ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

RAMAN-INDUCED SOLITONS IN OPTICAL POTENTIALS

M.Sc. THESIS

Merve KURT

Department of Mathematical Engineering

Mathematical Engineering Programme

SEPTEMBER 2019

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ISTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

OPTİK POTANSİYELLER ALTINDA RAMAN ETKİLİ SOLİTONLAR

YÜKSEK LİSANS TEZİ

Merve KURT
(509151213)

Matematik Mühendisliği Anabilim Dalı

Matematik Mühendisliği Programı

Tez Danışmanı: Prof. Dr. Nalan ANTAR

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Merve Kurt, a M.Sc. student of ITU Graduate School of ScienceEngineering and Technology 509151213 successfully defended the thesis entitled "RAMAN-INDUCED SOLITONS IN OPTICAL POTENTIALS", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Prof. Dr. Nalan ANTAR** Istanbul Technical University	....
Jury Members :	**Prof. Dr. Nalan ANTAR** Istanbul Technical University	....
	Assoc. Prof. Dr. İlkay BAKIRTAŞ AKAR Istanbul Technical University	....
	Assist. Prof. Dr. Mahmut BAĞCI Istanbul Bilgi University	....

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To my family, my dear friends
and my dear love Irshad,

FOREWORD

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September 2019 Merve Kurt

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ABBREVIATIONS

1D	**:**	One Dimensional
2D	**:**	Two Dimensional
Eq.	**:**	Equation
Fig.	**:**	Figure
NLS	**:**	Nonlinear Schrödinger
$\mathscr{P}\mathscr{T}$	**:**	Parity - Time

LIST OF FIGURES

RAMAN-INDUCED SOLITONS
IN OPTICAL POTENTIALS

SUMMARY

The optical solitons refer to special kind of wave packet that propagate a long distance without distortion due to the balance of the nonlinear and dispersive effects in the medium. Thus, the optical solitons have become a topic of current research in the long-distance fiber communication.

It is well known that propagation of short pulses can be described by the nonlinear Schrödinger equation. But in the femtosecond regime the standard Nonlinear Schrödinger equation becomes inadequate and several higher order effects such as the third order dispersion (TOD), self-steeping and the stimulated Raman scattering (SRS). The stimulated Raman scattering is the most important nonlinear effect that occur in the optical fibers. In nonlinear optic, the dispersion of light can be modelled by the Nonlinear Schrödinger equation with Raman effect. Raman effect is change in the magnitude of wave of light that happens when a light photon is redirected by molecules. A small light refraction is dispersed at optical energies and at the most of the times lower than the energy of the incident photons. This inelastic process of disperse of light is defined as the Raman Effect. In other words, Raman Scattering, can happen when molecule's energy changes as vibrational, rotational or electronically. If it is elastic process of disperse of light, it is called Rayleigh scattering. In 1928, V.C. Raman, Indian physicist, explored Raman scattering in the other words Raman Effect, later he deserved a Nobel prize for his study in 1930.

In this thesis, we demonstrate the existence and the stability properties of Raman-induced solitons governed by Nonlinear Schrödinger equation with the stimulated Raman effect and the external potentials. The external potentials can be assumed $\mathscr{PT}$-periodic-symmetric, $\mathscr{PT}$-symmetric and Non-$\mathscr{PT}$-symmetric potentials.

The Raman induced solitons are governed by the solutions of the NLS equation for dimensionless envelope $u(x,z)$ of the electromagnetic wave, including the stimulated Raman effect. The nonlinear Schrödinger equation with the Raman effect and the external potential can be given as

$$iu_z + u_{xx} + |u|^2u + \tau u\left(|u|^2\right)_x + V_x u = 0. \tag{1}$$

In given equation, $u(x,z)$ yields to the complex-valued function, u_{xx} yields to diffraction, τ is a complex constant which refers Raman Effect and V_x is an external potential.

As far as we know, the exact soliton solutions for the NLS equation including the Raman effects and periodic external potential have not been investigated up to now. By choosing the special forms of the external potentials possessing the $\mathscr{PT}$ or non-$\mathscr{PT}$ symmetry, we can obtain the Raman induced solitons as analytically. The $\mathscr{PT}$ and non-$\mathscr{PT}$ symmetric potentials can be given in the following from and analytical solutions of NLS equations:

$$\begin{aligned} V_{\mathscr{PT}}(x) &= V(x)+iW(x) = V_0 sech^2(x) + i\left[W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x)\right], \\ V_{\mathscr{PT}}(x) &= V(x)+iW(x) = \left[V_0 cos^2(x)\right] + i[W_0 sin(2x)], \\ V_{\mathscr{PT}}(x) &= V(x)+iW(x) = \left[V_0 sech^2(x) + V_1 sech^2(x) tanh(x)\right] \\ &\qquad + i\left[W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x)\right] \ , \end{aligned} \tag{2}$$

$$u(x,z) = \sqrt{2+\frac{{W_0}^2}{9} - V_0 sech(x) e^{i\left(z+\frac{W_0}{3} arctan(sinh(x))\right)}} \ , \tag{3}$$

by assuming the solution is the following form,

$$u(x,z) = f(x) e^{i[\mu z + g(x)]} \tag{4}$$

here $f(x)$ and $g(x)$ are real-valued functions.

In addition, in order to obtain the numerical solutions of NLS equation with Raman effect and with external potentials, we introduce the pseudo-spectral renormalization method [1] which based on the fixed point iteration [2]. Numerical and analytical solutions compared with each other and it is found that numerical solutions converge to analytical solutions. In order to obtain convergency, some conditions are found based on external potentials.

One dimensional NLS equation with Raman Effect without potential, it is observed that if the real part of τ of Raman Effect term is getting to zero value, soliton solutions convergency has also increased. It is also observed that if the complex part of τ of Raman Effect term is getting to zero value, and the real part of τ s equal to zero, soliton solutions are also convergent. But as we increased the value of the real part of τ and the value of the complex part of τ is equal to zero, then soliton solutions of NLS equation have not been found.

It is obtained that the imaginary part (W_0) of the external potential or the real part of τ needs to be different from zero, in order to get soliton solutions of NLS equation with the $\mathscr{PT}$-periodic symmetric potential, are consistent with the analytical solution. If the imaginary part of the external potential (W_0) and the real part of τ are taken as a zero, soliton solutions cannot be found. It is very easy to show that the existence of Raman-induced solitons depends on the propagation constant μ and the real part of the external potential (V_0). In order to get soliton solutions, the real part of the external potential (V_0) should be greater than or equal to the propagation constant μ.

We obtained the existence of Raman-induced soliton under $\mathscr{PT}$-symmetric potential, the real part of τ value should be taken as a zero. If the real part of τ value is not taken as a zero, $\mathscr{PT}$ symmetry conditions cannot be satisfied and the potential will not be $\mathscr{PT}$-symmetric, but becomes Non-$\mathscr{PT}$-symmetric.

The pseudospectral method has been used to analyze the nonlinear stability of obtained Raman-induced solitons. The effect of Raman term on the stability of the solitons is investigated.

It is seen that Raman Effect term has not much impact to the stability of solitons. If NLS equation has a higher-order nonlinearity, then the Raman Effect term will have much impact to stability of the soliton solutions. We observed that all Raman-induced solitons are nonlinearly stable.

OPTİK POTENSİYELLER ALTINDA
RAMAN ETKİLİ SOLİTONLAR

ÖZET

Optik solitonlar, ortamdaki doğrusal olmayan ve dağıtıcı etkilerin dengelenmesi sayesinde oluşan bozulmalar olmaksızın, uzun mesafede yayılan dalga paketinin özel bir türünü ifade eder. Bu sebeple, optik solitonlar, uzun mesafe fiber iletişimde mevcut araştırmaların konusu olmuştur.

Kısa sinyallerin yayılımlarının doğrusal olmayan Schrödinger denklemi ile tanımlanabileceği iyi bilinen bir gerçektir. Ancak, nanosayinenin milyonda biri düzeyinde, standart doğrusal olmayan Schrödinger denklemi yetersiz olur ve bazı yüksek mertebeden üçüncü mertebeli yayılım (ÜMY), kendi kendine dik posizyona gelmesi ve uyarlanmış Raman saçılması (URS) gibi etkileri olur. Uyarlanmış Raman saçılımı optik fiberlerde meydana gelen doğrusal olmayan en önemli etkidir. Doğrusal olmayan optikte, ışığın yayılması Raman etkili doğrusal olmayan Schrödinger denklemi ile modellenebilmektedir. Moleküller tarafından ışık fotonunun yönü değiştirilirken oluşan dalga boyundaki değişiklik, Raman etkisidir. Ufak miktarda ışığın kırılması optik enerjide yayılır ve çoğu zaman gelen ışının enerjisinden daha az olur. Işığın yayılmasının esnek olmayan bu süreci Raman Etkisi olarak tanımlanır. Bir diğer deyişle Raman saçılması, molekül enerjisinin titreşimsel, dönüşsel ve elektronik olarak değişmesi ile meydana gelebilir. Eğer ışık yayılım süreci esnek ise, buna Rayleigh saçılması denir. 1928'de hint fizikçi, V.C. Raman, Raman saçılması, diğer adı ile Raman etkisini keşfetmiştir, daha sonra 1930'da bu çalışmasından dolayı Nobel ödülünü almayı hak etmiştir.

Bu tez çalışmasında, uyarılmış Raman etkili ve dış potansiyelli doğrusal olmayan Schrödinger denkleminden türetilen Raman kaynaklı solitonların varlığını ve kararlılık özelliklerini gösterdik. Dış potansiyeller $\mathscr{PT}$-periyodik simetrik, $\mathscr{PT}$-simetrisi özelliği olan ve $\mathscr{PT}$-simetrisi özelliği olmayan potansiyeller olarak kabul edilebilir.

Raman kaynaklı solitonlar, uyarılmış Raman etkisini içererek, elektromanyetik dalganın boyutsuz sarmalı $u(x,z)$ için doğrusal olmayan Schrödinger denkleminin çözümleri olarak türetilir. Raman etkili ve dış potansiyelli doğrusal olmayan Schrödinger denklemi aşağıdaki gibi verilebilir

$$iu_z + u_{xx} + |u|^2 u + \tau u\left(|u|^2\right)_x + V_x u = 0. \tag{5}$$

Verilen denklemde, $u(x,z)$ karmaşık değerli fonksiyonu ifade etmektedir, u_{xx} yayılımı ifade etmektedir, τ Raman etkisini ifade eden karmaşık olan sabit bir sayıdır ve V_x dış potansiyeli temsil etmektedir.

Bildiğimiz kadarı ile, bu zamana kadar Raman etkisi ve periyodik potansiyel içeren doğrusal olmayan Schrödinger denklemi için tam soliton çözümleri bulunamamıştır. $\mathscr{PT}$ ve $\mathscr{PT}$ olmayan simetrik özelliği olan dış potansiyelinin özel bir formu seçilerek, analitik olarak Raman kaynaklı solitonlar elde edebiliriz. $\mathscr{PT}$ ve $\mathscr{PT}$ olmayan simetrik potansiyelleri ve doğrusal olmayan Schrödinger denkleminin analitik çözümü aşağıdaki formda verilebilir:

$$\begin{aligned} V_{\mathscr{PT}}(x) &= V(x)+iW(x) = V_0 sech^2(x) + i\left[W_0 sech(x)tanh(x) + W_1 sech^2(x)tanh(x)\right], \\ V_{\mathscr{PT}}(x) &= V(x)+iW(x) = \left[V_0 cos^2(x)\right] + i[W_0 sin(2x)], \\ V_{\mathscr{PT}}(x) &= V(x)+iW(x) = \left[V_0 sech^2(x) + V_1 sech^2(x)tanh(x)\right] \\ &\qquad + i\left[W_0 sech(x)tanh(x) + W_1 sech^2(x)tanh(x)\right] \ , \end{aligned} \tag{6}$$

$$u(x,z) = \sqrt{2+\frac{{W_0}^2}{9} - V_0 sech(x) e^{i\left(z+\frac{W_0}{3}arctan(sinh(x))\right)}} \ , \tag{7}$$

aşağıdaki gibi çözüm önerisi verilerek hesaplanmıştır,

$$u(x,z) = f(x)e^{i[\mu z + g(x)]} \tag{8}$$

burada $f(x)$ ve $g(x)$ gerçek değerli fonksiyonlardır.

Buna ek olarak, Raman etkili ve dış potansiyelli doğrusal olmayan Schrödinger denkleminin sayısal çözümlerini elde etmek için, sabit nokta iterasyonuna dayanan pseudo-spektral renormalizasyon yöntemini kullandık. Sayısal çözümler kesin çözümlerle karşılaştırılmış ve sayısal çözümlerin analitik çözümlere yakınsadığı bulunmuştur.

Potansiyel olmayan birinci derece Raman etkili NLS denkleminin, terim içerisinde bulunan τ karmaşık sayı sabitinin gerçek kısmının değeri sıfıra yaklaştıkça, yakınsaklığın arttığı gözlemlenmiştir. Eğer Raman etkisi terim içerisindeki τ sabitinin karmaşık kısmının değeri sıfıra yaklaştığında ve τ sabitinin değeri 0'a eşit olduğunda soliton çözümlerinin yakınsak olduğu gözlemlenmiştir. Ancak τ sabitinin gerçek kısmının büyük değerleri ve sanal kısmının sıfır değeri için soliton tipi çözümler bulunamamıştır.

$\mathscr{PT}$-periyodik simetrik potansiyelli ve Raman etkili NLS denkleminin soliton tipi çözümlerini elde etmek için dış potansiyelin sanal kısmının (W_0) veya τ sabitinin gerçek kısmının sıfırdan farklı olması gerektiği, böylece çözümlerin analitik çözümle uyumlu olduğu elde edilmiştir. Eğer dış potansiyelin sanal kısmı (W_0) ve τ sabitinin gerçek kısmı sıfır olarak alınırsa, soliton tipi çözümler bulunamamıştır. Raman kaynaklı solitonlar yayılım sabiti μ değerine ve dış potansiyelin gerçek kısmının (V_0) değerine bağlı olduğu kolaylıkla görülmektedir. Soliton tipi çözümler elde etmek için dış potansiyelin gerçek kısmının (V_0) yayılım sabiti μ değerinden daha büyük veya eşit olmalıdır.

