$$

$$b = -\int_{-\infty}^{\infty} [-\mathscr{F}\{(V + iW)w\}\hat{w}^* + (\mu + a{k_x}^2 + \beta {k_x}^3)|w|^2] dk. \tag{2.19}$$

For the convergence of the iteration, the appropriate soliton is $f(x) = \lambda(wx) = \lambda \mathscr{F}^{-1}(\hat{w})$.

3. EXACT SOLUTIONS OF CNLS EQUATION WITH THIRD ORDER DISPERSION AND A $\mathscr{PT}$-SYMMETRIC POTENTIAL

In this chapter, we will find the exact soliton type solution of the following cubic NLS equation with a $\mathscr{PT}$-symmetric external potential and 3OD gives as follows :

$$iu_z + au_{xx} + i\beta u_{xxx} + \alpha|u|^2 u + V_{PT} u = 0. \tag{3.1}$$

For $u = 0$ we find the trivial solution of Eq. (3.1). In order to find non-zero solutions, consider $u \neq 0$. Dividing Eq. (3.1) by u and by the use of Eq. (1.3) gives

$$i\frac{u_z}{u} + \frac{au_{xx}}{u} + i\beta\frac{u_{xxx}}{u} + \alpha|u|^2 + V + iW = 0. \tag{3.2}$$

In order to find non-zero stationary solitons, the following ansatz is used:

$$u(x,z) = f(x)e^{i(\mu z + g(x))} \tag{3.3}$$

where u is a function of x, z and to be determined, $f(x)$ and $g(x)$ are real-valued functions which are different than zero, and μ is the propagation constant. Derivatives of Eq. (3.3) with respect to z and x give the following:

$$\begin{aligned}
u_z &= f(x)i\mu e^{i(\mu z+g(x))} = i\mu u \\
u_x &= f'(x)e^{i(\mu z+g(x))} + ig' f e^{i(\mu z+g(x))} \\
u_{xx} &= [f''(x) + 2if'(x)g'(x) + if(x)g''(x) - f(x)(g'(x))^2]e^{i(\mu z+g(x))} \\
&= \frac{u}{f}[f''(x) + 2if'(x)g'(x) + if(x)g''(x) - f(x)(g'(x))^2] \\
u_{xxx} &= \frac{u}{f}[f'''(x) + 3if''(x)g'(x) + 3if'(x)g''(x) \\
&+ if(x)g'''(x) - 3f'(x)(g'(x))^2 - 3f(x)g'(x)g''(x) - if(g'(x))^3] \\
|u|^2 &= f(x)e^{i(\mu z+g(x))}f(x)e^{-i(\mu z+g(x))} = (f(x))^2
\end{aligned} \tag{3.4}$$

Substituting Eq. (3.4) into Eq. (3.2) yields

$$\begin{aligned}
&[-\mu + \tfrac{af''(x)}{f(x)} - a(g'(x)^2) - 3\beta\tfrac{f''(x)g'(x)}{f(x)} - 3\beta\tfrac{f'(x)g''(x)}{f(x)} - \beta g'''(x) + \beta(g'(x))^3 \\
&+\alpha|f|^2 + V(x)] + i[2a\tfrac{f'(x)g'(x)}{f(x)} + ag''(x) + \beta\tfrac{f'''(x)}{f(x)} \\
&-3\beta\tfrac{f'(x)g'(x)^2}{f(x)} - 3\beta g'(x)g''(x) + W(x)] = 0
\end{aligned} \tag{3.5}$$

The following ansatzs are used for investigation of soliton solutions:

$$f(x) = f_0 \sec h^p(x) \\ g'(x) = g_0 \sec h^q(x) \tag{3.6}$$

where f_0 and g_0 are non-zero real constants and $p \in N$. For the simplification of Eq. (3.5), calculating derivatives of f and g is an obligation. By using Eq. (3.6) we get

$$\begin{aligned}
f'(x) &= -fp\tanh(x) \\
f''(x) &= f[p^2 - (p^2+p)\sec h^2(x)] \\
f'''(x) &= fp\tan h(x)[(p^2+3p+2)\sec h^2(x) - p^2)] \\
g'(x) &= g_0 \mathrm{sech}^q(x) \\
g''(x) &= -g_0 q \sec h^q(x)\tanh(x) \\
g'''(x) &= g_0 q \sec h^q(x)[q-(q+1)\sec h^2(x)]
\end{aligned} \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (3.5) we get

$$\begin{aligned}
&-\mu + ap^2 - a(p^2+p)\sec h^2(x) - ag_0{}^2\sec h^{2q}(x) \\
&-3\beta[p^2-(p^2+p)\sec h^2(x)]g_0 \sec h^q(x) \\
&-3\beta pq\tanh(x) g_0 \sec h^q(x)\tanh(x) - \beta g_0 q\sec h^q(x)[q-(q+1)\sec h^2(x)] \\
&+\beta g_0{}^3 \sec h^{3q}(x) + \alpha f_0{}^2 \sec h^{2p}(x) + V(x) \\
&+i[3\beta g_0{}^2(p+q)\sec h^{2q}(x)\tanh(x) - ag_0(2p+q)\sec h^q(x)\tanh(x) \\
&+\beta p(p^2+3p+2)\sec h^2 \tanh(x) - \beta p^3 \tanh(x) + W(x)] = 0.
\end{aligned} \tag{3.8}$$

