ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

GEOMETRY OF WEYL SPACES WITH A SPECIAL CONNECTION

Ph.D. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Programme

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

ÖZEL KONEKSİYONA SAHİP WEYL UZAYLARININ GEOMETRİSİ

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To me,
who has never left me on my own

FOREWORD

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ABBREVIATIONS

co-vector	**:** Covariant vector
W $_{\mathbf{n}}$(g,w)	**:** n-dimensional Weyl Space with metric tensor g, and 1-form w
WS $_{\mathbf{n}}$(g,w, π, μ)	**:** n-dimensional Weyl Space with semi-symmetric recurrent metric connection with metric tensor g, and 1-forms w, π, μ
EW $_{\mathbf{n}}$	**:** n-dimensional Einstein Weyl space
EWS $_{\mathbf{n}}$	**:** n-dimensional Einstein Weyl space with semi-symmetric recurrent metric connection

SYMBOLS

g_{ij}	: Riemannian metric tensor
M_n	: n-dimensional Riemannian manifold
$\varphi : U \to V$	: Homeomorphism from U to V
$\bigcup_{\alpha\in A} U_\alpha$	: The union of the union open sets defined by set A
$\Gamma_{jk}^{\ i}$	: Connection coefficient of Riemannian space
$\Gamma_{(jk)}^{\ i}$	: The symmetric part of the connection coefficient of Riemannian space
$\Gamma_{[jk]}^{\ i}$	: The anti-symmetric part of the connection coefficient of Riemannian space
$[jk,h]$	: Christoffel symbol of the first kind
$\left\{ {i \atop jk} \right\}$	: Christoffel symbol of the second kind
$T_{li}^{\ \ h}$	: Tensor field of type (1,2)
$S_{kj}^{\ \ t}$	: Torsion Tensor
∇^g	: The Riemannian Connection
∇^g_k	: Covariant derivative of the Riemannian connection with respect to k
$R_{kji}^{\ \ \ h}$	: The curvature tensor of the Riemannian space
R_{kjil}	: The covariant curvature tensor of the Riemannian space
R_{ji}	: The Ricci tensor of Riemannian space
R	: The scalar curvature of Riemannian space
∇	: Torsion free (symmetric) connection on Weyl space
w	: 1-form complementary co-vector field
$\bar{g}$	: Re-normalized metric tensor
$\bar{w}$	: Re-normalized 1-form complementary co-vector field
$\mathbb{G}$	: Conformal structure
Λ^1	: Space of 1-forms
Ω	: Scalar function on Weyl space
$W_{kji}^{\ \ \ h}$	: The curvature tensor of Weyl space
W_{kjil}	: The covariant curvature tensor of Weyl space
W_{ji}	: The Ricci tensor of Weyl space
W	: The scalar curvature tensor of Weyl space
$W_{(ji)}$	: The symmetric part of the Ricci tensor of Weyl space
$W_{[ji]}$	: The anti-symmetric part of the Ricci tensor of Weyl space
π	: 1-form on Weyl Space with semi-symmetric metric connection
λ	: 1-form on Weyl Space with semi-symmetric metric connection
μ	: 1-form on Weyl Space with semi-symmetric metric connection
w	: 1-form on Weyl Space with semi-symmetric recurrent metric connection
∇^*	: Semi-symmetric recurrent metric connection
$\overline{\nabla}$	: Semi-symmetric recurrent metric connection on Weyl space
$\overline{R}_{kji}^{\ \ \ h}$	: The curvature tensor of Weyl Space with semi-symmetric recurrent metric connection
$\overline{R}_{kjil}$	: The covariant curvature tensor of Weyl space with semi-symmetric recurrent metric connection

$\overline{R}_{ji}$	**:** The Ricci tensor of Weyl space with semi-symmetric recurrent metric connection
$\overline{R}$	**:** The scalar curvature tensor of Weyl space with semi-symmetric recurrent metric connection
$\overline{R}_{(ji)}$	**:** The symmetric part of the Ricci tensor of Weyl space with semi-symmetric recurrent metric connection
$\overline{R}_{[ji]}$	**:** The anti-symmetric part of the Ricci tensor of Weyl space with semi-symmetric recurrent metric connection
θ	**:** Scalar Function
G_{l}^{j}	**:** Generalized Einstein Tensor of Weyl Space with semi-symmetric recurrent metric connection
σ	**:** Conformal transformation on Weyl space with semi-symmetric recurrent metric connection
$\mathbb{C}_{kjim}$	**:** The conformal curvature tensor of Weyl space with semi-symmetric recurrent metric connection
$\mathbb{P}_{kji}{}^{h}$	**:** The projective curvature tensor of Weyl space with semi-symmetric recurrent metric connection