$\mathscr{PT}$-simetrik potansiyel altında Raman kaynaklı solitonların varlığını elde ettik, τ sabitinin gerçek kısmı sıfır olarak alınmalıdır. Eğer τ sabitinin gerçek kısmı sıfır olarak alınmaz ise, $\mathscr{PT}$ simetri koşulları sağlanmaz ve potansiyel $\mathscr{PT}$-simetrik olmaz, ancak $\mathscr{PT}$ olmayan simetrik olur.

Elde edilen Raman kaynaklı solitonların doğrusal olmayan kararlılıkları pseudospektral yöntemi ile analiz edilmiştir. Raman teriminin solitonların kararlılıklarına olan etkisi sorgulanmıştır.

Raman etkisi teriminin solitonların kararlılıklarına çok büyük bir etkisi olmadığı görülmüştür. Eğer NLS denklemi yüksek dereceden bir doğrusal olmayan mertebesi var ise, Raman etkisi teriminin soliton çözümlerinin kararlılığına etkisi olacaktır. Raman kaynaklı tüm elde edilen solitonların doğrusal olmayan ve kararlı solitonlar olduklarını gözlemledik.

1. INTRODUCTION

In the past decades, nonlinear knowledge have been comprehensively improved to discover the riverting angle of nonlinear systems. Nonlinear knowledges are not new issue of science, even though they gives considerably a new notion and outstanding conclusions. It is easy to analyse nonlinear phenomena in quite variant systems with the suitable experimental equipments. The all scope of nonlinear science can be considered into some categories, such as fractals, solitons, complex systems etc.

Solitons are nonlinear waves which localized and take place in several scopes of physics [1]. The basic comprehension of complicated nonlinear systems have been figured out by soliton features. Nonlinear Schrödinger equation (NLS) is described by the nonlinear dynamics of waves. In 1927, Erwin Schrödinger discovered NLS equation [2]. Many amount of studies have been made about NLS equations with different nonlinearities and different analytical and numerical methods have been used to solve the problems.

In optical science, the dispersion distance z occurs of the time parameter t of quantum mechanics. From this point of view, NLS type of equations are used in order to pattern $\mathscr{PT}$-symmetric structures. Many studies have been made about the diffusion of electromagnetic waves in photorefractive substances. For this purpose, optical solitons investigated on $\mathscr{PT}$-symmetric lattices as a solution of two-dimensional NLS equation with cubic and quintic terms [3].

The most of the photons are elastically dispersed when the light is dispersed from the cyrstal or the molecule. The dispersed photons have the same frequency and the same length of wave like the incident photons. A small light refraction is dispersed at optical energies and at the most of the times lower than the energy of the incident photons. This inelastic process of disperse of light is defined as the Raman Effect. In other words, Raman Scattering, can happen when molecule's energy changes as vibrational, rotational or electronically. If it is elastic process of disperse of light, it is called Rayleigh scattering [4]. In 1928, V.C. Raman, Indian physicist, explored

Raman scattering in the other words Raman Effect, later he deserved a Nobel prize for his study in 1930 [5].

Many researches have been proven higher-order dissipative and nonlinear impacts in NLS equation [6]. In nonlinear optics, Raman scattering which is a term in NLS equation as $u\left(|u|^2\right)_x$, has a significant role among the higher-order nonlinear impacts. Raman effects on solitons which generated by the Raman scattering term in NLS equation, were experientially obtained in 1985 [7]. It became a phenomenon as soliton self-frequency rotation and then investigations have been widely made about Raman scattering and higher-order nonlinear impacts [7]. Raman scattering induces to deceleration of the soliton forwarder frequency when the pulse spectrum occurs so prevalent which defines a energy flow from the high-frequency components of a pulse to the low frequency components of the identical pulse through raman amplification. It is proven that if NLS equation has Raman effect term, then pulse-like solutions cannot be obtained and Raman result has a delayed nature.

1.1 Purpose of Thesis

In this thesis, investigating of the existence of the Raman-induced solitons and their nonlinear stability analysis of NLS equation with Raman effect and periodic, $\mathscr{PT}$-symmetric and Non-$\mathscr{PT}$-symmetric potential is aimed.

1.2 Literature Review

Solitons are obtained as a result of solutions of nonlinear dissipative partial differential equations which defining pyhsical systems [8]. Soliton theory has been developed since 'soliton' was defined by Zabusky and Krusal in 1965 in [9].

The nonlinear optics is one of the best area to investigate the optical solitons. Optical solitons, temporal optical solitons, spatial optical solitons are derived from the balance between nonlinearity and dispersion or diffraction. It is well-known that the perturbed nonlinear Schrödinger equation can be used to describe the propagation of the short optical pulses. The perturbed terms include the higher-order effects, the third-order dispersion self-steeping and the stimulated Raman scattering. The Raman scattering effect is the most important effect in the optical fiber communication.

Since the refractive index of the optical solitond can be complex, it is very important to investigate the light propagation in optics governed by the NLS equation with real external potentials or gain and loss distributions [10]- [13].

In optics, optical mode which means soliton, indicates to any optical field which does not evolve during dispersion by reason of a delicate equilibrium between linear and nonlinear impacts in the medium.

Bender and Boettcher showed that necessary but not sufficient condition for Hamiltonians to be $\mathscr{PT}$ symmetric is that real part of $\mathscr{PT}$ potentials should be even functions of position and the imaginary part of the potentials should be odd [14]. In optics, there have been a growing interest in $\mathscr{PT}$ symmetry because of $\mathscr{PT}$ optical complex potentials can be seen both theorically and experimentally.

In 2008, Musslimani et al, studied the optical solitons in $\mathscr{PT}$ symmetric optical potential theoretically [15]. The existence and stability of $\mathscr{PT}$-symmetric optical solitons have been widely studied.

Spectral renormalization method is a form of Fourier iteration method. This method was asserted by Petviashvili in [16]. In [17], numerical approximations of localized solitons in periodic potential is shown by spectral renormalization method.

Spectral renormalization method has improved to pseudospectral renormalization method for using in various nonlinearities in [18]. Pseudospectral renormalization method is based on inverse fourier transform for nonlocal terms in NLS. Method has applied to (2+1) dimensional NLS with the qubic and quintic nonlinear terms.

It is explained that in [19], the $\mathscr{PT}$-symmetric nonlinear lattices supported existence of localized solitons. In [20], by using spectral renormalization method, the solutions of saturable NLS equation with an external periodic and Penrose type potentials are obtained. It was investigated that the existence of solutions and stability of solitons in periodic and quasicrystal lattices.

It is investigated the existence of exact solutions for bright and dark solitons in weakly nonlocal media and with the cubic and quintic nonlinear terms in [21]. It is shown that when the solitons are unstable in local media, nonlocal effects can make them stable.

In [22], it is observed bright and dark solitons for the higher-order NLS with Raman Effect and with the cubic, quintic and septic terms. Raman effect and self-steepening terms investigated seperately, and it is proven that Raman Effect term is more dominant than self-steepening term.

It is reported conclusions of the analytical and numerical works for the modulational instability of continuous wave in NLS with the pseudo stimulated Raman scattering term in [6] and found that the modulation is able to control the multi-soliton patterns which are found by modulation instability in stable or unstable form. Modulation instability defines exponential of perturbations joint to continuous wave because of the nonlinearity.

1.3 Hypothesis

The existence of Raman-induced solitons and the linear and nonlinear stability properties are investigated. We found that all Raman-induced solitons are nonlinearly stable.

2. SOLITARY WAVES AND SOLITONS

The soliton means as a defition of a word that special kind of wave which spreads unvaried for long distances. Because of the cancelation of dissipative and nonlinear impacts in the medium. John Scott Russel observed solitons in 1834, in Edinburgh canal where a horse was pulling a boat after it stopped, a water wave created. Scott Russel followed that created water wave and he noticed that the wave was moving with a constant speed and conserving its magnitude. He lost the wave after it moved couple of miles in the turning of the canal [23]. After this observation, Scott Russel did some experiment and he figured out following results [24]. It is called Solitary waves if the wave holds following features. A nonlinear solitary waves called solitons with second feature below that they conserves their structure, even after being in interaction with other soliton [25].

- Waves are not dispersive which means they conserves their shapes and sizes
- After collision of two diferrent waves, they still conserves their shapes and sizes
- Velocity of the waves are constant even after collision

There are several examples for the traceable solitons, such as solitary waves in water, on the surface of the sea or deep inside of the water and for atmospheric solitons as an example Morning Glory clouds. Korteweg and de Vries who are two Physicists in Holland, 1895 invented the KdV equation that defines the dynamics of solitary waves in water [26]. In 1965, Zabusky and Kruskal numerically calculated the solution of the KdV equation [9]. They obtained that calculated numerical solutions came into collision one another and conserved their shapes and velocity after the collision. Therefore, Zabusky and Kruskal defined the waves as solitons.

Zakharov and Shabat who are russian scientists, discovered solitons in optical fibers in 1971 [27]. In 1973, they calculated the Nonlinear Schrödinger Equation (NLS) by using the inverse scattering method [28]. Hasegwa and Tappert noticed that NLS conducts the pulse diffusing in optical fibers in 1973 [29]. Moreover, Mollenauer and Smith experientially found the same kind of solitons in 1988. They transferred the soliton pulses more than 2500 miles by using Raman effect in order to ensure optical gain in the fiber [30].

In several subjects of physics, scientists comprehensively work on solitons such as optics, plasmas, condensed matter physics, uid mechanics, particle physics and even astrophysics. Optical transmission systems which are based on solitons, can take advantage of over range of many thousands of miles with enormous information conveying capacity by using optical amplifiers. Solitons come in the one of the essential technologies in the present transmission reformation since it is noticed that the transfer of data is faster without any decreasement. Moreover, solitons are significant in optics because they are implemented to communication systems in order to obtain high velocity data transmission and optical changing.

2.1 Optical Solitons

The surround of light waves for which the nonlinear Schrödinger equation (NLS) defined some fundamental features, is called optical solitons [31]. The optical solitons have been theorically and experimentally studied since they are helpfully applied in the area of fiber-optic communications. Optical solitons are developed from a nonlinear change in the refractive list of a substance caused by the light field. This change in the refractive list of a substance because of an applied field is called optical Kerr effect. The Kerr effect, the density subjection of the refractive list, induces to nonlinear impacts responsible for soliton generation in an optical medium.

A bunch of a optical wave, naturally tends to disperse in a medium, either because of chromatic diffusion or because of spatial refraction. Such a bunch of wave, in time or in space or both, is called an optical soliton.

2.2 Spatial and Temporal Solitons

The optical solitons can be categorized as spatial or temporal depending on the restriction of light in space or time during diffusion. If the electromagnetic field is localized in time, then solitons are temporal, and the dispersion in a medium is because of chromatic diffusion. If the optical field is localized in the transverse directions, then solitons are spatial, and the dispersion in a medium is because of spatial refraction. The temporal solitons' dynamics can be defined by NLS equation. The spatial solitons' dynamics can be defined by normalized NLS equation. Hasegawa and Tappert are made researches about the existence of temporal solitons [29], and Mollenauer experimented with temporal solitons [30]. Ashkin and Bjorkholm made an experiment about optical spatial solitons in 1974 [32].

The spatial self-focusing (or self-defocusing) of optical rays and temporal self-phase modulation (SPM) of pulses nonlinearly effect to the improvement of spatial and temporal solitons in a nonlinear optical medium. When self-focusing of an optical ray exactly atone the propagating because of diffraction, it results to the generation of spatial soliton, and a temporal soliton is generated when SPM balances the effect of dispersion-caused expanding of an optical pulse. The wave diffused without any differences in its shape and is called as self-trapped. Chiao, Garmire and Townes invented the spatial soliton named as self-trapping of optical ray in a nonlinear medium in 1964 [33]. McCall and Hahn observed the temporal soliton named as self-caused trapping of optical pulses in nonlinear medium [34].

In NLS equation, there are the lower order velocity diffraction term and nonlinear cubic term, these terms give balance to optical solitons. Optical solitons have a inclination to disperse either by reason of chromatic or spacial diffraction.

2.3 Bright Solitons

It is mentioned before that solitons are special form of the solutions of NLS equation. Standard NLS equation can be written as

$$iu_t + u_{xx} \pm |u|^2 u = 0 \ , \tag{2.1}$$

where if the sign of nonlinear term is $(+)$, it is self-focusing nonlinearity, if the sign is $(-)$, it is self-defocusing nonlinearity. Solutions of NLS equation can be calculated as bright soliton for the self-focusing case of NLS. It is called bright soliton that it vanishes to background status at infinity. The common form of bright soliton of NLS equation as follows [35]:

$$u(x,z) = asech[a(t - vx)]e^{i\left(vt + \left(a^2 - v^2\right)x/2\right)} \ . \tag{2.2}$$

a refers to the magnitude of soliton and v represents the speed of diffusing soliton. The bright soliton for NLS equation can be seen in Fig.(2.1) for some values of a and v.

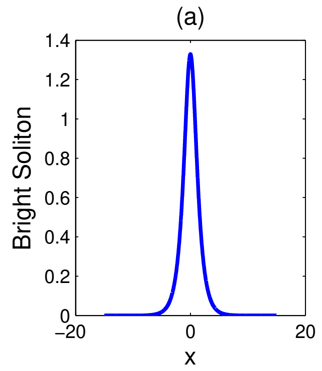

Figure 2.1 : (a) Bright Soliton of NLS equation

2.4 Dark Solitons

Solutions of NLS equation can be calculated as dark soliton for the self-focusing case of NLS. It is called dark soliton that it does not vanish to background at infinity. The common form of dark soliton of NLS equation as follows [35]:

$$u(x,z) = u_0[\beta tanh(u_0\beta(t - au_0x))ia]e^{-iu_0{}^2x} \quad . \tag{2.3}$$

u_0 refers to continuos-wave background and $a^2 + \beta^2 = 1$, here $a = sin\phi$ and $\beta = cos\phi$ and ϕ is single constant that 2ϕ angle refers to the total degree of rotation across the dark soliton. The dark soliton for NLS equation is shown in Fig.(2.2) for some values of u_0, a, β and ϕ.