In order to get real and imaginary parts of the $\mathscr{PT}$-symmetric potential, we split Eq. (3.8) as:

Real Part

The real part of the Eq. (3.8) can be expressed as,

$$\begin{aligned}
&-\mu + ap^2 - a(p^2+p)\sec h^2(x) - ag_0{}^2\sec h^{2q}(x) \\
&-3\beta g_0 p^2 \sec h^q(x) + 3\beta g_0(p^2+p)\sec h^{q+2}(x) - 3\beta g_0 pq\sec h^q(x) \\
&+3\beta g_0 pq\sec h^{q+2}(x) - \beta g_0 q^2 \sec h^q(x) + \beta g_0 q(q+1)\sec h^{q+2}(x) \\
&+\beta g_0{}^3 \sec h^{3q}(x) + \alpha f_0{}^2 \sec h^{2p}(x) + V(X) = 0.
\end{aligned} \tag{3.9}$$

The real part of the $\mathscr{P}\mathscr{T}$-symmetric potential is found as

$$V(x) = V_0 + V_1 \text{sech}^2(x) + V_2 \text{sech}^q(x) + V_3 \text{sech}^{2q}(x)$$
$$+ V_4 \text{sech}^{q+2}(x) + V_5 \text{sech}^{3q}(x) + V_6 \text{sech}^{2p}(x) \tag{3.10}$$

where

$$\begin{aligned}
V_0 &= \mu - ap^2 \\
V_1 &= a(p^2 + p) \\
V_2 &= \beta g_0[(3p^2 + 3pq + q^2)] \\
V_3 &= a{g_0}^2 \\
V_4 &= -\beta g_0[3(p^2 + p) + 3pq + q(q+1)] \\
V_5 &= -\beta {g_0}^3 \\
V_6 &= -\alpha {f_0}^2
\end{aligned} \tag{3.11}$$

we can see in the following form that $V(x)$ is an even function

$$\begin{aligned}
V(-x) &= V_0 + V_1 \text{sech}^2(-x) + V_2 \text{sech}^q(-x) + V_3 \text{sech}^{2q}(-x) \\
&+ V_4 \text{sech}^{q+2}(-x) + V_5 \text{sech}^{3q}(-x) + V_6 \text{sech}^{2p}(-x) \\
&= V_0 + V_1 \text{sech}^2(x) + V_2 \text{sech} q(x) + V_3 \text{sech}^{2q}(x) + V_4 \text{sech}^{q+2}(x) \\
&+ V_5 \text{sech}^{3q}(x) + V_6 \text{sech}^{2p}(x) \\
&= V(x).
\end{aligned} \tag{3.12}$$

$V(x)$ can be expressed by the powers of $sech(x)$. For $p = q = 1$ Eq. (3.10) can be rewritten as:

$$V(x) = \mu - a + 2a \sec h^2(x) + 7\beta g_0 \sec h(x) + a{g_0}^2 \sec h^2(x)$$
$$-11\beta g_0 \sec h^3(x) - \beta {g_0}^3 \sec h^3(x) - \alpha {f_0}^2 \sec h^2(x) \tag{3.13}$$

$V(x)$ can be written as

$$V(x) = \mu - a + 7\beta g_0 \sec h(x)(2a + a{g_0}^2 - \alpha {f_0}^2) \sec h^2(x)$$
$$+(-11\beta g_0 - \beta {g_0}^3) \sec h^3(x) \tag{3.14}$$

where

$$V_0 = \mu - a$$
$$V_1 = 7\beta g_0$$
$$V_2 = 2a + a{g_0}^2 - \alpha {f_0}^2 \tag{3.15}$$
$$V_3 = -\beta g_0(11 + {g_0}^2)$$

The real part of the $\mathscr{PT}$-symmetric potential is obtained as,

$$V(x) = \mu - a + 7\beta g_0 \text{sech}(x) + (2a + a{g_0}^2 - \alpha {f_0}^2)\text{sech}^2(x) \tag{3.16}$$
$$-\beta g_0(11 + {g_0}^2)\text{sech}^3(x).$$

Imaginary Part

The complex part of the Eq. (3.8) can be expressed as

$$3\beta {g_0}^2(p+q)\text{sech}(x)^{2q}\tanh(x) - a g_0(2p+q)\text{sech}^q(x)\tanh(x) \tag{3.17}$$
$$+\beta p(p^2+3p+2)\text{sech}^2(x)\tanh(x) - \beta p^3\tanh(x) + W(x) = 0$$

The imaginary part of the $\mathscr{PT}$-symmetric potential is obtained as

$$W(x) = W_0\text{sech}^{2q}(x)\tanh(x) + W_1\text{sech}^q(x)\tanh(x) \tag{3.18}$$
$$+W_2 \sec h^2(x)\tanh(x) + W_3\tanh(x)$$

where

$$W_0 = -3pg_0{}^2(p+q)$$
$$W_1 = a g_0(2p+q)$$
$$W_2 = -\beta p(p^2+3p+2) \tag{3.19}$$
$$W_3 = \beta p^3$$

W(x) is an odd function; as a result of the following form:

$$W(-x) = W_0\text{sech}^2 q(-x)\tanh(-x) + W_1\text{sech}^q(-x)\tanh(-x)$$
$$+W_2\text{sech}^2(-x)\tanh(-x) + W_3\tanh(-x) \tag{3.20}$$
$$= W_0\text{sech}^2 q(x)(-\tanh(x)) + W_1\text{sech}^q(x)(-\tanh(x))$$
$$+W_2\text{sech}^{2q}(x) + W_3(-\tanh(x)) = -W(x).$$