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GEOMETRY OF WEYL SPACES WITH A SPECIAL CONNECTION

SUMMARY

In this thesis, we investigate the semi-symmetric recurrent metric connection on Weyl manifolds and obtain some properties of curvatures, and curvature related quantities; give a necessary and sufficient condition for an Einstein Weyl manifold to be an Einstein Weyl manifold with semi symmetric recurrent metric connection; obtain geometric structures of semi-symmetric recurrent metric connection on Weyl manifolds under conformal, and projective transformations and construct some examples on this manifold having zero and constant scalar curvatures. In addition to all of above, comparing changes of geodesic equations with Riemannian, Weyl, and Weyl space with semi-symmetric recurrent metric, and to solve these equations for the specific example is given.
In the first chapter, the geometric quantities in Riemannian space, such as the curvature, curvature related properties and theorems are given primarily, and then Weyl Space is examined with similar analogy.
Two Riemannian metrics g and $\bar{g}$ are conformal if they coincide up to a factor which is positive function, i.e. $\bar{g} = e^{2\lambda} g$. This is an equivalence relation, each class $\mathbb{G}$ being called a conformal structure. A Weyl structure is a map $w : \mathbb{G} \to \Lambda^1(W)$ satisfying $w(e^{2\lambda} g) = w(g) + 2d\lambda$. A n-dimensional manifold with a Weyl structure is called a n-dimensional Weyl manifold denoted by $W_n(g,w)$. For $W_n(g,w)$, there exists an unique torsion-free connection ∇ that preserves the conformal class $\mathbb{G}$. Preserving the conformal class means that for any $g \in \mathbb{G}$ there exists 1-form w such that

$$\nabla g = 2w \otimes g.$$

The relation between the Weyl connection ∇, and the Riemannian connection ∇^g is

$$\nabla_X Y = \nabla^g_X Y - w(X)Y - w(Y)X + g(X,Y)\psi,$$

where X, Y are vector fields on $W_n(g,w)$ and ψ is the dual vector field to w such that $w(X) = g(X,\psi)$. In local coordinates, it can be given by

$$\Gamma_{ji}{}^{l} = \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} - (w_j \delta^l_i + w_i \delta^l_j - w^l g_{ji}),$$

where Γ^l_{ji} are the coefficients of the Weyl connection. The curvature tensor W of ∇ is given by

$$W(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,,$$

also, in local coordinates,

$$W_{kji}{}^{l} = R_{kji}{}^{l} - \delta^l_j w_{ki} + \delta^l_k w_{ji} + \delta^l_i (w_{jk} - w_{kj}) + g^{ls}(g_{ji} w_{ks} - g_{ki} w_{js}) \,,$$

where

$$w_{jk} = \nabla_j w_k + w_j w_k - \frac{1}{2} g_{jk} w_t w^t ,$$

and $R_{kji}{}^{l}$ represents the Riemannian curvature tensor.
Let (M_n, g), $(n > 2)$ be an n-dimensional differentiable manifold with metric tensor g, and let ∇ be Riemannian connection. A linear connection $\overset{\circ}{\nabla}$ on M_n, whose coefficients are $\overset{\circ}{\Gamma}{}_{jk}^{i}$, is said to be semi-symmetric if the torsion tensor S of $\overset{\circ}{\nabla}$ satisfies

$$S_{jk}{}^{i} = \overset{\circ}{\Gamma}{}_{jk}^{i} - \overset{\circ}{\Gamma}{}_{kj}^{i} = \pi_k \delta_j^i - \pi_j \delta_k^i ,$$