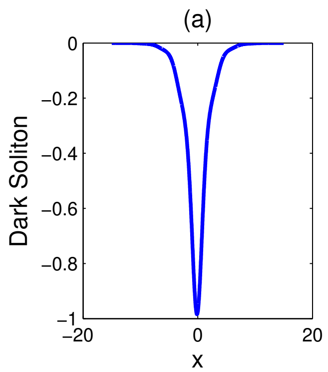

Figure 2.2 : (a) Dark Soliton of NLS equation

3. OPTICAL LATTICES

3.1 $\mathscr{P}\mathscr{T}$-Symmetry

All the phsical mesurements rely on a real quantity. In quantum mechanics, measurements match up with eigenvalues of operators. Therefore, all the eigenavalues of operators require to be real.

Take the Hamiltonian operator $\widehat{H}$:

$$\widehat{H} = \widehat{T} + \widehat{V} = \frac{\widehat{p}^2}{2m} + \widehat{V}(\widehat{x}) \tag{3.1}$$

where $\widehat{T}$ is the sum of the kinetic energy operator, $\widehat{V}$ is potential energy operator, $\widehat{p}$ is the momentum operator, m is the mass and $\widehat{x}$ is the position operator.

Real eigenvalues of Eq.(3.1) conform with a real energy spectrum. In order to assure a real spectrum, it was supposed that all measurements conformed with eigenvalues of Hermitian (i.e. selfjoint) operators by looking back the result from linear algebra that Hermitian matrices have real spectra. In fact, a Hermitian Hamiltonian provides a real energy spectrum. Nervetheless, analyzed non-Hermitian Hamiltonians and discovered that the most of them have completely real spectra given that they are named the parity-time ($\mathscr{P}\mathscr{T}$) symmetry property [14]. Moreover, they indicated in the most of the cases a threshold value above which spectrum becomes complex.

$\mathscr{P}\mathscr{T}$-Symmetry is defined by the parity operator $\widehat{\mathscr{P}}$ and the time operator $\widehat{\mathscr{T}}$ whose actions are given below:

$$\begin{aligned} &\mathscr{P} : \widehat{p} \to -\widehat{p}, \widehat{x} \to -\widehat{x} && (\mathscr{P}(a\psi + b\phi))(\mathbf{x}) = a\psi(-\mathbf{x}) + b\phi(-\mathbf{x}) \\ &\mathscr{T} : \widehat{p} \to -\widehat{p}, \widehat{x} \to -\widehat{x}, i \to -i && (\mathscr{T}(a\psi + b\phi))(\mathbf{x}) = a^*\psi^*(\mathbf{x}) + b^*\phi^*(\mathbf{x}) \end{aligned} \tag{3.2}$$

[36, 37]. A Hamiltonian is called $\mathscr{P}\mathscr{T}$-symmetric is it has the same eigenfunctions as

the $\mathscr{P}\mathscr{T}$ operator and satisfies as follows:

$$\widehat{\mathscr{P}}\widehat{\mathscr{T}}\widehat{H} = \widehat{H}\widehat{\mathscr{P}}\widehat{\mathscr{T}}. \tag{3.3}$$

Firstly,

$$\begin{aligned}
(\mathscr{P}\mathscr{T}H)(f(\mathbf{x},t)) &= (\mathscr{P}\mathscr{T})\left(\frac{p^2}{2m}f(\mathbf{x},t) + V(\mathbf{x})f(\mathbf{x},t)\right) \\
&= \mathscr{P}\left(\frac{(-p)^2}{2m}f^*(\mathbf{x},t) + V^*(\mathbf{x})f^*(\mathbf{x},t)\right) \\
&= \frac{p^2}{2m}f^*(-\mathbf{x},t) + V^*(-\mathbf{x})f^*(-\mathbf{x},t)
\end{aligned} \tag{3.4}$$

and secondly,

$$\begin{aligned}
(H\mathscr{P}\mathscr{T})(f(\mathbf{x},t)) &= (H\mathscr{P})(f^*(\mathbf{x},t)) \\
&= H(f^*(-\mathbf{x},t)) \\
&= \frac{p^2}{2m}f^*(-\mathbf{x},t) + V^*(\mathbf{x})f^*(-\mathbf{x},t).
\end{aligned} \tag{3.5}$$

The necessary but not sufficient condition Eq.(3.3) states

$$\left.\begin{aligned}
\widehat{H}\widehat{\mathscr{P}}\widehat{\mathscr{T}} &= \frac{\widehat{p}^2}{2m} + V(\mathbf{x}) \\
\widehat{\mathscr{P}}\widehat{\mathscr{T}}\widehat{H} &= \frac{\widehat{p}^2}{2m} + V^*(-\mathbf{x})
\end{aligned}\right\} \Rightarrow V(\mathbf{x}) = V^*(-\mathbf{x}). \tag{3.6}$$

Let us consider the complex potential as

$$V_{\mathscr{P}\mathscr{T}}(\mathbf{x}) = V(\mathbf{x}) + iW(\mathbf{x}) \quad , \quad V,W \in \mathbb{R}^n. \tag{3.7}$$

And

$${V_{\mathscr{P}\mathscr{T}}}^*(-\mathbf{x}) = V^*(-\mathbf{x}) - iW^*(-\mathbf{x}) = V(-\mathbf{x}) - iW(-\mathbf{x}), \tag{3.8}$$

the real part of the potential, $V(\mathbf{x})$ needs to be an even function and the complex part of the potential, $W(\mathbf{x})$ needs to be an odd function so that Eq.(4.6) satisfies [38].

3.2 Optical Lattices with $\mathscr{PT}$ and Non-$\mathscr{PT}$ Symmetry

In this thesis, we will consider the optical lattice of the following form:

$$\begin{aligned} V(x) &= V_0 sech^2(x) + V_1 sech^2(x) tanh(x) \\ W(x) &= W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x) \end{aligned} \tag{3.9}$$

with the real parameters V_0 , V_1 , W_0 and W_1. Here V_i $(i = 0, 1)$ and W_i $(i = 0, 1)$ refer to the depths of the real and complex part of the optical lattice. $V(x)$ and $W(x)$ describe the real-valued external lattice and gain-loss distribution respectively. According to values of parameters, we will define even and odd functions of potential, moreover; it will be defined $\mathscr{PT}$ and Non-$\mathscr{PT}$ Symmetric potentials.

3.2.1 $\mathscr{PT}$-Symmetric Optical Lattices

In this section, the $\mathscr{PT}$ symmetric optical lattice $(V(x) + iW(x))$ requires that the real and imaginary parts should satisfy $V(x) = V(-x)$ and $W(-x) = -W(x)$. In order to obtain the $\mathscr{PT}$ symmetry optical lattice for NLS equation with Raman effect, it will be proved that it must be $V_1 = 0$. In this case, $V(x)$ is an even function and $W(x)$ is an odd function which satisfy the $\mathscr{PT}$-symmetry requirements, hence $\mathscr{PT}$-symmetric potential is in the following form:

$$\begin{aligned} V(x) &= V_0 sech^2(x) \\ W(x) &= W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x) \end{aligned} \tag{3.10}$$

In Fig. (3.1), we plotted the real-valued external lattice and the gain-loss distribution $W(x)$.

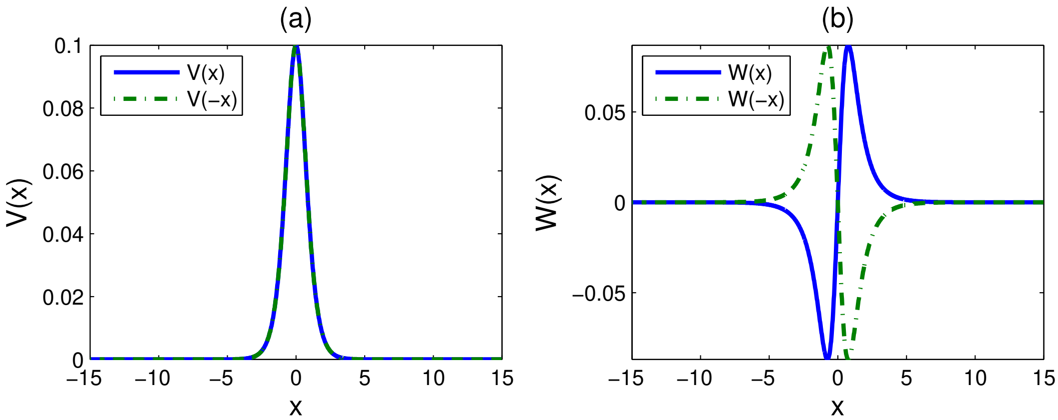

Figure 3.1 : (a) Real part of the $\mathscr{PT}$-Symmetric Potential, (b) Imaginary part of the $\mathscr{PT}$-Symmetric Potential, Eq.(3.9) with $V_0 = W_0 = W_1 = 0.1$.

3.2.2 Non-$\mathscr{PT}$-Symmetric Optical Lattices

In this section, we will consider non-$\mathscr{PT}$-symmetric potential.

$$V_{\mathscr{PT}}(x) = V(x) + iW(x) = \left[V_0 sech^2(x) + V_1 sech^2(x) tanh(x)\right] + i\left[W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x)\right] \quad (3.11)$$

where $V_i(i = 0, 1)$ and $W_j(j = 0, 1)$ refer to depths of the real and complex parts of the potentials, respectively.

If we take $V_1 \neq 0$ in the (3.10), $V(x)$ will not become an even function. Hence, the potential will not be $\mathscr{PT}$-Symmetric potential, but it will be Non-$\mathscr{PT}$-Symmetric Potential.

The $\mathscr{PT}$ symmetry require $V(x)$ should be an even function. In order to show Eq.(3.10) do not satisfy the $\mathscr{PT}$ symmetry requirement we plotted the real valued of the optical lattice, $V(x)$ and gain-loss distribution $W(x)$. As it is seen from this Fig. (3.2), $V(x)$ is not an even function, but the $W(x)$ is still an odd function.

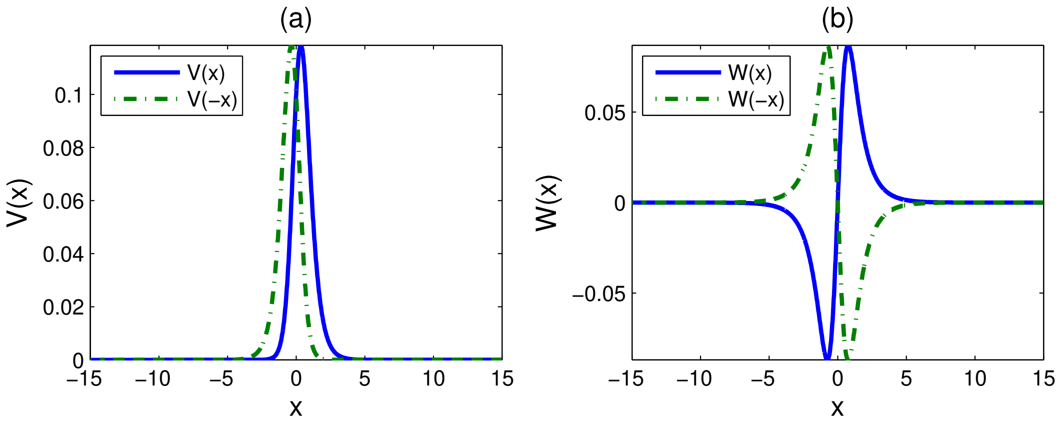

Figure 3.2 : (a) Real part of the Non-$\mathscr{PT}$-Symmetric Potential, (b) Imaginary part of the Non-$\mathscr{PT}$-Symmetric Potential, Eq.(3.10) with $V_0 = V_1 = W_0 = W_1 = 0.1$.

3.3 $\mathscr{PT}$-Periodic Symmetric Lattices

In this section, we will consider the following $\mathscr{PT}$-periodic symmetric potential

$$V_{\mathscr{PT}}(x) = V(x) + iW(x) = \left[V_0 cos^2(x)\right] + i[W_0 sin(2x)] \tag{3.12}$$

where V_0 and W_0 refer to depths of the real and complex parts of the potentials, respectively. It is easy to see that from the Fig. (3.3), V is an even real-valued function and W is an odd real-valued function.

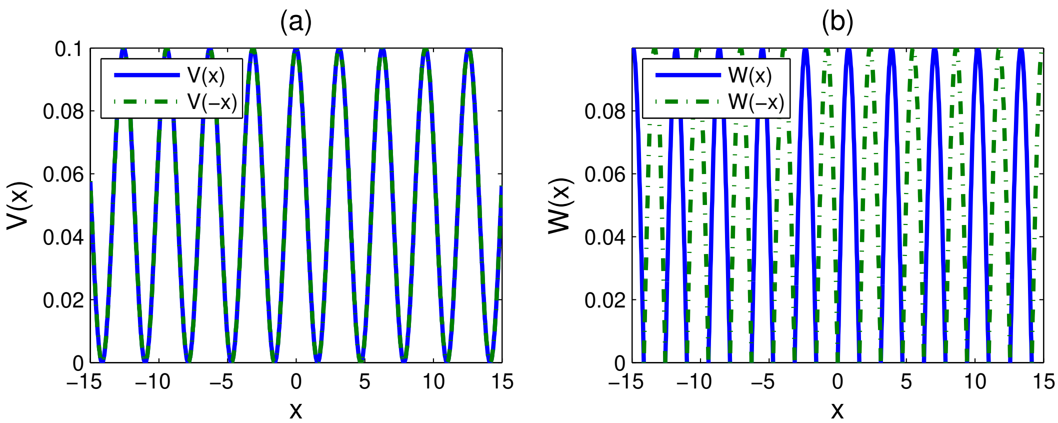

Figure 3.3 : (a) Real part of the $\mathscr{PT}$-Periodic Symmetric Potential, (b) Imaginary part of the $\mathscr{PT}$-Periodic Symmetric Potential, Eq.(3.11) with $V_0 = W_0 = 0.1$.

4. SPECTRAL METHODS

4.1 Pseudo-Spectral Renormalization Method

Self-localized solutions of many nonlinear system can be found by using different computational techniques such as shooting, self-consistency, relaxation and Newton-Conjugate-Gradient method, Squared-Operator Iteration methods [39] , Imaginary-Time Evolution methods [40] or different variational procedure [41]. One of the most useful method is Petviashvili's method. This method based on a fixed-point algorithm. In this method governing nonlinear equations are transformed into Fourier space. Patviashvili's method appeared to generate numerically lump solitary wave of the KP-I equation, a converge factor is determined from the algebric equation. In this section, we use a fixed-point pseudo-spectral renormalization method to solve the equation,

$$iu_z + u_{xx} + |u|^2 u + \tau |u|^2{}_x u + [V(x) + iW(x)]u = 0 \tag{4.1}$$

where z is the propopation direction of optical pulse, x is the transverse coordinate, i denotes the imaginary number and u is the complex amplitude of the optical pulse, and $V(x)$ is the optical lattice, τ is the raman effect.