For $p = q = 1$, then we can transform Eq. (3.18) as following form,

$$W(x) = -6\beta {g_0}^2 \sec h^2(x)\tanh(x) + 3ag_0 \sec h(x)\tanh(x)$$
$$-6\beta \sec h^2(x)\tanh(x) + \beta \tanh(x) \tag{3.21}$$

where

$$W_0 = \beta$$
$$W_1 = 3ag_0 \tag{3.22}$$
$$W_2 = -6\beta({g_0}^2 + 1)$$

The equations (3.15), (3.22) puts a constrain on the potential depths: $V_2 < 2 + {W_1}^2/9$ or $\alpha = 1$ and $a = 1$ For the case of $p = q = 1$ the analytical solution of the problem can begin with

$$u(x,z) = f_0 \sec h(x) e^{i[\mu z + g_0 \arctan h(x) \sinh(x)]}. \tag{3.23}$$

where $f_0 = \sqrt{2 + {W_1}^2/9 - V_2}$ and $g_0 = W_1/3$ Hence, Eq. (1.2), with the real and the imaginary parts can be given as

$$V_{PT} = [V_0 + V_1\text{sech}(x) + V_2\text{sech}^2(x) + V_3\text{sech}^3(x)]$$
$$+i[W_2\text{sech}^2(x)\tanh(x) + W_1\text{sech}(x)\tanh(x) + W_0\tanh(x)]. \tag{3.24}$$

Eq. (3.24) can be seen as extension of the Scarf II potential for a Kerr medium [34]. In Fig. 3.1, the soliton numerically obtained by the SR method which is explained in Chapter 2 is plotted with dashed green solid line while analytically obtained soliton which is explained above is plotted with red solid line. It is seen from the figure that, two solitons overlap and obtained numerical solution satisfies Eq. (2.1) with an absolute error 10^{-7}. Therefore, it shows that SR method is suitable for solving this equation and analytical solution is proved to be correct.

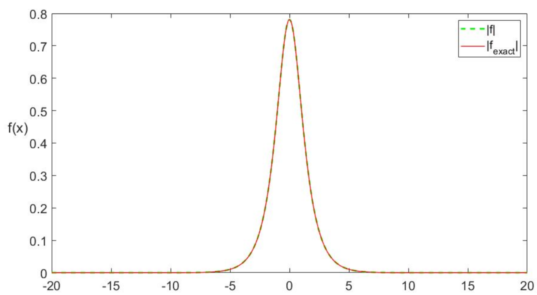

Figure 3.1 : Analytically and numerically obtained soliton for $\mu = 1$, $V_2 = 1.4$ and $W_1 = 0.3$ with $\beta = -0.1$

In Fig. 3.2, all the obtained solitons are shown for the cubic NLS equation with $\mathscr{PT}$-symmetric potential and 3OD for varying potential depths $V_2 - W_1$. In this figure the contrait curve $V_2 = 2 + {W_1}^2/9$ is depicted by dashed line.

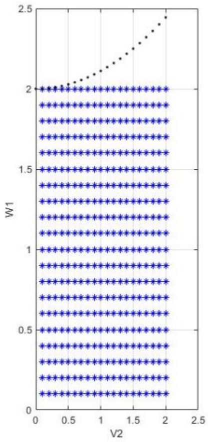

Figure 3.2 : Existence region of cubic NLS equation with $\mathscr{PT}$-symmetric potential and 3OD for varying potential depths

4. STABILITY ANALYSIS

4.1 Split-Step Fourier Method

Split-Step Fourier Method is one of various types of evolution methods. In his book, Yang detailed the Split-Step Fourier Method for solving wave equations in [35]. The method is based on splitting the evolution equation into several pieces. Although the idea of this method has come up for a long time ago, the application in NLS type of equation has been investigated for recent years.

4.2 Nonlinear Stability Analysis

In field of optics, nonlinearly stable means that a soliton conserves its shape, position and the maximum amplitude during propagation. We used Split-Step Fourier method to study the nonlinear stability properties of obtained solitons in Chapter 3. For this purpose we employed the Split-Step Fourier Method that is explained in detail by Göksel while investigating (2+1)D NLS in [36]. To study the nonlinear stability we computed obtained solitons over a long distance. For this thesis $z = 40$ found to be adequate to decide whether a soliton is nonlinearly stable or not.

Consider Eq. (1.2) which can be rewritten as

$$u_z = iau_{xx} - (\beta u_{xxx}) + i\alpha|u|^2u + iV_{\mathscr{P}\mathscr{T}}u \tag{4.1}$$

and hence can be split as in Eq.(4.1) with the linear operator $M = i(a\partial_{xx} + i\beta\partial_{xxx})$ and the operator $N = i(\alpha|u|^2 + V_{\mathscr{P}\mathscr{T}})$.