where $\pi_j = \lambda_j + \mu_j$ is defined by arbitrary 1-forms λ_j and μ_j. In addition, if a semi-symmetric metric connection has recurrency condition $\overset{\circ}{\nabla}_k g_{ij} = 2\mu_k g_{ij}$ then the connection $\overset{\circ}{\nabla}$ is said to be semi-symmetric recurrent-metric connection and μ is called the recurrent covariant vector field.
Theorem. Let $WS_n(g, w, \pi, \mu)$ be an n-dimensional Weyl manifold equipped with the semi-symmetric recurrent-metric connection $\overline{\nabla}$ associated with 1-forms, w, π, and μ, respectively. Then, there exists a unique connection $\overline{\nabla}$ on $WS_n(g, w, \pi, \mu)$ given by

$$\overline{\nabla}_X Y = \nabla_X Y - \mu(X)Y - \mu(Y)X + g(X, Y)\xi + \pi(Y)X - g(X, Y)\eta ,$$

where ξ , η are dual vector fields such that

$$\mu(X) = g(X, \xi), \qquad \pi(X) = g(X, \eta) .$$

In local coordinates, it is obtained as

$$\overline{\Gamma}{}_{ik}^{l} = \begin{Bmatrix} l \\ ik \end{Bmatrix} - (w_i \delta_k^l + w_k \delta_i^l - w^l g_{ik}) + (\lambda_k \delta_i^l - \mu_i \delta_k^l - \lambda^l g_{ik}) ,$$

where $\lambda_k = \pi_k - \mu_k$.
Theorem. The curvature tensor of WS_n is obtained as

$$\overline{R}_{kjim} = W_{kjim} + Q_{kjim} - g_{mk}\alpha_{ij} + g_{mj}\alpha_{ik} - g_{ij}\alpha_{mk} + g_{ik}\alpha_{mj} ,$$

where

$$\begin{aligned} Q_{kjim} &= (g_{mj}\lambda_{ki} - g_{mk}\lambda_{ji} + \lambda_{jm} g_{ki} - \lambda_{km} g_{ji}) + 2g_{mi}\nabla_{[j}\mu_{k]} , \\ \alpha_{ij} &= (\lambda_i w_j + \lambda_j w_i - g_{ij} w_l \lambda^l) . \end{aligned}$$

Theorem. The curvature tensor of WS_n satisfies the following symmetry relations:

$$\begin{aligned} \overline{R}_{kjil} + \overline{R}_{jkil} &= 0 , \\ \overline{R}_{kjil} + \overline{R}_{kjli} &= W_{kjil} + W_{kjli} + Q_{kjil} + Q_{kjli} \\ &= 4g_{il}(\nabla_{[j}\mu_{k]} + \nabla_{[j}w_{k]}) . \end{aligned}$$

Theorem. The curvature tensor of WS_n satisfies the following the first and the second Bianchi identities for WS_n, respectively,

$$\begin{aligned} \overline{R}_{kji}{}^{l} + \overline{R}_{jik}{}^{l} + \overline{R}_{ikj}{}^{l} &= 2(\delta_j^l \nabla_{[k}\pi_{i]} + \delta_i^l \nabla_{[j}\pi_{k]} + \delta_k^l \nabla_{[i}\pi_{j]}) , \\ \overline{\nabla}_l \overline{R}_{kji}{}^{t} + \overline{\nabla}_j \overline{R}_{lki}{}^{t} + \overline{\nabla}_k \overline{R}_{jli}{}^{t} &= 2(\pi_l \overline{R}_{kji}{}^{t} + \pi_j \overline{R}_{lki}{}^{t} + \pi_k \overline{R}_{jli}{}^{t}) . \end{aligned}$$

The Ricci curvature, the scalar curvature for WS_n are

$$\overline{R}_{ji} = W_{ji} + Q_{ji} - (n-2)(\lambda_j w_i + \lambda_i w_j) + 2(n-2) g_{ji} w_l \lambda^l \ ,$$
$$\overline{R} = R + 2(n-1)(\nabla_t w^t - \nabla_t \lambda^t) - (n-1)(n-2)(w_t - \lambda_t)(w^t - \lambda^t) \ .$$

Theorem. WS_n and W_n have same curvature tensors if and only if, the recurrent covariant vector field μ_k of $\overline{\nabla}$ is gradient vector, and the following equation holds

$$\lambda_{ij} + \alpha_{ji} = 0,$$

where

$$\lambda_{ij} = \nabla_i \lambda_j - \lambda_i \lambda_j + \frac{1}{2} g_{ij} \lambda_t \lambda^t \ .$$

Also, we give an example of 3-dimensional WS_n by the metric as

$$ds^2 = \frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (r > 0 \ , \ 0 \leq \theta < \pi, \ 0 \leq \phi < 2\pi) \ ,$$

where $1 - \kappa r^2 > 0$, and κ is an arbitrary constant, and find out in which condition the scalar curvature of WS_n becomes the same as the scalar curvature of Riemannian space.