We look for the self-localized solutions to Eq.(4.1) in the form:

$$u = q(x)e^{i\mu z} \tag{4.2}$$

Taking the derivatives of u and substituting Eq.(4.2) into Eq.(4.1) we get the following equation

$$-\mu q + q_{xx} + |q|^2 q + V(x)q - (\tau_r + i\tau_i)q + |q|^2{}_x q = 0 \tag{4.3}$$

where μ is the propopation constant.

Let us define the 1D Fourier transform of q as

$$\hat{q}(k) = \mathscr{F}[q(x)] = \int_{-\infty}^{\infty} q(x)e^{ikx}dx \tag{4.4}$$

and inverse 1D Fourier transform as follows:

$$q(x) = \mathscr{F}^{-1}[\hat{q}(k)] = \int_{-\infty}^{\infty} \hat{q}(k) e^{-ikx} dk. \tag{4.5}$$

Applying Fourier and inverse Fourier transform to q_{xx} and substitute in Eq.(4.3) then we get the following equation:

$$-\mu q - \mathscr{F}^{-1}\left[|k|^2 \hat{q}\right] + [V(x) + iW(x)]q + (\tau_r + i\tau_i) q \mathscr{F}^{-1}\left[\mathscr{F}\left[ik|q|^2\right]\right] + |q|^2 q = 0 \tag{4.6}$$

Let us define q such as

$$-\mu q = -\mu \mathscr{F}^{-1}[\mathscr{F}[q]]. \tag{4.7}$$

Substituting Eq.(4.7) into Eq.(4.6) then we get

$$-\mathscr{F}^{-1}\left[\left(\mu + |k|^2\right)\hat{q}\right] + [V(x) + iW(x)]q + (\tau_r + i\tau_i) q \mathscr{F}^{-1}\left[\mathscr{F}\left[ik|q|^2\right]\right] + |q|^2 q = 0. \tag{4.8}$$

Solving $\hat{q}$ from Eq.(4.8), we obtain

$$\hat{q}(k) = \frac{\mathscr{F}\left[(\tau_r + i\tau_i) q \mathscr{F}^{-1}\left[ik\mathscr{F}\left[|q|^2\right]\right] + |q|^2 q + [V(x) + iW(x)]q\right]}{\mu + |k|^2}. \tag{4.9}$$

In order to find the self-localized solutions of the Eq.(4.1), we use fixed-point iteration method. However the iterations of Eq.(4.9) may grow unboundadly or it may tend to 0. In order to get convergent fixed-point iteration method, we introduce a new field variable

$$q(x) = \lambda w(x) \ , \ \lambda \neq 0 \tag{4.10}$$

λ is called a converging factor to be determined in each iteration step substituting (4.10) into the function $w(x)$ satisfy the following equation

$$-\mathscr{F}^{-1}\left[\left(\mu + |k|^2\right)\hat{w}\right] + (\tau_r + i\tau_i) w \mathscr{F}^{-1}\left[ik|\lambda|^2 \mathscr{F}\left[|w|^2\right]\right] + |\lambda|^2 |w|^2 w + [V(x) + iW(x)]w = 0. \tag{4.11}$$

In order to find $|\lambda|$, we multiply Eq.(4.1) by w and integrate we obtain an algebric equation for the convergence factor $|\lambda|$:

$$|\lambda|^2 = -\frac{S_1}{S_2}. \tag{4.12}$$

Where S_1 and S_2 are defined by

$$S_1 = \int_{-\infty}^{\infty} w\left[[V(x)+iW(x)]w - \mathscr{F}^{-1}\left[\left(\mu+|k|^2\right)\hat{w}\right]\right]dx,$$
$$S_2 = \int_{-\infty}^{\infty} w\left[|w|^2 w + (\tau_r + i\tau_i)w\mathscr{F}^{-1}\left[ik\mathscr{F}\left[|w|^2\right]\right]\right]dx. \tag{4.13}$$

The fixed-point pseudo-spectral iteration scheme for w is given as follows:

$$\hat{w}_{n+1} = \frac{\mathscr{F}\left[(\tau_r+i\tau_i)w_n\mathscr{F}^{-1}\left[ik|\lambda_n|^{3/2}\mathscr{F}\left[|w_n|^2\right]\right] + |\lambda_n|^{3/2}w_n|w_n|^2 + |\lambda_n|^{1/2}[V(x)+iW(x)]w_n\right]}{\left(\mu+|k|^2\right)} \tag{4.14}$$

which subject to the additional constraint where $Im(\lambda_n) = 0$.

It was been found that pseudo-spectral iretative method prevents the numerical scheme from diverging. This the self-localized solutions of Eq.(4.1) can be obtained fro convergent iterative scheme. The initial condition for w is typically choosen to be as a Gaussian such as

$$w(x,0) = e^{-(x-x_0)^2}. \tag{4.15}$$

The iteration continuous until the relative error as follows:

$$\lambda_{error} = \left|\frac{\lambda_{n+1}}{\lambda_n} - 1\right|. \tag{4.16}$$

5. STABILITY ANALYSIS

5.1 Nonlinear Stability

Nonlinearly stable soliton is proven that it preserves the shape, position and maximum magnitude. In order to investigate nonlinear stability of solitons, calculated soliton solutions will be computed for long distances in z. For this purpose, Runge-Kutta method and Pseudospectral method is used for nonlinear stability.

5.1.1 Runge-Kutta Method

The Runge-Kutta method is a well-known numerical method in order to solve ODE and PDE systems. Consider systems of ODEs as follows:

$$F\left(x, y, y', \ldots, y^{(n-1)}\right) = y^{(n-1)} \tag{5.1}$$

where y is the vector valued function,

$$y : \mathbb{R} \to \mathbb{R}^m, \tag{5.2}$$

in x and $y^{(n)}$ represents nth derivative of y.

In this section, in order to solve Eq.(5.1), Runge-Kutta method will be used with the vector valued functions.

Consider following initial value problem written as

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{5.3}$$

The Runge-Kutta method for given inital value problem Eq.(5.3)

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4), \\ t_{n+1} &= t_n + h, \end{aligned} \tag{5.4}$$

where y_{n+1} represents approximation of $y(t_{n+1})$,

$$
\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \\
k_3 &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right) \\
k_4 &= f(t_n + h, y_n + hk_3).
\end{aligned}
\tag{5.5}
$$

The next value y_{n+1} will be found by the value of y_n and the size of interval h. [42] In Eq.(5.5), k_1 represents the slope at the first step of interval, k_2 represents the slope at the middle step of interval, k_3 represents the slope at the middle step of interval, k_4 represents the slope at the last step of interval, and the avareage of the slopes represents approximately numerical solution of ODE Eq.(5.3) as follows: [42]

$$
y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{5.6}
$$

Eq.(5.6) is called 4th order Runge-Kutta method result which refers the error in each step on the order of h^5, and the total error approximately is h^4. Runge-Kutta method formulae are applicable for scalar-valued and vector-valued functions. [42]

5.1.2 Pseudospectral Method

The pseudospectral method is one of the first spectral method found for wave equations. In order to explain the method, let define the NLS equation as

$$
iu_t + u_{xx} + 2|u|^2 u = 0. \tag{5.7}
$$

Discretize Eq.(5.7) in space,

$$
u_{n,t} = i\left(u_{n,xx} + 2|u_n|^2 u_n\right), \tag{5.8}
$$

where u_n refers the solution on x_n grid point. The fundamental meaning of the pseudospectral method is that using discrete Fourier transform to obtain the spatial derivative $u_{n,xx}$, and use and proper scheme such as Runge-Kutta to advance in time. [39] For this computation, we used the fourth-order Runge-Kutta method. $u_{n,xx}$ obtained by discrete Fourier transform as

$$
u_{n,xx} = \mathscr{F}^{-1}\left[(ik)^2 \mathscr{F}(u_n)\right], \tag{5.9}
$$

where $\mathscr{F}$ refers to the discrete Fourier, $\mathscr{F}^{-1}$ refers to the inverse Fourier transforms and k is the wave number.

In this thesis, we used pseudospectral method for stability of the wave equation. The presicion of this calculation is spectral, and the error is smaller than Δx which is power of spatial difference. [39]

After $u_{n,xx}$ is found, Eq.(5.8) can be calculated by Runge-Kutta method which is a time-stepping scheme. Stability of numerical solution calculated by pseudospectral method by taking diferent time-step value Δt and spatial difference Δx. It is found that the numerical solution became unstable for large time-step value. This is stated that the pseudospectral method has a condition for stability on the time-step size Δt and this condition is enough for the stable numerical solution. [39]

$$\frac{\Delta t}{\Delta x^2} \leq \frac{2\sqrt{2}}{\pi^2} \tag{5.10}$$

Eq.(5.10) is the necessary and sufficient condition for stability, for the pseudospectral method on the NLS equation Eq.(5.7).

5.2 Linear Stability

Linear stability will be calculated by obtaining and analyzing the linear spectrum or the linear evolution.

5.2.1 Linear Evolution

In order to examine the linear stability of the equation below,

$$\begin{aligned} iu_z(x,z) + u_{xx}(x,z) + |u(x,z)|^2 u(x,z) + (\tau_r + i\tau_i) u(x,z) \left(|u(x,z)|^2 \right)_x \\ + V_{\mathscr{PT}}(x,z) u(x,z) = 0, \end{aligned} \tag{5.11}$$

the soliton solution of Eq.(5.11) is perturbed as:

$$u(x,z) = \left[u(x) + \varepsilon \tilde{U}(x,z) \right] e^{i\mu z} \tag{5.12}$$

where $\varepsilon << 1$ and $\tilde{U}$ refers to governing equation of the NLS. Substitute solution form to Eq.(5.11) as follows:

$$\begin{aligned}
u_z &= i\mu\left[u(x)+\varepsilon\tilde{U}(x,z)\right]e^{i\mu z}+\left[\tilde{U}_z(x,z)\right]e^{i\mu z}\\
u_{xx} &= \left[u_{xx}+\varepsilon\tilde{U}_{xx}\right]e^{i\mu z}\\
|u|^2u &= u\cdot u^*\cdot u = u^2\cdot u^* = \left[|u|^2u+\varepsilon\left(u^2\tilde{U}^*+2|u|^2\tilde{U}\right)\right]e^{i\mu z}\\
\left(|u|^2\right)_x u &= (u_xu^*+uu_x^*)u = u_x|u|^2+u^2u_x^*\\
&= \left[u^2u_x^*+|u|^2u_x+\varepsilon\left(2uu_x^*\tilde{U}+u^2\tilde{U}_x^*+u_x\left(u^*\tilde{U}+u\tilde{U}^*\right)+\tilde{U}_x|u|^2\right)\right]e^{i\mu z}.
\end{aligned} \tag{5.13}$$

When we substitute to Eq. (5.11) and seperate $\theta(\varepsilon^0)$ terms, we obtain

$$\begin{aligned}
&-\mu u+u_xx+|u|^2u+\tau|u|^2u_x+\tau u^2u_x^*+V_{\mathscr{PT}}u=0,\\
&-\mu u+u_{xx}+|u|^2u+\tau\left(|u|^2\right)_x u+V_{\mathscr{PT}}u=0.
\end{aligned} \tag{5.14}$$

When we seperate $\theta(\varepsilon)$ terms as follows:

$$\begin{aligned}
-\mu\tilde{U}+i\tilde{U}_z+\tilde{U}_{xx}+\left(u^2\tilde{U}^*+2|u|^2\tilde{U}\right)+\\
\tau\left(u^*u_x\tilde{U}+u\tilde{U}^*u_x+|u|^2\tilde{U}_x+2uu_x^*\tilde{U}+u^2\tilde{U}_x^*\right)+V_{\mathscr{PT}}u=0.
\end{aligned} \tag{5.15}$$

Rewrite the Eq.(5.15), then obtain the Linear Evolution,

$$\begin{aligned}
i\tilde{U}_z+\tilde{U}_{xx}+\tilde{U}\left[2|u|^2+\tau u^*u_x-\mu+2\tau uu_x^*+V_{\mathscr{PT}}\right]+\tilde{U}_x\left[|u|^2\right]\\
+\tilde{U}^*\left[u^2+\tau uu_x\right]+\tilde{U}_x^*\left[\tau u^2\right]=0.
\end{aligned} \tag{5.16}$$

5.2.2 Linear Spectrum

Linear spectrum is the eigenvalues of linear stability operator of soliton. Concened eigenvalues will give an idea about the linear stability of the soliton.

Consider the following NLS equation with Raman effect:

$$\begin{aligned}
iu_z(x,z)+u_{xx}(x,z)+|u(x,z)|^2u(x,z)+(\tau_r+i\tau_i)u(x,z)\left(|u(x,z)|^2\right)_x\\
+V_{\mathscr{PT}}(x,z)u(x,z)=0.
\end{aligned} \tag{5.17}$$

Soliton solutions of Eq.(5.17) form is $u(x,t)=f(x)e^{i\mu t}$.