The linear step $u_z = Mu$ is solved by means of Fourier transform. Taking the Fourier transform of both sides of

$$u_z = ia\partial_{xx} - \beta\partial_{xxx} \tag{4.2}$$

gives

$$\hat{u}_z = ia(ik_x)^2\hat{u} - \beta(ik_x)^3\hat{u} = -i(ak_x{}^2 - \beta k_x{}^3)\hat{u}\ . \tag{4.3}$$

This is nothing but an ordinary differential equation (ODE) of $\hat{u}$ and its exact solution is given by

$$\hat{u} = \hat{C}_1 e^{-i(ak_x{}^2 - i\beta k_x{}^3)z} \quad \Rightarrow \quad u = \mathscr{F}^{-1}\left(\hat{C}_1 e^{-i(ak_x{}^2 - i\beta k_x{}^3)z}\right) . \tag{4.4}$$

The step $u_z = Nu$, i.e.

$$u_z = i(\alpha|u|^2 + V_{\mathscr{P}\mathscr{T}})u \tag{4.5}$$

has the exact solution

$$u = C_2 e^{i(\alpha|u|^2 + V_{\mathscr{P}\mathscr{T}})z} . \tag{4.6}$$

Having found solutions to both parts, the Split-Step Fourier method can now be employed for the Eq.(1.2) equation by using any splitting scheme.

In Fig.(4.1), the nonlinear evolution of solitons are represented for $\mu = 1$, $W_1 = 0.3$ and $V_2 = 1.4$ with a $\mathscr{P}\mathscr{T}$-symmetric potential and for $\beta = -0.1, \beta = 0$ and $\beta = 0.1$ respectively. It can be seen that β does not have a major effect on the nonlinear stability of the system of the solitons for the potential depths $V_2 = 1.4$ and $W_1 = 0.3$.

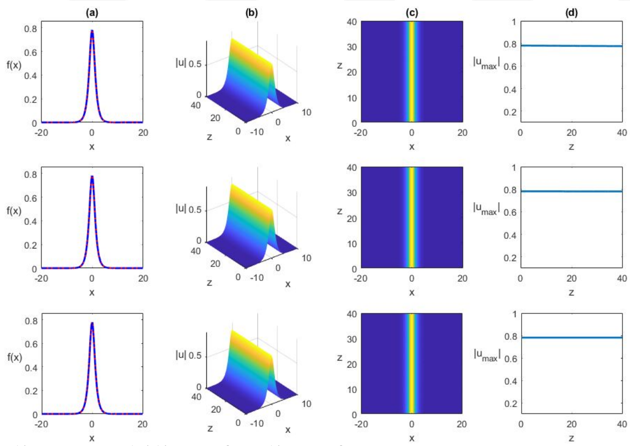

Figure 4.1 : Nonlinear stability of soliton for $\mu = 1$, $V_2 = 1.4$ and $W_1 = 0.3$ with a $\mathscr{P}\mathscr{T}$-symmetric potential and for $\beta = -0.1, \beta = 0$ and $\beta = 0.1$ respectively; (a) Numerically produced soliton (blue dashes) on top of the soliton after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In Fig.(4.2), the nonlinear evolution of solitons are shown for $\mu = 1$, $W_1 = 1.8$ and $V_2 = 1.8$ with a $\mathscr{P}\mathscr{T}$-symmetric potential and for $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$ respectively. The potential depth values V_2, W_1 are chosen from the instability region. It can be seen this figure that for this specific potential depth values the maximum amplitude of the soliton decreases during the evolution; moreover, the soliton for $\beta = 0.1$ is deteriorated around $= 40$.

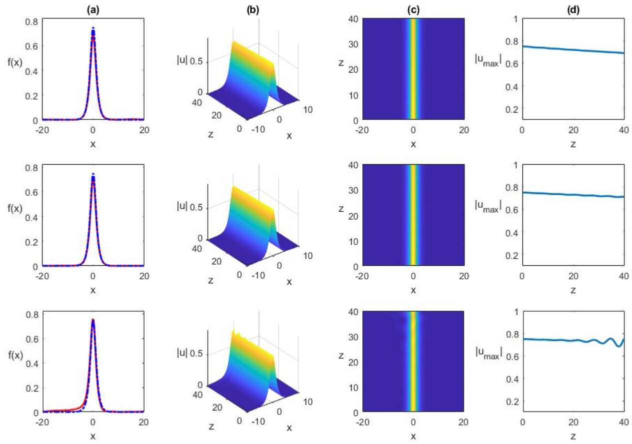

Figure 4.2 : Nonlinear instability of soliton for $\mu = 1$, $V_2 = 1.8$ and $W_1 = 1.8$ with a $\mathscr{PT}$-symmetric potential and for $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$ respectively; (a) Numerically produced soliton (blue dashes) on top of the soliton after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In Fig. (4.3) the nonlinear evolution of solitons are represented for $\mu = 1$, $W_1 = 0.2$ and $V_2 = 0.5$ with a $\mathscr{PT}$-symmetric potential and for $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$ respectively. As is seen from the figure that for these specific values of the potential depths while negative 3OD causes nonlinear instability of the soliton, both removing the negative 3OD and adding a positive 3OD to the system stabilizes the soliton by preventing the maximum amplitude from getting smaller during the evolution.