Theorem. For a n-dimensional Einstein Weyl manifold EW_n to be an Einstein Weyl manifold with the semi-symmetric recurrent-metric connection EWS_n the necessary and sufficient condition is

$$\nabla_j \lambda_i + \nabla_i \lambda_j + 2(w_i \lambda_j + w_j \lambda_i + \lambda_j \lambda_i) = \beta g_{ij}$$

for a scalar function β defined on WS_n.

Theorem. In an EWS_n, coefficient β is not constant in general.

Theorem. Generalized Einstein tensor for WS_n is prolonged covariant constant.

Theorem. Any isotropic WS_n can be locally mapped conformally to EWS_n.

Theorem. The conformal curvature tensor $\mathbb{C}_{kjim}$ of WS_n is obtained as in the form

$$\begin{aligned}
\mathbb{C}_{kjim} = {} & \overline{R}_{kjim} - \overline{R}_{mj}\left(\frac{(3n+2)g_{ik}}{2n(1+n)}\right) - \overline{R}_{ik}\left(\frac{(3n+2)g_{jm}}{2n(1+n)}\right) - \overline{R}_{jm}\left(\frac{(3n+2)g_{ki}}{2n(1+n)}\right) \\
& + \overline{R}_{jk}\left(\frac{1-g_{im}}{2(1+n)}\right) + \overline{R}_{kj}\left(\frac{g_{im}}{2(1+n)}\right) + \overline{R}_{km}\left(\frac{1-g_{ji}}{2(1+n)}\right) + \overline{R}_{mk}\left(\frac{g_{ji}}{2(1+n)}\right) - \overline{R}_{ki}\left(\frac{g_{mj}}{2(1+n)}\right) \\
& + \overline{R}_{mi}\left(\frac{1-g_{kj}}{2(1+n)}\right) + \overline{R}_{im}\left(\frac{g_{kj}}{2(1+n)}\right) + \overline{R}_{ij}\left(\frac{1-g_{mk}}{2(1+n)}\right) - \overline{R}_{ji}\left(\frac{g_{mk}}{2(1+n)}\right) \\
& + \overline{R}\left(\frac{1}{(n-1)(n+1)}\right)[2g_{jm}g_{ki} - g_{ji}g_{km} - g_{kj}g_{mi}] \ .
\end{aligned}$$

Theorem. The projective curvature tensor $\mathbb{P}_{ikh}^{\ \ \ l}$ of WS_n is obtained as in the form

$$\begin{aligned}
\mathbb{P}_{ikh}^{\ \ \ l} = {} & \overline{R}_{ikh}^{\ \ \ l} + \mathbb{Q}_{ikh}^{\ \ \ l} - \mathbb{Q}_{kih}^{\ \ \ l} \\
& + \frac{1}{n^2-1}[(n(\overline{R}_{ih} + \mathbb{Q}_{ih} - \mathbb{Q}_{hi}) + \overline{R}_{hi} + \mathbb{Q}_{hi} - \mathbb{Q}_{ih})\delta_k^l - (n(\overline{R}_{ik} + \mathbb{Q}_{ik} - \mathbb{Q}_{ki}) + \overline{R}_{ki} + \mathbb{Q}_{ki} - \mathbb{Q}_{ik})\delta_h^l] \\
& + \frac{1}{n+1}(\overline{R}_{kh} - \overline{R}_{hk} + 2(\mathbb{Q}_{kh} - \mathbb{Q}_{hk}))\delta_i^l \ .
\end{aligned}$$

Theorem. The sufficient condition for having the same geodesic equations of Riemannian Space and WS_n is given by $\mu_k + w_k = 0$. Here, w, and μ are 1-forms.

Also, we examine the geodesics of the hyperbolic plane with given metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$ where $y > 0$ on WS_n as an example.