Substitute solution form to Eq.(5.17) as follows:

$$\begin{aligned}
u_z &= fe^{i\mu z}i\mu\\
u_{xx} &= f_{xx}e^{i\mu z}\\
|u|^2 &= u\cdot u^* = fe^{i\mu z}\cdot f^*e^{-i\mu z} = f\cdot f^* = |f|^2\\
\left(|u|^2\right)_x &= 2|f||f|_x,
\end{aligned} \tag{5.18}$$

and multiplying by $e^{-i\mu t}$ results

$$-\mu f + f_{xx} + |f|^2 f + (\tau_r + i\tau_i) f \left(|f|^2\right)_x + V_{\mathscr{P}\mathscr{T}} f = 0. \tag{5.19}$$

In order to obtain linear stability, perturb the soliton solution as

$$u(x,t) = \left[f(x) + \varepsilon\left(g(x)e^{\lambda t} + h^*(x)e^{\lambda^* t}\right)\right]e^{i\mu t} \tag{5.20}$$

where $\varepsilon << 1$ and $\varepsilon^2 \approx 0$, g and h are perturbation eigenfunctions and λ is the eigenvalue.

$$\begin{aligned} u_t &= \left[\varepsilon\left(\lambda g e^{\lambda t} + \lambda^* h^* e^{\lambda^* t} + g e^{\lambda t} + h^* e^{\lambda^* t}\right) + i\mu f\right]e^{i\mu t} \\ u_{xx} &= \left[f_{xx} + \varepsilon\left(g_{xx}e^{\lambda t} + h^*_{xx}e^{\lambda^* t}\right)\right]e^{i\mu t} \end{aligned} \tag{5.21}$$

$$|u|^2 u = \left[|f|^2 f + \varepsilon\left(|f|^2 g e^{\lambda t} + |f|^2 h^* e^{\lambda^* t} + e^{\lambda t}\left(f^2 h + |f|^2 g\right) + e^{\lambda^* t}\left(f^2 g^* + |f|^2 h^*\right)\right)\right]e^{i\mu t} \tag{5.22}$$

$$\begin{aligned} \left(|u|^2\right)_x u = \left(|f|^2\right)_x f e^{i\mu t} &+ \varepsilon e^{\lambda t}\left(f f_x h + h_x f^2 + f f^*_x g + g_x |f|^2 + \left(|u|^2\right)_x g\right)e^{i\mu t} \\ &+ \varepsilon e^{\lambda^* t}\left(f f_x g^* + f^2 g^*_x + f f^*_x h^* + |f|^2 h^*_x + \left(|u|^2\right)_x h^*\right)e^{i\mu t} \end{aligned} \tag{5.23}$$

Substituting Eq.(5.21), (5.22) and (5.23) into Eq.(5.17) gives

$$\begin{aligned} & i\left(\left[\varepsilon\left(\lambda g e^{\lambda t} + \lambda^* h^* e^{\lambda^* t} + g e^{\lambda t} + h^* e^{\lambda^* t}\right) + i\mu f\right]\right)e^{i\mu t} \\ & + \left[f_{xx} + \varepsilon\left(g_{xx}e^{\lambda t} + h^*_{xx}e^{\lambda^* t}\right)\right]e^{i\mu t} \\ & + \left[|f|^2 f + \varepsilon\left(|f|^2 g e^{\lambda t} + |f|^2 h^* e^{\lambda^* t} + e^{\lambda t}\left(f^2 h + |f|^2 g\right) + e^{\lambda^* t}\left(f^2 g^* + |f|^2 h^*\right)\right)\right]e^{i\mu t} \\ & + (\tau_r + i\tau_i)e^{i\mu t}\left\{\left(|f|^2\right)_x f + \varepsilon e^{\lambda t}\left(f f_x h + h_x f^2 + f f^*_x g + g_x |f|^2 + \left(|u|^2\right)_x g\right)\right. \\ & \left. + \varepsilon e^{\lambda^* t}\left(f f_x g^* + f^2 g^*_x + f f^*_x h^* + |f|^2 h^*_x + \left(|u|^2\right)_x h^*\right)\right\} \\ & + V_{\mathscr{P}\mathscr{T}}\left(f + \varepsilon\left(g e^{\lambda t} + h^* e^{\lambda^* t}\right)\right)e^{i\mu t} = 0. \end{aligned} \tag{5.24}$$

Classifying the terms and multiplying by $e^{-i\mu t}$

$$
\begin{aligned}
&\left[-\mu f + f_{xx} + |f|^2 f + (\tau_r + i\tau_i)\left(|f|^2\right)_x f + V_{\mathscr{PT}} f\right] \\
&+ \varepsilon e^{\lambda t}\Big\{ -\mu g + i\lambda g + g_{xx} + |f|^2 g + f^2 h + |f|^2 g \\
&+ (\tau_r + i\tau_i)\Big(f f_x h + h_x f^2 + f f_x^* g + g_x |f|^2 + \left(|f|^2\right)_x g\Big) + V_{\mathscr{PT}} g\Big\} \\
&+ \varepsilon e^{\lambda^* t}\Big\{ -\mu h^* + i\lambda^* h^* + h_{xx}^* + |f|^2 h^* + f^2 g^* + |f|^2 h^* \\
&+ (\tau_r + i\tau_i)\Big(f f_x g^* + f^2 g_x^* + f f_x^* h^* + |f|^2 h_x^* + \left(|f|^2\right)_x h^*\Big) + V_{\mathscr{PT}} h^*\Big\} = 0.
\end{aligned} \tag{5.25}
$$

First line of Eq.(5.25) is equal to 0, since f is a solution, it is seem from Eq.(5.19). In order to make Eq.(5.25) equal to 0, the coefficients of the exponentials must be 0. Therefore, first coefficient of $e^{\lambda t}$,

$$
\begin{aligned}
&-\mu g + i\lambda g + g_{xx} + |f|^2 g + f^2 h + |f|^2 g \\
&+ (\tau_r + i\tau_i)\Big(f f_x h + h_x f^2 + f f_x^* g + g_x |f|^2 + \left(|f|^2\right)_x g\Big) + V_{\mathscr{PT}} g = 0
\end{aligned} \tag{5.26}
$$

and second coefficient of $e^{\lambda^* t}$

$$
\begin{aligned}
&-\mu h^* + i\lambda^* h^* + h_{xx}^* + |f|^2 h^* + f^2 g^* + |f|^2 h^* \\
&+ (\tau_r + i\tau_i)\Big(f f_x g^* + f^2 g_x^* + f f_x^* h^* + |f|^2 h_x^* + \left(|f|^2\right)_x h^*\Big) + V_{\mathscr{PT}} h^* = 0.
\end{aligned} \tag{5.27}
$$

Eq.(5.26) and Eq.(5.27) can be written as

$$
\begin{aligned}
&g_{xx} + \Big(-\mu + |f|^2 + |f|^2 + (\tau_r + i\tau_i)\Big(f f_x^* + \left(|f|^2\right)_x\Big) + V_{\mathscr{PT}}\Big) g \\
&\quad + \big(f^2 + f f_x(\tau_r + i\tau_i)\big) h + (\tau_r + i\tau_i) f^2 h_x + (\tau_r + i\tau_i)|f|^2 g_x = -i\lambda g,
\end{aligned} \tag{5.28}
$$

$$
\begin{aligned}
&h_{xx}^* + \Big(-\mu + |f|^2 + |f|^2 + (\tau_r + i\tau_i)\Big(f f_x^* + \left(|f|^2\right)_x\Big) + V_{\mathscr{PT}}\Big) h^* \\
&\quad + \big(f^2 + f f_x(\tau_r + i\tau_i)\big) g^* + (\tau_r + i\tau_i) f^2 g_x^* + (\tau_r + i\tau_i)|f|^2 h_x^* = -i\lambda^* h^*.
\end{aligned} \tag{5.29}
$$

Taking the conjugate of Eq.(5.29) as follows:

$$
\begin{aligned}
&h_{xx} + \Big(-\mu + |f|^2 + |f|^2 + (\tau_r - i\tau_i)\Big(f^* f_x + \left(|f|^2\right)_x^*\Big) + V_{\mathscr{PT}}^*\Big) h \\
&\quad + \Big(\left(f^2\right)^* + f^* f_x^*(\tau_r - i\tau_i)\Big) g + (\tau_r - i\tau_i)\left(f^2\right)^* g_x + (\tau_r - i\tau_i)|f|^2 h_x = i\lambda h.
\end{aligned} \tag{5.30}
$$

Multiplying Eq.(5.30) by -1

$$
\begin{aligned}
&-h_{xx} - \Big(-\mu + |f|^2 + |f|^2 + (\tau_r - i\tau_i)\Big(f^* f_x + \left(|f|^2\right)_x^*\Big) + V_{\mathscr{PT}}^*\Big) h \\
&\quad - \Big(\left(f^2\right)^* + f^* f_x^*(\tau_r - i\tau_i)\Big) g - (\tau_r - i\tau_i)\left(f^2\right)^* g_x - (\tau_r - i\tau_i)|f|^2 h_x = -i\lambda h.
\end{aligned} \tag{5.31}
$$

Writing Eq.(5.28) and Eq.(5.31) in the matrix form

$$i\begin{bmatrix} L_1 & L_2 \\ -L_2{}^* & -L_1{}^* \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} = \lambda \begin{bmatrix} g \\ h \end{bmatrix} \tag{5.32}$$

where

$$\begin{aligned} L_1 &= \frac{\partial^2}{\partial x^2} + (\tau_r + i\tau_i)|f|^2 \frac{\partial}{\partial x} + 2|f|^2 + (\tau_r + i\tau_i)\left(f f_x^* + \left(|f|^2\right)_x \right) - \mu + V_{\mathscr{P}\mathscr{T}} \\ L_2 &= (\tau_r + i\tau_i) f^2 \frac{\partial}{\partial x} + f^2 + (\tau_r + i\tau_i) f f_x. \end{aligned} \tag{5.33}$$

The eigenvalues λ can be numerically obtained. According to obtained eigenvalues, linear stability can be investigated.

6. SOLITONS OF THE NLS EQUATION WITH RAMAN EFFECT

6.1 NLS Equation with Raman Effect with a $\mathscr{PT}$-Symmetric Potential

First of all, we study the analytical solution of NLS equation with Raman effect and $\mathscr{PT}$ symmetric and non-$\mathscr{PT}$ symmetric optical lattices.

6.1.1 Analytical Solutions

In this section, we concentrate on the analytical self-localized soliton solutions of the NLS equation for $\mathscr{PT}$-Symmetric optical lattices. NLS equation with Raman effect can be given as follows:

$$\begin{aligned} iu_z(x,z)+u_{xx}(x,z)+|u(x,z)|^2u(x,z)+(\tau_r+i\tau_i)u(x,z)\left(|u(x,z)|^2\right)_x \\ +[V(x)+iW(x)]u(x,z)=0 \end{aligned} \tag{6.1}$$

It is clear to see that if $u(x,z)=0$ in Eq.(6.1), it will be trivial solution. In order to obtain non-zero solutions, it will be set $u(x,z)\neq 0$. Dividing Eq.(6.1) by $u(x,z)$ as follows:

$$i\frac{u_z}{u}+\frac{u_{xx}}{u}+|u|^2+(\tau_r+i\tau_i){|u|^2}_x+[V(x)+iW(x)]=0. \tag{6.2}$$

To get solution, the following form of solution will be used:

$$u(x,z)=f(x)e^{i[\mu z+g(x)]} \tag{6.3}$$

where $f(x)$ and $g(x)$ are realed valued and non-zero functions. Substituting this along with

$$\begin{aligned} u_z &= i\mu f e^{(\mu z+g)} = i\mu u \\ u_{xx} &= \left(f''+2if'g'+if'g''-f(\dot{g'})^2\right)e^{(\mu z+g)} \\ |u|^2 &= fe^{i(\mu z+g)}fe^{-i(\mu z+g)} = f^2 \\ \left(|u|^2\right)_x &= 2|f||f|_x \end{aligned} \tag{6.4}$$

into Eq.(6.1) results

$$\left[-\mu+\frac{f''}{f}-\left(g'\right)^2+|f|^2+2\tau_r|f||f|_x+V\right]+i\left[\frac{2f'g'}{f}+g''+2\tau_i|f||f|_x+W\right]=0. \tag{6.5}$$

Seperating real and imaginary parts of the equation (6.5), we get the following coupled system of equations for f and g. In order to get the soliton solutions, following solutions for f and g will be used

$$\begin{aligned} f(x)&=f_0sech^p(x) \quad \text{where} \quad f_0\in\mathbb{R}/\{0\} \text{ and } p\in\mathbb{N}\ , \\ g'(x)&=g_0sech^q(x) \quad \text{where} \quad g_0\in\mathbb{R}/\{0\} \text{ and } q\in\mathbb{N}\ . \end{aligned} \tag{6.6}$$

Substituting

$$\begin{aligned} f'&=f_0psech^{p-1}(x)(-sech(x)tanh(x))=f(-ptanh(x)) \\ f''&=f'(-ptanh(x))+f\left(-psech^2(x)\right)=f\left[p^2-\left(p^2+p\right)sech^2(x)\right] \\ g''&=g_0qsech^{q-1}(x)(-sech(x)tanh(x))=-g_0qsech^q(x)tanh(x) \end{aligned} \tag{6.7}$$

into Eq.(6.5) results

$$\begin{aligned} &-\mu+p^2-\left(p^2+p\right)sech^2(x)-{g_0}^2sech^{2q}(x)+{f_0}^2sech^{2p}(x) \\ &-2p\tau_r{f_0}^2sech^{2p}(x)tanh(x)+V(x) \\ &+i\left[-(2p+q)g_0sech^q(x)tanh(x)-2p\tau_i{f_0}^2sech^{2p}(x)tanh(x)+W(x)\right]=0 \end{aligned} \tag{6.8}$$

Let us divide Eq.(6.8) into real and imaginary parts as follow:

Real Part of the Analytic Solution

We can rewrite the real part of Eq.(6.8) as,

$$\begin{aligned} -\mu+p^2-\left(p^2+p\right)sech^2(x)-{g_0}^2sech^{2q}(x)+{f_0}^2sech^{2p}(x)& \\ -2p\tau_r{f_0}^2sech^{2p}(x)tanh(x)+V(x)&=0. \end{aligned} \tag{6.9}$$

$V(x)$ should be given in a form as,

$$V(x)=V_0+V_1sech^2(x)+V_2sech^{2q}(x)+V_3sech^{2p}(x)+V_4sech^{2p}(x)tanh(x) \tag{6.10}$$

where

$$V_0=\mu-p^2\ ,\ V_1=p^2+p\ ,\ V_2={g_0}^2\ ,\ V_3=-{f_0}^2\ ,\ V_4=2p\tau_r{f_0}^2\ . \tag{6.11}$$

In order to make the real part of the potential simple, let us set $\mu = p^2$ to get rid of V_0. $V(x)$ needs to be an even function for $\mathscr{PT}$-symmetry as

$$\begin{aligned} V(-x) &= V_1 sech^2(-x) + V_2 sech^{2q}(-x) + V_3 sech^{2p}(-x) + V_4 sech^{2p}(-x) tanh(-x) \\ &= V_1 sech^2(x) + V_2 sech^{2q}(x) + V_3 sech^{2p}(x) - V_4 sech^{2p}(x) tanh(x) \neq V(x) \ . \end{aligned} \tag{6.12}$$