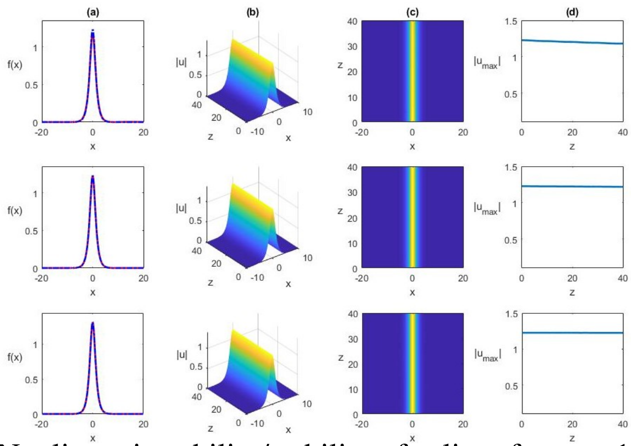

Figure 4.3 : Nonlinear instability/stability of soliton for $\mu = 1$, $V_2 = 0.5$ and $W_1 = 0.2$ with a $\mathscr{PT}$-symmetric potential and for $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$ respectively; (a) Numerically produced soliton (blue dashes) on top of the soliton after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

Fig. (4.4) (a) shows the numerical/analytical existence points of the Eq.(1.2) for $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$. It is found that 3OD does not have an effect on the soliton existence region for the interval $-0.1 \leq \beta \leq 0.1$ We observed that the nonlinear stability increases with existence of β. It can be easily concluded that as β increases from -0.1 to 0.1 the nonlinear stability region is enlarged. As a result, one can conclude that adding a positive 3OD term to the system helps to improve the nonlinear stability of the soliton.

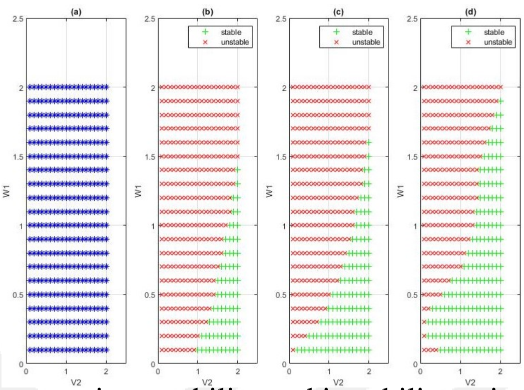

Figure 4.4 : (a) Existence points, stability and instability points for (b) $\beta = -0.1$ (c) $\beta = 0$ (d) $\beta = 0.1$

4.3 Linear Stability

Linear stability will be investigated by acquiring and analyzing the linear spectrum of the obtained solitons.

4.3.1 Linear Spectrum

Linear stability spectrum or short, linear spectrum are the eigenvalues of the linear stability operator of a soliton. These eigenvalues give information about the linear stability of a soliton. Consider the following equation having general type of nonlinearities where $F(\cdot) \in \mathbb{R}$ and $F(0) = 0$:

$$\begin{aligned} iu_z(x,z) + au_{xx}(x,z) + i\beta u_{xxx}(x,z) + \alpha F(|u(x,z)|^2)u(x,z) \\ + V_{\mathscr{PT}}(x)u(x,z) = 0 \end{aligned} \tag{4.7}$$

Eq. (4.7) admits soliton solutions of the form $u(x,z) = f(x)\, e^{i\mu z}$. Substituting

$$\begin{aligned} u_z &= i\mu f e^{i\mu z} \\ u_{xx} &= f_{xx}\, e^{i\mu z} \\ u_{xxx} &= f_{xxx}\, e^{i\mu z} \\ |u|^2 &= uu^* = fe^{i\mu z} f^* e^{-i\mu z} = ff^* = |f|^2 \end{aligned} \tag{4.8}$$

in Eq. (4.7) and multiplying by $e^{-i\mu z}$ gives

$$-\mu f + a f_{xx} + i\beta f_{xxx} + \alpha F(|f|^2) f + V_{\mathscr{PT}} f = 0. \tag{4.9}$$

To analyze the linear stability, the soliton solution is perturbed as follows

$$u(x,z) = \left[f(x) + g(x) e^{\sigma z} + h^*(x) e^{\sigma^* z} \right] e^{i\mu z} \tag{4.10}$$

where g and h are perturbation eigenfunctions and σ is the eigenvalue.

$$\begin{aligned}
u_z &= \left(\sigma g e^{\sigma z} + \sigma^* h^* e^{\sigma^* z} + i\mu f + i\mu g e^{\sigma z} + i\mu h^* e^{\sigma^* z} \right) e^{i\mu z} \\
u_{xx} &= \left(f_{xx} + g_{xx} e^{\sigma z} + h^*_{xx} e^{\sigma^* z} \right) e^{i\mu z} \\
u_{xxx} &= \left(f_{xxx} + g_{xxx} e^{\sigma z} + h^*_{xxx} e^{\sigma^* z} \right) e^{i\mu z}
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
|u|^2 = uu^* &= \left(f + g e^{\sigma z} + h^* e^{\sigma^* z} \right) e^{i\mu z} \left(f^* + g^* e^{\sigma^* z} + h e^{\sigma z} \right) e^{-i\mu z} \\
&= f f^* + f g^* e^{\sigma^* z} + f h e^{\sigma z} + f^* g e^{\sigma z} + g g^* e^{(\sigma+\sigma^*)z} \\
&\quad + g h e^{2\sigma z} + f^* h^* e^{\sigma^* z} + g^* h^* e^{2\sigma^* z} + h h^* e^{(\sigma+\sigma^*)z} \\
&\simeq |f|^2 + \left(g^* e^{\sigma^* z} + h e^{\sigma z} \right) f + \left(g e^{\sigma z} + h^* e^{\sigma^* z} \right) f^*
\end{aligned} \tag{4.12}$$