ÖZEL KONEKSİYONA SAHİP WEYL UZAYLARININ GEOMETRİSİ

ÖZET

Bu tezde, özel koneksiyona sahip Weyl manifoldu üzerinde bazı belirgin geometrik yapıları ve büyüklükleri inceledik. Özel olarak, yarı-simetrik rekürant metrik koneksiyona sahip uzay tanımlanarak, geometrik yapısı üzerinde duruldu ve Riemann uzayı ile bu yeni yapıya sahip Weyl uzayı arasındaki farklılıklara ve benzerliklere değinildi.

Genel bir çerçeve verilecek olursa; Weyl manifoldu üzerinde yarı-simetrik rekürant metrik koneksiyon tanımlanarak, bu manifold üzerinde eğriler, eğrilere bağlı özellikler ve büyüklükler incelendi; yarı-simetrik rekürant metrik koneksiyona sahip Einstein Weyl manifoldunun, Einstein Weyl manifoldu olması için gerek ve yeter koşul ispatlandı, kesitsel eğriliği incelendi; bu yapıya sahip Weyl manifoldu üzerindeki geometrik yapıların dönüşümlerini saptamak adına konformal ve projektif dönüşümler altındaki eğrilikleri hesaplandı; sabit ve sıfır skaler eğriliklere sahip Weyl manifoldu inşa edilerek, ilgili eğrisel hesaplamalar üzerinde çalışıldı. Tüm bunlara ek olarak, Riemann, Weyl ve yarı-simetrik rekürant metrik koneksiyona sahip Weyl uzayları arasında, ilgili jeodezik denklemlerin değişimlerini karşılaştırmak için, koşulları özel olarak seçilen bir örnek kurularak, çözümü yapıldı.

İlk bölümde, öncelikli olarak, Riemann uzayındaki geometrik büyüklükler; eğriler, eğrilerle ilgili özellikler ve teoremler verildi. Sonrasında, benzer anoloji kurularak, Weyl uzayındaki yapılar araştırıldı.

n-boyutlu Riemann uzayında, verilen herhangi bir x^i $(i = 1, \cdots, n)$, koordinatları için, birbirine çok yakın sonsuz küçük mesafedeki x^i ve $x^i + dx^i$ noktaları

$$ds^2 = g_{ij} dx^i dx^j, \quad (i, j = 1, 2, \cdots, n)$$

şeklinde tanımlanır. Burada, g_{ij} katsayısı, x^i 'nin koordinat fonksiyonunu temsil eder ve Riemann metrik tensörü olarak adlandırılır.

Levi-Civita koneksiyonuna göre g_{ij} metrik tensörünün kovaryant türevi

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \begin{Bmatrix} a \\ kj \end{Bmatrix} g_{ai} - \begin{Bmatrix} a \\ ki \end{Bmatrix} g_{aj} = 0$$

şeklindedir. Genel olarak, M_n uzayında, eğrilik tensörü

$$R_{kji}{}^h = \partial_k \Gamma_{ji}^{\ h} - \partial_j \Gamma_{ki}^{\ h} + \Gamma_{kt}^{\ h} \Gamma_{ji}^{\ t} - \Gamma_{jt}^{\ h} \Gamma_{ki}^{\ t}, \quad (\partial_k = \frac{\partial}{\partial x^k})$$

olarak tanımlanır. Burada, $\Gamma_{ji}^{\ h}$ 'ya koneksiyon katsayısı denir.

g ve $\bar{g}$ iki Riemann metriği olmak üzere, eğer bu iki metrik birbirinin, pozitif tanımlanmış bir fonksiyon katı kadar ise, yani aralarında $\bar{g} = e^{2\lambda} g$ ilişkisi mevcutsa, bu iki metrik birbirine konformaldir, denir. Buradaki denklik ifadesinde yer alan her

bir $\mathbb{G}$ sınıfı konformal bir yapıdır.
Weyl yapısı, $w(e^{2\lambda}g) = w(g) + 2d\lambda$ koşulunu sağlayan $w : \mathbb{G} \to \Lambda^1(W)$ ile tanımlı bir dönüşümdür. Böyle bir yapıya sahip olan n-boyutlu manifolda da Weyl manifoldu denir ve $W_n(g,w)$ ile gösterilir. n-boyutlu Weyl manifoldu $W_n(g,w)$ için, $\mathbb{G}$ konformal sınıfları koruyan, burulmasız tek bir ∇ koneksiyonu vardır. Bahsi geçen, konformal sınıfları koruyan ifadesi, herhangi bir $g \in \mathbb{G}$ için öyle bir w 1-form vardır ki