To make $V(x)$ an even function, it has to be taken as $V_4 = 0$. Since, $V_4 = 2p\tau_r {f_0}^2 = 0$, as given $f_0 \neq 0$ and $p \neq 0$. Hence, in order to make the potential $\mathscr{PT}$-symmetric, it has to be taken as $\tau_r = 0$. Now,

$$\begin{aligned} V(-x) &= V_1 sech^2(-x) + V_2 sech^{2q}(-x) + V_3 sech^{2p}(-x) \\ &= V_1 sech^2(x) + V_2 sech^{2q}(x) + V_3 sech^{2p}(x) = V(x) \ . \end{aligned} \tag{6.13}$$

Imaginary Part of the Analytic Solution

Now, we can rewrite the imaginary part of Eq.(6.8) as,

$$-(2p+q) g_0 sech^q(x) tanh(x) - 2p\tau_i {f_0}^2 sech^{2p}(x) tanh(x) + W(x) = 0. \tag{6.14}$$

The imaginary part of the $\mathscr{PT}$-symmetric potential is obtained in a form as,

$$W(x) = W_0 sech^q(x) tanh(x) + W_1 sech^{2p}(x) tanh(x) = 0 \tag{6.15}$$

where

$$W_0 = (2p+q) g_0 \ , \ \ W_1 = 2p\tau_i {f_0}^2 \ . \tag{6.16}$$

$W(x)$ needs to be an odd function for $\mathscr{PT}$-symmetry as

$$\begin{aligned} W(-x) &= W_0 sech^q(-x) tanh(-x) + W_1 sech^{2p}(-x) tanh(-x) \\ &= -W_0 sech^q(x) tanh(x) - W_1 sech^{2p}(x) tanh(x) = -W(x) \ . \end{aligned} \tag{6.17}$$

As a result, the general soliton solution of Eq.(6.1) with

$$\begin{aligned} V(x) &= \left(p^2 + p\right) sech^2(x) + {g_0}^2 sech^{2q}(x) - {f_0}^2 sech^{2p}(x) + 2p\tau_r {f_0}^2 sech^{2p}(x) tanh(x) \\ W(x) &= (2p+q) g_0 sech^q(x) tanh(x) + 2p\tau_i {f_0}^2 sech^{2p}(x) tanh(x) \end{aligned} \tag{6.18}$$

is given as

$$u(x,z) = f_0 sech^p(x) e^{i\left[p^2 z + g_0 \int sech^q(x) dx\right]} \ . \tag{6.19}$$

Now, we can make $V(x)$ simple by equating $sech(x)$ powers. The three powers $2p$,$2q$ and 2 can be equated in only on way:

1)$\{2 = 2p = 2q\} \Rightarrow p = q = 1$:

$$\begin{aligned} V(x) &= \left(2 + {g_0}^2 - {f_0}^2\right) sech^2(x) \\ W(x) &= 3g_0 sech(x) tanh(x) + 2\tau_i {f_0}^2 sech^2(x) tanh(x) \\ u(x,z) &= f_0 sech(x) e^{i(z + g_0 arctan(sinh(x)))} \quad . \end{aligned} \tag{6.20}$$

We will take the $\mathscr{PT}$-symmetric potential in the case.

$$\begin{aligned} V(x) &= V_0 sech^2(x) \\ W(x) &= W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x) \end{aligned} \tag{6.21}$$

where

$$\begin{aligned} V_0 &= 2 + {g_0}^2 - {f_0}^2 \\ W_0 &= 3g_0 \\ W_1 &= 2\tau_i {f_0}^2 \quad . \end{aligned} \tag{6.22}$$

This results will give the potential Eq.(3.10) along with the exact solution to Eq.(6.1). It follows from Eq.(6.21) that

$$\begin{aligned} f_0 &= \sqrt{2 + \frac{{W_0}^2}{9} - V_0} \\ g_0 &= \frac{W_0}{3} \end{aligned} \tag{6.23}$$

and the exact solution can be given as follow:

$$u(x,z) = \sqrt{2 + \frac{{W_0}^2}{9} - V_0} sech(x) e^{i\left(z + \frac{W_0}{3} arctan(sinh(x))\right)} \quad . \tag{6.24}$$

6.1.2 Numerical Solutions

Numerical solutions to Eq.(6.1) are investigated with the four different cases of potentials by means of Pseudo-Spectral Renormalization Method. The propagation constant can be taken as $\mu = 1$ by the choice of the potential. To determine the potentials, we get the different values of potential depths V_0, V_1, W_0 and W_1. For the numerical results, we set different values of τ_r and τ_i.

6.1.2.1 Numerical Solutions of the Optical Soliton without Potential

First of all, NLS equation is considered without any potential. The real and imaginary part of the optical solitons and error analysis can be seen in Fig. 6.1 for $\tau_r = 0.1$ and $\tau_i = 0.1$.

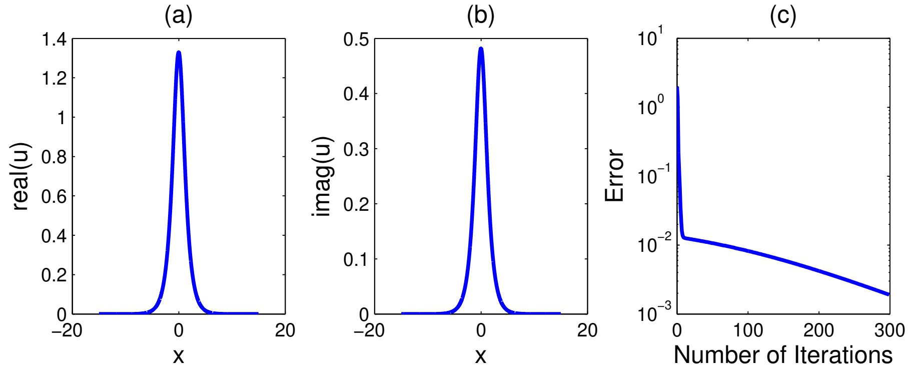

Figure 6.1 : (a) Real part, (b) Imaginary part of optical solitons without potential for $\tau_r = 0.1$ and $\tau_i = 0.1$, (c) Error of convergency.

It is seen from Fig. 6.1. the convergency cannot be achieved. When τ_r value will be decreased as $\tau_r = 0.001$, it is easy to notice that by comparing Fig. 6.1 and Fig. 6.2, the error started to decrease. Therefore, it is found out that while τ_r value is

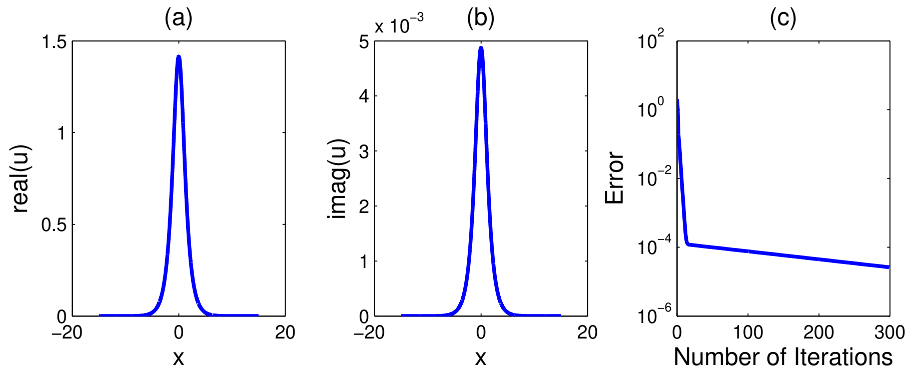

Figure 6.2 : (a) Real part, (b) Imaginary part of optical solitons without potential $\tau_r = 0.001$ and $\tau_i = 0.1$, (c) Error of convergency.

approaching to 0 ($\tau_r = 0.0$), numerical solution of optical solitons without potential can be obtained in Fig. 6.3.

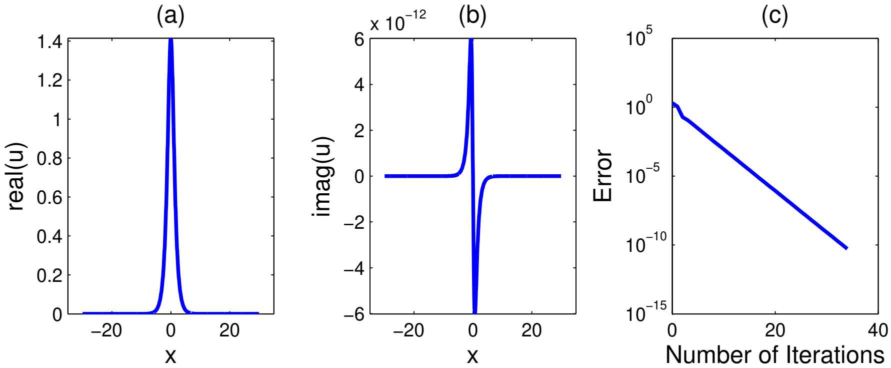

Figure 6.3 : (a) Real part,(b) Imaginary part of optical solitons without potential $\tau_r = 0.0$ and $\tau_i = 0.1$, (c) Error of convergency.

When $\tau_r = 0.0$ and we decrease the value of τ_i as 0.001, it is observed that optical solitons without potential is convergent. It is shown in Fig. 6.4.

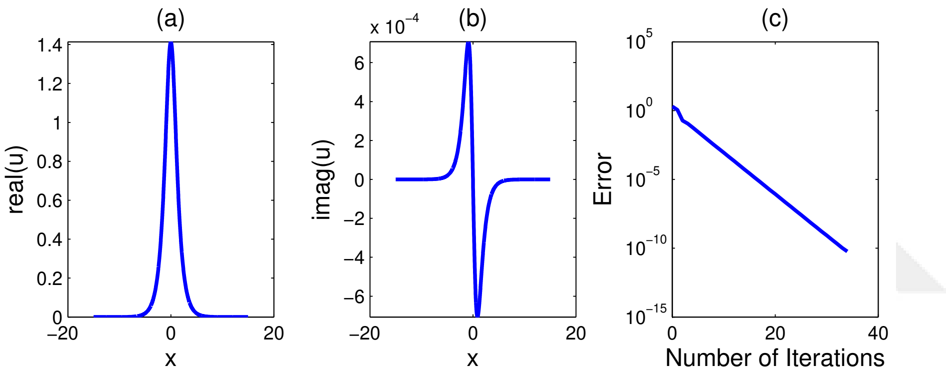

Figure 6.4 : (a) Real part,(b) Imaginary part of optical solitons without potential $\tau_r = 0.0$ and $\tau_i = 0.001$, (c) Error of convergency.

If we assume both the value of τ_r and τ_i is 0, then we get the optical soliton and an error of order 10^{-10} in 30 iteration, approximately in Fig. 6.5. Moreover, the order of the pseudospectral method is linear.

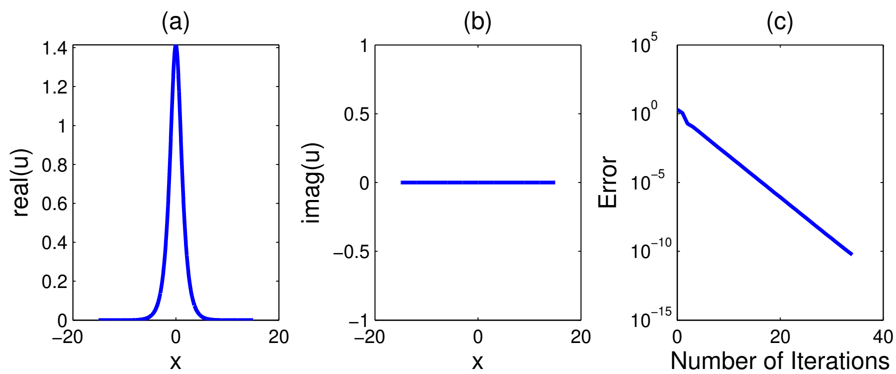

Figure 6.5 : (a) Real part,(b) Imaginary part of optical solitons without potential $\tau_r = 0.0$ and $\tau_i = 0.0$, (c) Error of convergency.

If we set $\tau_r = 0.1$ and $\tau_i = 0.0$, we found that soliton solutions cannot be found since, convergency cannot be achieved in Fig. 6.6.

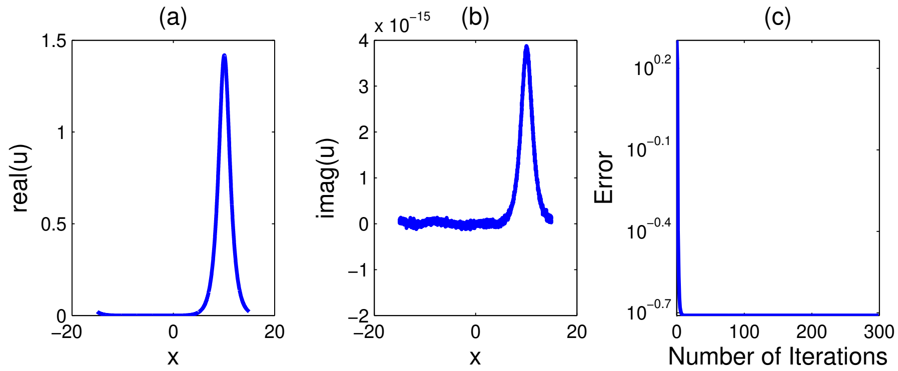

Figure 6.6 : (a) Real part,(b) Imaginary part of optical solitons without potential $\tau_r = 0.1$ and $\tau_i = 0.0$, (c) Error of convergency.

If we change $\tau_r = 0.2$, it is shown in Fig. 6.7. that solitons are not still convergent.

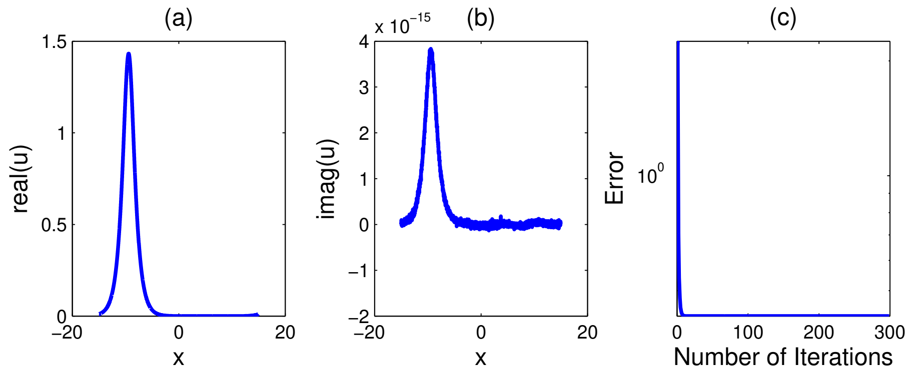

Figure 6.7 : (a) Real part,(b) Imaginary part of optical solitons without potential $\tau_r = 0.2$ and $\tau_i = 0.0$, (c) Error of convergency.