Using linear Taylor expansion $F(x+h) = F(x) + hF'(x) + O(h^2)$,

$$\begin{aligned}
F(|u|^2) &= F\left(|f|^2 + \left[\left(g^* e^{\sigma^* z} + h e^{\sigma z} \right) f + \left(g e^{\sigma z} + h^* e^{\sigma^* z} \right) f^* \right] \right) \\
&\simeq F(|f|^2) + \left[\left(g^* e^{\sigma^* z} + h e^{\sigma z} \right) f + \left(g e^{\sigma z} + h^* e^{\sigma^* z} \right) f^* \right] F'(|f|^2) \,.
\end{aligned} \tag{4.13}$$

Hence,

$$\begin{aligned}
& F(|u|^2) u e^{-i\mu z} \\
&= F(|f|^2) f + \left[\left(g^* e^{\sigma^* z} + h e^{\sigma z} \right) f^2 + \left(g e^{\sigma z} + h^* e^{\sigma^* z} \right) |f|^2 \right] F'(|f|^2) \\
&+ F(|f|^2) g e^{\sigma z} + F(|f|^2) h^* e^{\sigma^* z} \\
&+ \left[\left(g g^* e^{(\sigma+\sigma^*)z} + g h e^{2\sigma z} \right) f + \left(g^2 e^{2\sigma z} + g h^* e^{(\sigma+\sigma^*)z} \right) f^* \right] F'(|f|^2) \\
&+ \left[\left(g^* h^* e^{2\sigma^* z} + |h|^2 e^{(\sigma+\sigma^*)z} \right) f + \left(g h^* e^{(\sigma+\sigma^*)z} + (h^*)^2 e^{2\sigma^* z} \right) f^* \right] F'(|f|^2) \\
&\simeq F(|f|^2) \left[f + g e^{\sigma z} + h^* e^{\sigma^* z} \right] \\
&+ F'(|f|^2) \left[\left(f^2 h + |f|^2 g \right) e^{\sigma z} + \left(f^2 g^* + |f|^2 h^* \right) e^{\sigma^* z} \right]
\end{aligned} \tag{4.14}$$

Substituting Eq. (4.10), (4.11) and (4.14) into Eq. (4.7) gives

$$\begin{aligned}
& i\left(\sigma g e^{\sigma z}+\sigma^* h^* e^{\sigma^* z}+i\mu f+i\mu g e^{\sigma z}+i\mu h^* e^{\sigma^* z}\right) e^{i\mu z} \\
& +a\left(f_{xx}+g_{xx} e^{\sigma z}+h_{xx}^* e^{\sigma^* z}\right) e^{i\mu z} \\
& +i\beta\left(f_{xxx}+g_{xxx} e^{\sigma z}+h_{xxx}^* e^{\sigma^* z}\right) e^{i\mu z} \\
& +\alpha\left\{\begin{array}{l}
F(|f|^2)\left[f+g e^{\sigma z}+h^* e^{\sigma^* z}\right] \\
+F'(|f|^2)\left[\left(f^2 h+|f|^2 g\right) e^{\sigma z}+\left(f^2 g^*+|f|^2 h^*\right) e^{\sigma^* z}\right]
\end{array}\right\} e^{i\mu z} \\
& +V_{\mathscr{P}\mathscr{T}}\left(f+g e^{\sigma z}+h^* e^{\sigma^* z}\right) e^{i\mu z}=0\ .
\end{aligned} \tag{4.15}$$

Grouping the terms and multiplying by $e^{-i\mu z}$ yields

$$\begin{aligned}
& \left[-\mu f+f_{xx}+i\beta f_{xxx}+\alpha F(|f|^2) f+V_{\mathscr{P}\mathscr{T}} f\right] \\
& +\left[i\sigma g-\mu g+a g_{xx}+i\beta g_{xxx}+\alpha F(|f|^2) g+\alpha\left(f^2 h+|f|^2 g\right) F'(|f|^2)+V_{\mathscr{P}\mathscr{T}} g\right] e^{\sigma z} \\
& +\left[i\sigma^* h^*-\mu h^*+a h_{xx}^*+i\beta h_{xxx}^*+\alpha F(|f|^2) h^*\right] e^{\sigma^* z} \\
& +\left[\left(f^2 g^*+\alpha|f|^2 h^*\right) F'(|f|^2)+V_{\mathscr{P}\mathscr{T}} h^*\right] e^{\sigma^* z}=0
\end{aligned} \tag{4.16}$$

Here, the first bracket is identically zero as f is a solution (see Eq. (4.9)). For Eq.(4.16) to hold true, the factors of the exponentials must be zero simultaneously. Hence, one has on one hand

$$\begin{aligned}
& i\sigma g-\mu g+a g_{xx}+i\beta g_{xxx}+\alpha F(|f|^2) g+\alpha\left(f^2 h+|f|^2 g\right) F'(|f|^2)+V_{\mathscr{P}\mathscr{T}} g \\
& =0
\end{aligned} \tag{4.17}$$

which can be rewritten as

$$\begin{aligned}
& a g_{xx}+i\beta g_{xxx}+\left[\alpha F(|f|^2)+\alpha F'(|f|^2)|f|^2-\mu+V_{\mathscr{P}\mathscr{T}}\right] g+\alpha F'(|f|^2) f^2 h \\
& =-i\sigma g
\end{aligned} \tag{4.18}$$

and on the other hand

$$\begin{aligned}
& i\sigma^* h^*-\mu h^*+a h_{xx}^*+i\beta h_{xxx}^*+\alpha F(|f|^2) h^* \\
& +\alpha\left(f^2 g^*+|f|^2 h^*\right) F'(|f|^2)+V_{\mathscr{P}\mathscr{T}} h^*=0
\end{aligned} \tag{4.19}$$

which can be rewritten as

$$\begin{aligned}
& a h_{xx}^*+i\beta h_{xxx}^*+\left[\alpha F(|f|^2)+\alpha F'(|f|^2)|f|^2-\mu+V_{\mathscr{P}\mathscr{T}}\right] h^*+\alpha F'(|f|^2) f^2 g^* \\
& =-i\sigma^* h^*
\end{aligned} \tag{4.20}$$