$$\nabla g = 2w \otimes g.$$

eşitliğini sağlar manasındadır. Weyl koneksiyonu ∇ ve Riemann koneksiyonu ∇^g arasında

$$\nabla_X Y = \nabla^g_X Y - w(X)Y - w(Y)X + g(X,Y)\psi,$$

ilişkisi mevcuttur. Burada, X, Y, W_n üzerinde vektör alanları ve ψ de, $w(X) = g(X,\psi)$ ifadesine sağlayan w'ya dual vektördür. Lokal koordinatlarda ise

$$\Gamma_{ji}^{\ \ l} = \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} - (w_j\delta^l_i + w_i\delta^l_j - w^l g_{ji}),$$

olarak verilir; $\Gamma_{ji}^{\ \ l}$, Weyl koneksiyonunun katsayılarını belirtir. Ayrıca,

$$\left\{ \begin{matrix} l \\ ji \end{matrix} \right\} = \frac{1}{2} g^{lm}(\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ji}).$$

Levi-Civita koneksiyonu, ∇^g 'nin katsayılarıdır.
∇ koneksiyonuna sahip Weyl uzayının, W eğrilik tensörü

$$W(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

ile verilir. Lokal koordinatlar da ise,

$$W_{kji}^{\ \ \ l} = R_{kji}^{\ \ \ l} - \delta^l_j w_{ki} + \delta^l_k w_{ji} + \delta^l_i(w_{jk} - w_{kj}) + g^{ls}(g_{ji}w_{ks} - g_{ki}w_{js}),$$

öyle ki

$$w_{jk} = \nabla_j w_k + w_j w_k - \frac{1}{2} g_{jk} w_t w^t$$

şeklindedir ve $R_{kji}^{\ \ \ l}$ Riemann eğrilik tensörünü temsil eder.
(M_n, g), $(n > 2)$ için, g metrik tensörüne sahip, n-boyutlu türevlenebilir bir manifold ve ∇ ise Riemann koneksiyon olsun. Koneksiyon katsayıları $\overset{\circ}{\Gamma}{}_{jk}^{i}$ ile verilen M_n üzerinde tanımlı $\overset{\circ}{\nabla}$ lineer koneksiyonu yarı-simetrik ise,

$$S_{jk}^{\ \ i} = \overset{\circ}{\Gamma}{}_{jk}^{i} - \overset{\circ}{\Gamma}{}_{kj}^{i} = \pi_k\delta^i_j - \pi_j\delta^i_k,$$

$\overset{\circ}{\nabla}$ 'ya ait S burulma tensörü yukarıdaki koşulu sağlar. Burada, $\pi_j = \lambda_j + \mu_j$ ile tanımlı olmak üzere, λ_j ve μ_j keyfi 1-formlardır. Buna ek olarak, yarı-simetrik koneksiyon $\overset{\circ}{\nabla}_k g_{ij} = 2\mu_k g_{ij}$ ve μ rekürant kovaryant vektör alanı olmak üzere, bu rekürantlık koşuluna da sahipse, $\overset{\circ}{\nabla}$ koneksiyonuna, yarı-simetrik rekürant metrik koneksiyon denir. $\overset{\circ}{\nabla}$ lineer koneksiyonuna ait koneksiyon katsayısı, 1-formlar aracılığıyla,

$$\overset{\circ}{\Gamma}{}_{ji}^{h} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} + (\delta_j^h \lambda_i - \delta_i^h \mu_j - g_{ji} \lambda^h)$$

ile ifade edilir.