In conclusion, NLS equation with Raman effect without potential the error of order changes, according to different τ values. When τ_r value is closing to 0, and by choosing different values of τ_i, soliton solutions are obtained. But when we get $\tau_i = 0$ and nonzero values of τ_r, we could not find any solitons.

6.1.2.2 Numerical Solutions of the Optical Soliton with $\mathscr{PT}$-Periodic-Symmetric Potential

Secondly, NLS equation is considered with $\mathscr{PT}$-periodic-symmetric potential as follows:

$$V(x) = V_0 cos^2(x) + iW_0 sin(2x) \quad . \tag{6.25}$$

For this potential, there is no analytical solution for NLS equation given in Eq.(6.1). The propagation constant is fixed to $\mu = 1$, it is investigated different potentials by setting τ_r, τ_i, V_0 and W_0 different values. The real and imaginary part of the optical solitons and error analysis can be seen for $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.0$ in Fig. 6.8.

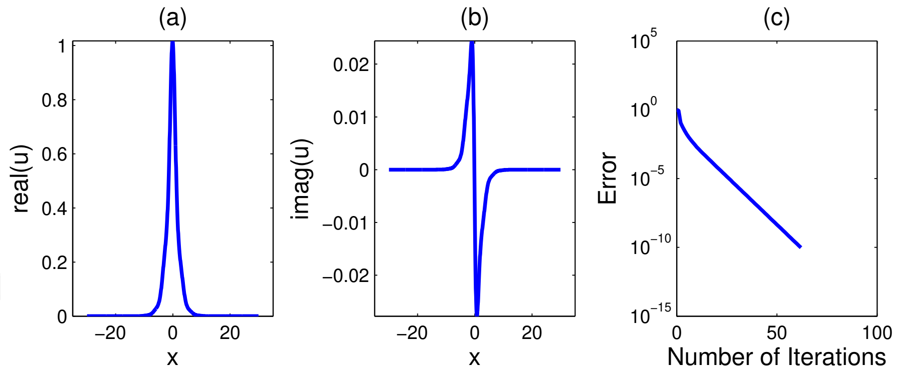

Figure 6.8 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{PT}$-periodic-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.0$, (c) Error of convergency.

When W_0 value will be increased as $W_0 = 0.3$, it is easy to see that by comparing Fig. 6.8 and Fig. 6.9, soliton shapes are different from each other. In Fig. 6.9, solution type is dark solitons and soliton solution is not convergent.

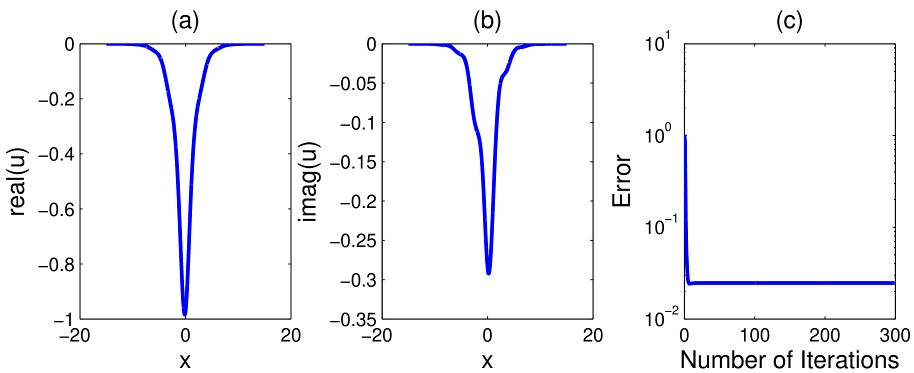

Figure 6.9 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{PT}$-periodic-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$, (c) Error of convergency.

Moreover, as soon as τ_r value is approaching to 0, independently from the other values by comparing Fig. 6.9 and Fig 6.10, $\tau_r = 0.0$, numerical solution of optical solitons with $\mathscr{PT}$-periodic-symmetric potential is converging. It is shown in Fig. 6.10.

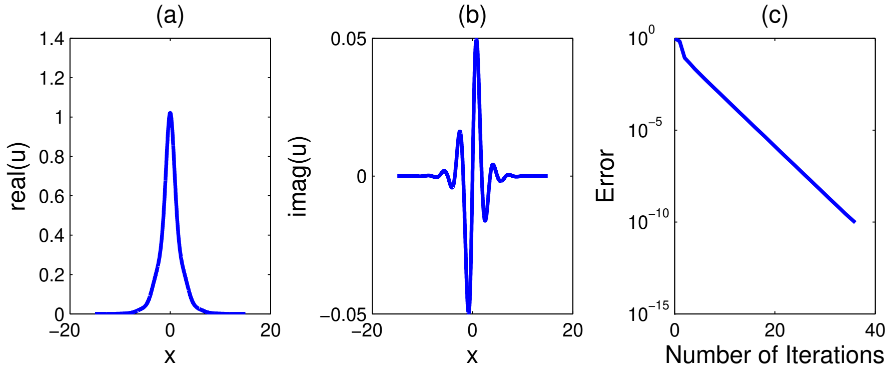

Figure 6.10 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{PT}$-periodic-symmetric potential $\tau_r = 0.0$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$, (c) Error of convergency.

If we only change the depth of the real part of the external potential $V_0 = 2$ and the other values of τ_r, τ_i and W_0 keep the same, we could not find any soliton solution (see in Fig. 6.11.).

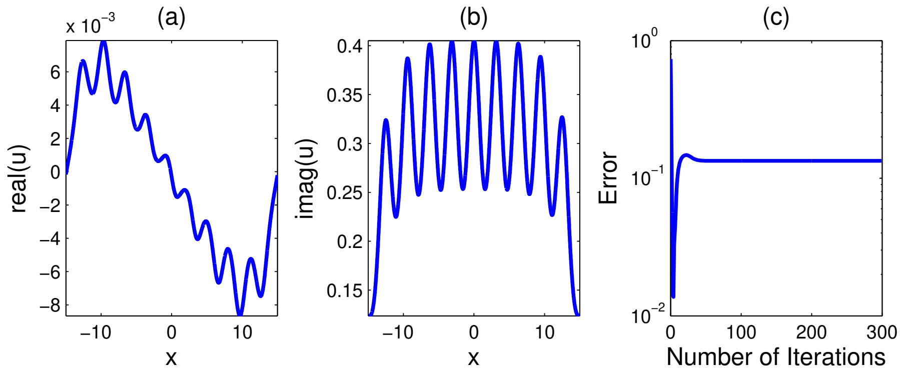

Figure 6.11 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{PT}$-periodic-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 2$, $W_0 = 0.0$ and $\mu = 1$, (c) Error of convergency.

When we increase the propagation constant, μ as 2, the convergency can be achieved in 30 iterations in Fig 6.12.

If we take $V_0 = 3$ and $\mu = 2$, we could not find soliton form of solutions in Fig. 6.13.

When we choose the values of $\mu = 3$ and $V_0 = 3$, it can be seen in Fig. 6.14, soliton form of the solution exists.

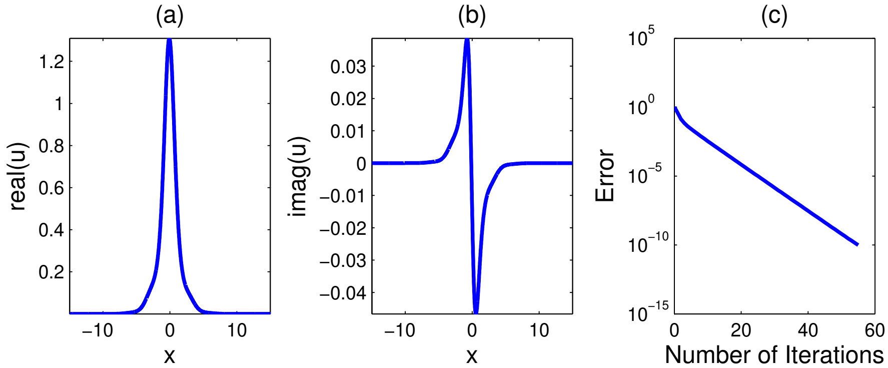

Figure 6.12 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{P}\mathscr{T}$-periodic-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 2$, $W_0 = 0.0$ and $\mu = 2$, (c) Error of convergency.

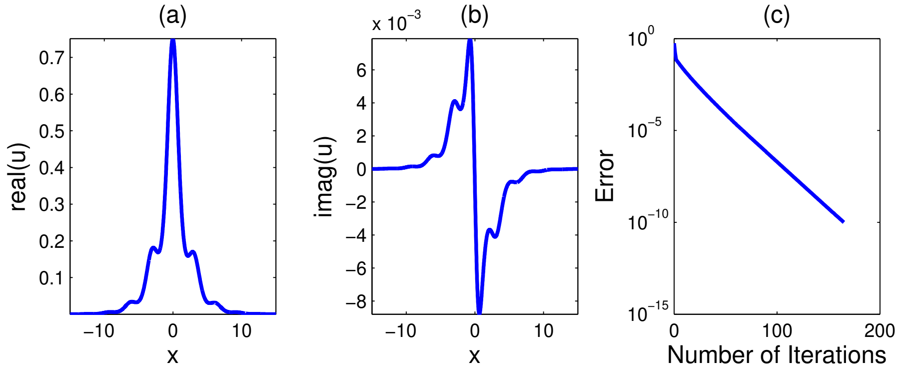

Figure 6.13 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{P}\mathscr{T}$-periodic-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 3$, $W_0 = 0.0$ and $\mu = 2$, (c) Error of convergency.

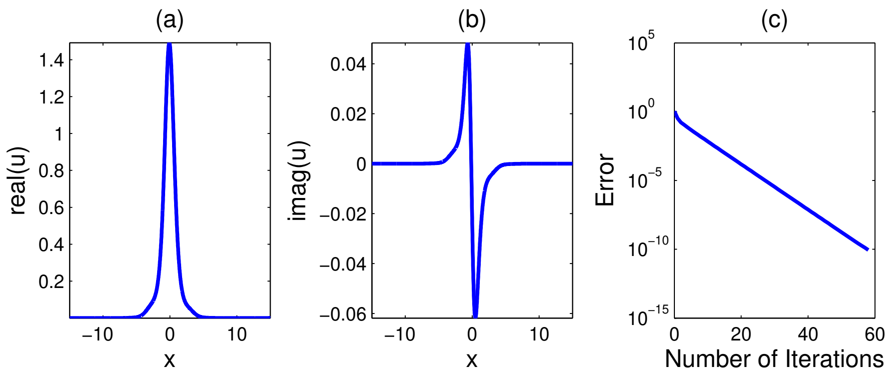

Figure 6.14 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{P}\mathscr{T}$-periodic-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 3$, $W_0 = 0.0$ and $\mu = 3$, (c) Error of convergency.

As a result of the numerical observations, we can conclude that the existence of Raman-induced solitons depend on the propagation constant, the depth of the real part of the potential and τ parameters. In order to find the Raman-induced solitons, the propagation constant (μ) should be equal to or greater than the depth of the real part of the potential (V_0).

6.1.2.3 Numerical Solutions of the Optical Soliton with $\mathscr{PT}$-Symmetric Potential

In this part of the thesis, we numerically demonstrate the soliton solutions of NLS equation with Raman effect and $\mathscr{PT}$-symmetric potential for different values of τ (the coefficients of Raman effect) and the depth of the real and the imaginary part of the potential.

$\mathscr{PT}$ symmetry condition means that the real part of potential should be even function of the position and the imaginary part of the potential should be odd. It is necesaary to take $\tau_r = 0$ in order to satisfy $\mathscr{PT}$ symmetry condition. Let us consider the following $\mathscr{PT}$-symmetric potential:

$$\begin{aligned} V(x) &= V_0 sech^2(x) \\ W(x) &= W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x) \end{aligned} \tag{6.26}$$

First, we show numerical soliton solutions of NLS equaiton with Raman effect for $\tau_r = 0$ in Fig. 6.15.

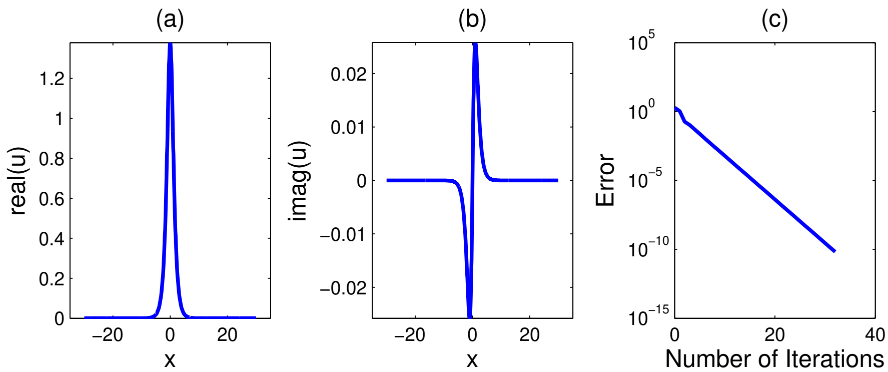

Figure 6.15 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{PT}$-symmetric potential $\tau_r = 0.0$, $\tau_i = 0.1$, $V_0 = 0.1$ and $W_0 = 0.1$, (c) Error of convergency

We show the analytical soliton solution and soliton solution obtained numerically in Fig. 6.16. As can be seen from this figure the pseudospectral renormalization method converges to the analytical solution for the parameters $\tau_r = 0.0$, $\tau_i = 0.1$, $V_0 = 0.1$ and $W_0 = 0.1$.