Taking the conjugate of Eq. (4.20) gives

$$\begin{aligned}
& a h_{xx}+i\beta h_{xxx}+\left[\alpha F(|f|^2)+\alpha F'(|f|^2)|f|^2-\mu+V_{\mathscr{P}\mathscr{T}}^*\right] h \\
& +\alpha F'(|f|^2)\left(f^2\right)^* g=i\sigma h.
\end{aligned} \tag{4.21}$$

Multiplying Eq. (4.21) by -1 gives

$$
\begin{aligned}
& -ah_{xx} - i\beta h_{xxx} - \left[\alpha F(|f|^2) + \alpha F'(|f|^2)|f|^2 - \mu + V^*_{\mathscr{PT}}\right] h \\
& -\alpha F'(|f|^2)\left(f^2\right)^* g = -i\sigma h
\end{aligned}
\tag{4.22}
$$

Writing Eq. (4.18) and (4.22) in matrix form yields

$$
i\begin{bmatrix} L_1 & L_2 \\ -L_2{}^* & -L_1{}^* \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix} = \sigma \begin{bmatrix} g \\ h \end{bmatrix}
\tag{4.23}
$$

where

$$
\begin{aligned}
& L_1 = a\partial_{xx} + i\beta\partial_{yy} + \alpha F(|f|^2) + \alpha F'(|f|^2)|f|^2 - \mu + V_{\mathscr{PT}} \\
& L_2 = \alpha F'(|f|^2) f^2 \; .
\end{aligned}
\tag{4.24}
$$

For the cubic nonlinearity,

$$
\begin{aligned}
& F(x) = Ax + Bx^2 \\
& F'(x) = A + 2Bx \; .
\end{aligned}
\tag{4.25}
$$

Using Eq. (4.25) in Eq. (4.24) yields

$$
\begin{aligned}
& L_1 = a\partial_{xx} + i\beta\partial_{xxx} + 2A|f|^2 - \mu + V_{\mathscr{PT}} \\
& L_2 = Af^2 \; .
\end{aligned}
\tag{4.26}
$$

by taking $B = 0$. If the soliton is real, Eq. (4.26) becomes

$$
\begin{aligned}
& L_1 = a\partial_{xx} + i\beta\partial_{xxx} + 2Af^2 - \mu + V_{\mathscr{PT}} \\
& L_2 = Af^2 \; .
\end{aligned}
\tag{4.27}
$$

Linear spectrum of numerically obtained solitons are found for the various values of β in order to examine the impact of the third order dispersion term to linear stability of solitons of Eq.(1.2).

It can be seen from the figures Fig. (4.5) - Fig. (4.7) that, the existence of β has a positive effect on linear stability of the solitons for the given potential depths since the first eigenvalue with a nonzero real part appears to be larger or even zero for nonzero 3OD cases(both negative and positive 3OD).

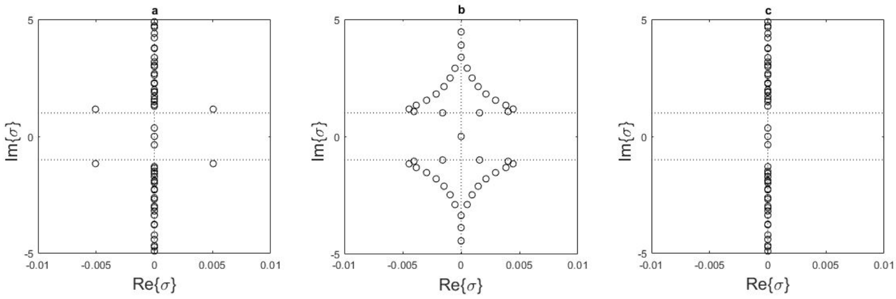

Figure 4.5 : Linear spectrum of CNLS equation for $\mu = 1$, $V_2 = 2.0$ and $W_1 = 0.4$ with $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$

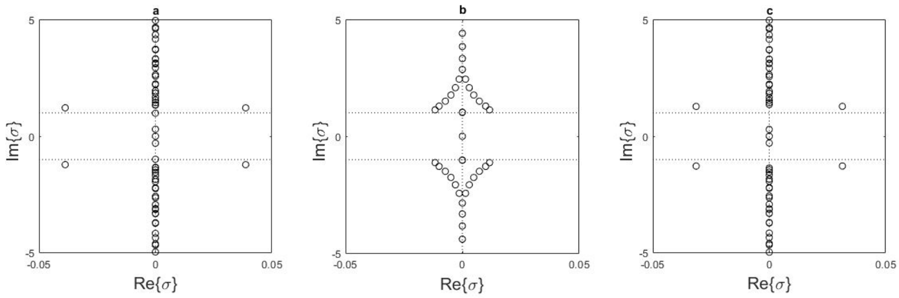

Figure 4.6 : Linear spectrum of CNLS equation for $\mu = 1$, $V_2 = 1.2$ and $W_1 = 1.4$ with $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$