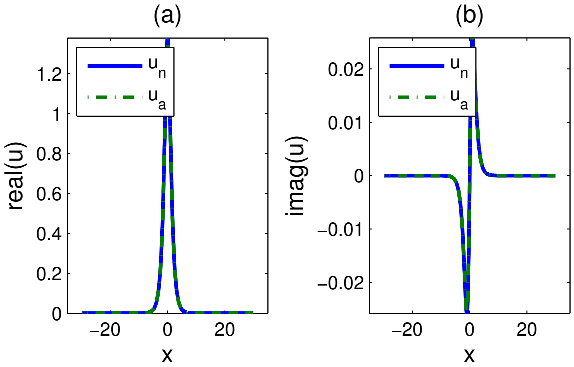

Figure 6.16 : (a) Real part, (b) Imaginary part of optical solitons with $\mathscr{PT}$-symmetric potential $\tau_r = 0.0$, $\tau_i = 0.1$, $V_0 = 0.1$ and $W_0 = 0.1$, u_n represents numerical solution and u_a represents analytical solution

6.1.2.4 Numerical Solutions of the Optical Soliton with Non-$\mathscr{PT}$-Symmetric Potential

So far, we have assumed that the external potential used in this thesis, are the periodic and $\mathscr{PT}$-symmetric potential. Now we will consider the Non-$\mathscr{PT}$-symmetric potential which requires $\tau_r \neq 0$ (the real part of the coefficient of the Raman effect). Let us consider the following Non-$\mathscr{PT}$-symmetric potential:

$$V_{\mathscr{PT}}(x) = V(x) + iW(x) = \left[V_0 sech^2(x) + V_1 sech^2(x) tanh(x)\right] + i\left[W_0 sech(x) tanh(x) + W_1 sech^2(x) tanh(x)\right] \quad (6.27)$$

It is concluded that even if potential is not $\mathscr{PT}$-symmetric, soliton solution can be found and it can be seen in Fig. 6.17. An error of order is 10^{-10} in 30 iterations, approximately.

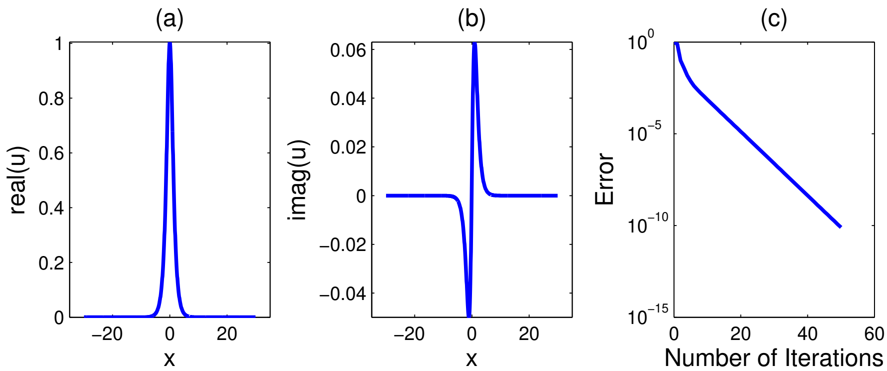

Figure 6.17 : (a) Real part, (b) Imaginary part of optical solitons with Non-$\mathscr{PT}$-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$, (c) Error of convergency.

It is shown that even if potential is Non-$\mathscr{PT}$-symmetric, numerical solution is converging to analytical solution in Fig. 6.18.

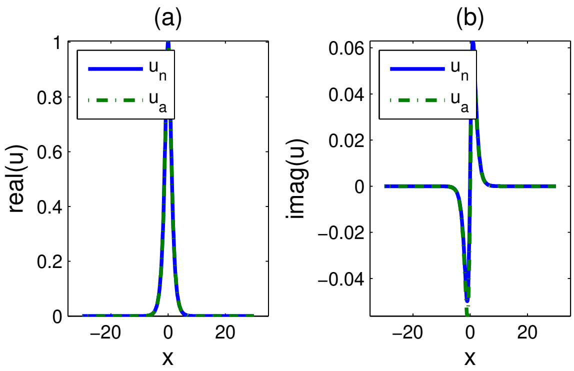

Figure 6.18 : (a) Real part, (b) Imaginary part of optical solitons with Non-$\mathscr{PT}$-symmetric potential $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$, u_n represents numerical solution and u_a represents analytical solution

6.1.3 Nonlinear Stability

In this section, we will numerically show how the Raman scattering effect nonlinear stability properties. In order to study nonlinear stability, obtained Raman-induced solitons are computed over a long-distance, for this purpose, pseudospectral method and Runge-Kutte method are used to advance in z.

First, we took Raman-induced soliton without external potential ($V_{PT}(x)$) obtained by pseudo-spectral renormalization method and evolved it for $z = 10$. The numerical results are shown in Fig 6.19. This figure shows that Raman-induced soliton is nonlinearly stable as it preserves its maximum amplitude during the evolution.

Nonlinear stability of Raman-induced solitons with $\mathscr{PT}$-periodic potential are given in Fig. 6.20 and Fig. 6.21. It can be seen from these figures all solitons obtained numerically are nonlinearly stable for the parameters ($\tau_r = 0$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$) and ($\tau_r = 0$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$), respectively.

Nonlinear stable Raman-induced soliton is demonstrated in Fig. 6.22. It is proved that if the potential is $\mathscr{PT}$-symmetric potential, the soliton is stable.

In Fig. 6.23, we take $\tau_r \neq 0$ and Non-$\mathscr{PT}$-symmetric potential. We observed from the figure that the maximum amplitude of soliton remains the same during the evolution. It means that this soliton is nonlinearly stable.

As a conclusion, we proved that $\mathscr{PT}$-symmetry is not necessary condition for soliton solutions to become stable.

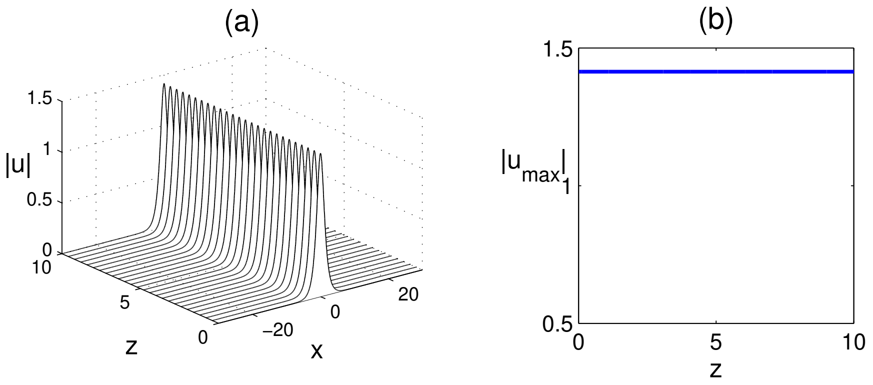

Figure 6.19 : (a) Nonlinear evolution of the soliton without potential for $\tau_r = 0$ and $\tau_i = 0.1$, (b) Maximum magnitude as a function of the distance z

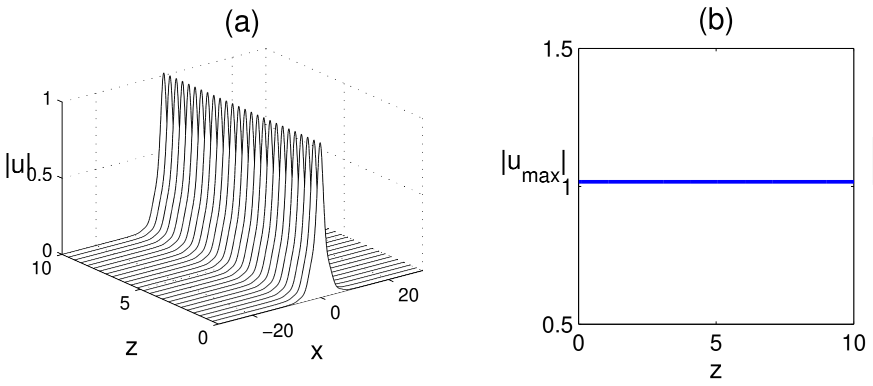

Figure 6.20 : (a) Nonlinear evolution of the soliton with $\mathscr{PT}$-periodic potential for $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0$, (b) Maximum magnitude as a function of the distance z

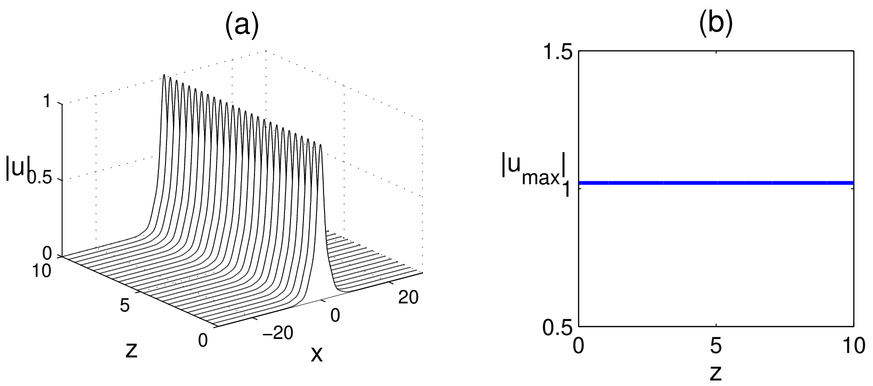

Figure 6.21 : (a) Nonlinear evolution of the soliton with $\mathscr{PT}$-periodic potential for $\tau_r = 0$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$, (b) Maximum magnitude as a function of the distance z

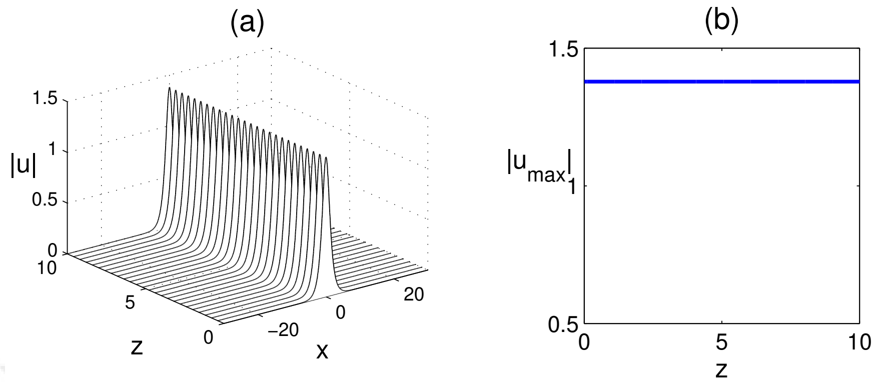

Figure 6.22 : (a) Nonlinear evolution of the soliton with $\mathscr{P}\mathscr{T}$-symmetric potential for $\tau_r = 0$, $\tau_i = 0.1$, $V_0 = 0.1$ and $W_0 = 0.1$, (b) Maximum magnitude as a function of the distance z

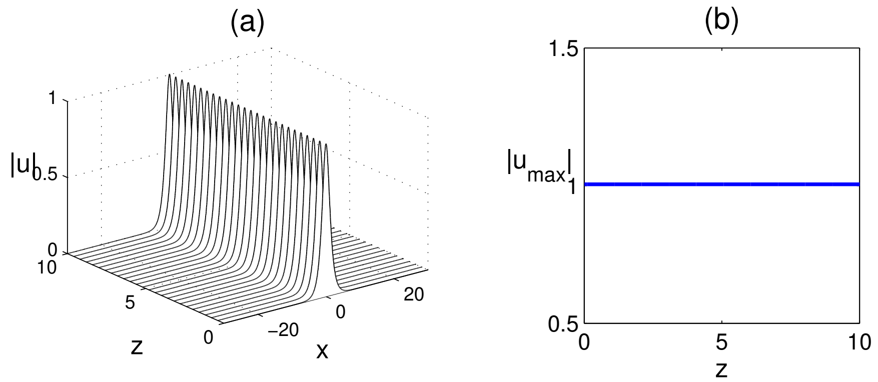

Figure 6.23 : (a) Nonlinear evolution of the soliton with Non-$\mathscr{P}\mathscr{T}$-symmetric potential for $\tau_r = 0.1$, $\tau_i = 0.1$, $V_0 = 1$ and $W_0 = 0.3$, (b) Maximum magnitude as a function of the distance z

7. CONCLUSION

In this thesis, we have investigated the existence of the Raman-induced optical solitons and their nonlinear stability properties governed by NLS equation with the Raman effect and periodic, $\mathscr{PT}$-symmetric and Non-$\mathscr{PT}$-symmetric potentials. Firsty, the Raman-induced solitons are found numerically by means of the pseudospectral method for varies τ values and the real and the imaginary potentials. The obtained numerical results are compared the analytical solutions for the case $\mathscr{PT}$-symmetric and Non-$\mathscr{PT}$-symmetric potentials. It is shown that numerical and analytic results are in good agreement.

As a result of the numerical observation, we concluded that

(i) In order the get the optical solitons, the propagation constant should be equal or greater than the depth of the real external potential.

(ii) It is shown that NLS equation with Raman effect and without potential, convergency is investigated according to different τ values. It is obtained that for the smaller values of real part of τ, numerical solutions converge to analytical solutions for NLS equation with Raman effect without potential. When the real part of τ is approaching to zero, and letting the imaginary part of τ as constant value, soliton solutions have obtained and the convergency has been achieved.

(iii) The solution of NLS equation with Raman effect and periodic potential are obtained for the different V_0, W_0 and τ values. It is found that W_0 or real part of τ value must be zero in order to get the convergency. If both of W_0 or real part of τ value are not equal to zero, bright soliton solutions have not been found.

(iv) It is necessary but not sufficient condition for the $\mathscr{PT}$-symmetry requires that the real and imaginary parts of the external potential are even and odd functions of the position respectively. In this thesis, in order to get the $\mathscr{PT}$-symmetric potential, the real part of the coefficient τ_r must be zero.

(v) The Raman-induced solitons have been also obtained for the Non-$\mathscr{PT}$-symmetric potential.

(vi) We shown that obtained Raman-induced solitons are nonlinearly stable for all the external potentials including periodic, $\mathscr{PT}$, and Non-$\mathscr{PT}$-symetric potentials.

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CURRICULUM VITAE

Name Surname: Merve KURT

Place and Date of Birth: Eminönü, 25 November 1992

Adress: Şemsipaşa Mah. 54. Sokak No:3/5 34250 Küçükköy, Gaziosmanpaşa/IST.

E-Mail: kurtm@itu.edu.tr

EDUCATION:

- **B.Sc.:** 2015, Istanbul Technical University, Faculty of Science and Letters, Department of Mathematical Engineering
- **M.Sc.:** 2019, Istanbul Technical University, Graduate School of Science, Engineering and Technology, Department of Mathematical Engineering