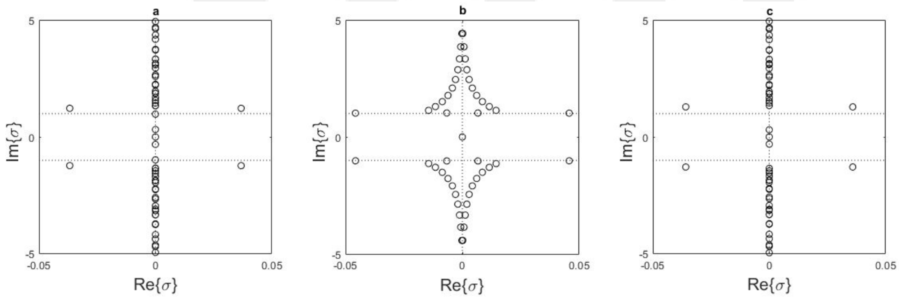

Figure 4.7 : Linear spectrum of CNLS equation for $\mu = 1$, $V_2 = 1.8$ and $W_1 = 1.8$ with $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$

On the other hand, for the specific potential depths $V_2 = 0.5$ and $W_1 = 0.2$ the soliton is not linearly stable for either $\beta = -0.1$ or $\beta = 0.1$ but soliton is linearly stable without 3OD.

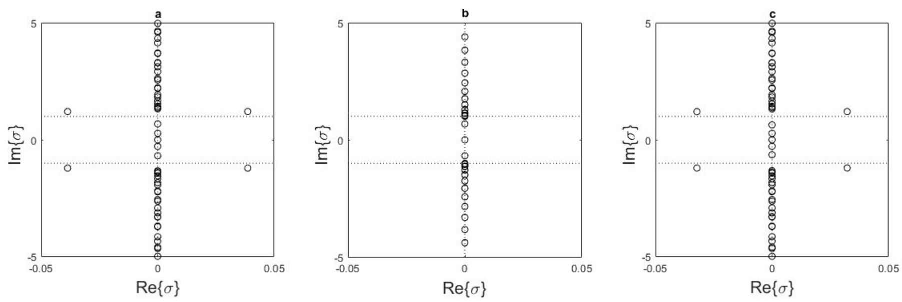

Figure 4.8 : Linear spectrum of CNLS equation for $\mu = 1$, $V_2 = 0.5$ and $W_1 = 0.2$ with $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$

5. CONCLUSION

In this thesis, we have explored NLS equation with an external $\mathscr{PT}$-symmetric potential and third order dispersion term. First, we have employed the well-know Spectral Renormalization method to the model equation and obtained numerical solutions. We have also found exact solutions of this equation by introducing an ansatz and specifying the structure of the $\mathscr{PT}$-symmetric potential. In this thesis, the $\mathscr{PT}$-symmetric potential is considered as an extension of Scarf II potential.

In order to prove that the analytical and the numerical solutions overlap we have depicted a figure comparing both aforementioned solitons. By the use of the Spectral Renormalization method, the numerical existence region for this model equation is plotted for varying potential depths V2, W1 and for various values of 3OD. It is observed that additional 3OD term either for negative or positive coefficient β does not have an effect on the existence region.

We have also investigated the linear and nonlinear stability properties of the obtained solitons and by using the Split-Step Fourier method. It is found that the positive 3OD coefficient enlarges the nonlinear stability region. For varying potential depths the nonlinear stability and/or instability of some specific solitons are shown and discussed with some figures.

In the last part of this thesis, the linear stability properties of the obtained solitons are also discussed. We studied the linear stability by analysing the linear spectrum. The results are illustrated by some figures. It is concluded that for certain region of potential depths adding a 3OD term to the system improves the linear stability of the obtained solitons. For future studies, considering the forth order dispersion (4OD) in addition to the third order dispersion (3OD) would be a more realistic model for data transmission in nonlinear optical models. One can also take Ramman effect into account for a more extended variant of NLS equation with this rich complex potentials.

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APPENDICES

APPENDIX A.1 : Fourier Transform

APPENDIX A.1

Fourier Transform

For a continuous, smooth and absolutely integrable function $f(x)$, the integral transform

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(k_x)x} dx \tag{A.1}$$

is called *the Fourier transform of f (x)* and conversely, the transform

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k_x)x} \mathrm{d}x \tag{A.2}$$

is called *the inverse Fourier transform of* $F(k_x)$.

The Fourier transform of f is denoted by $\mathscr{F}(f) = \hat{f}$, the inverse Fourier transform of $\hat{f}$ is denoted by $\mathscr{F}^{-1}(\hat{f})$ and clearly $\mathscr{F}^{-1}(\hat{f}) = \mathscr{F}^{-1}(\mathscr{F}(\hat{f}))$.
Integral transform methods are very useful for solving partial differential equations because of their properties such as linearity, shifting, scaling, etc.
Suppose that $f(x)$ tends to zero as x tends to infinity. Then,

$$\mathscr{F}\left(f'(x)\right) = ik_x \mathscr{F}(f(x)) \tag{A.3}$$

This result can be extended to obtain the differentiation property of the Fourier transform:

$$\mathscr{F}(f^n(x)) = (ik_x)^n (f(x)) = (ik_x)^n \hat{f}, \qquad n \in N \tag{A.4}$$

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