Teorem. w, π ve μ 1-formlarıyla ilişkili $\overline{\nabla}$ yarı-simetrik rekürant metrik koneksiyonuna sahip n-boyutlu Weyl manifoldu $WS_n(g,w,\pi,\mu)$ tanımlı olsun. Öyleyse, $WS_n(g,w,\pi,\mu)$ üzerinde tek tip tanımlı $\overline{\nabla}$ koneksiyonu

$$\overline{\nabla}_X Y = \nabla_X Y - \mu(X)Y - \mu(Y)X + g(X,Y)\xi + \pi(Y)X - g(X,Y)\eta$$

ile verilir. Burada, ξ , η

$$\mu(X) = g(X,\xi), \qquad \pi(X) = g(X,\eta)$$

ifadelerini sağlayan dual vektör alanlarıdır. Lokal koordinatlarda ise

$$\overline{\Gamma}{}_{ik}^{l} = \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} - (w_i \delta_k^l + w_k \delta_i^l - w^l g_{ik}) + (\lambda_k \delta_i^l - \mu_i \delta_k^l - \lambda^l g_{ik}) ,$$

$\lambda_k = \pi_k - \mu_k$ koşulu ile tanımlıdır.

Teorem. WS_n manifolduna ait eğrilik tensörü

$$\overline{R}_{kjim} = W_{kjim} + Q_{kjim} - g_{mk}\alpha_{ij} + g_{mj}\alpha_{ik} - g_{ij}\alpha_{mk} + g_{ik}\alpha_{mj}$$

şeklinde saptanır, burada Q_{kjim} ve α_{ij}

$$\begin{aligned} Q_{kjim} &= (g_{mj}\lambda_{ki} - g_{mk}\lambda_{ji} + \lambda_{jm}g_{ki} - \lambda_{km}g_{ji}) + 2g_{mi}\nabla_{[j}\mu_{k]} , \\ \alpha_{ij} &= (\lambda_i w_j + \lambda_j w_i - g_{ij} w_l \lambda^l) \end{aligned}$$

ile tanımlıdır.

Teorem. WS_n manifolduna ait eğrilik tensörü aşağıda verilen simetri özelliklerini sağlar:

$$\begin{aligned} \overline{R}_{kjil} + \overline{R}_{jkil} &= 0 , \\ \overline{R}_{kjil} + \overline{R}_{kjli} &= W_{kjil} + W_{kjli} + Q_{kjil} + Q_{kjli} \\ &= 4g_{il}(\nabla_{[j}\mu_{k]} + \nabla_{[j}w_{k]}) . \end{aligned}$$

Teorem. WS_n manifolduna ait eğrilik tensörü, sırasıyla aşağıda verilen birinci ve ikinci Bianchi özdeşliklerini sağlar:

$$\begin{aligned} \overline{R}_{kji}{}^{l} + \overline{R}_{jik}{}^{l} + \overline{R}_{ikj}{}^{l} &= 2(\delta_j^l \nabla_{[k}\pi_{i]} + \delta_i^l \nabla_{[j}\pi_{k]} + \delta_k^l \nabla_{[i}\pi_{j]}) , \\ \overline{\nabla}_l \overline{R}_{kji}{}^{t} + \overline{\nabla}_j \overline{R}_{lki}{}^{t} + \overline{\nabla}_k \overline{R}_{jli}{}^{t} &= 2(\pi_l \overline{R}_{kji}{}^{t} + \pi_j \overline{R}_{lki}{}^{t} + \pi_k \overline{R}_{jli}{}^{t}) . \end{aligned}$$

WS_n için; Ricci eğriliği, bu eğriliğe ait simetri özellikleri ve skaler eğriliği sırasıyla,

$$\overline{R}_{ji} = W_{ji} + Q_{ji} - (n-2)(\lambda_j w_i + \lambda_i w_j) + 2(n-2) g_{ji} w_l \lambda^l ,$$

$$\overline{R}_{(ji)} = W_{(ji)} - \frac{(n-2)}{2}[(\nabla_j\lambda_i + \nabla_i\lambda_j) + 2\lambda_i\lambda_j + 2(\lambda_j w_i + \lambda_i w_j)] - g_{ji}[2(n-2)\lambda_t\lambda^t + \nabla_t\lambda^t] ,$$

$$\overline{R}_{[ji]} = n\nabla_{[j}w_{i]} - (n-2)\nabla_{[j}\lambda_{i]} + 2\nabla_{[j}\mu_{i]} ,$$

$$\overline{R} = R + 2(n-1)(\nabla_t w^t - \nabla_t\lambda^t) - (n-1)(n-2)(w_t - \lambda_t)(w^t - \lambda^t)$$

şeklindedir.
Teorem. WS_n ve W_n uzaylarının aynı eğrilik tensörlerine sahip olması için gerek ve yeter koşul; $\overline{\nabla}$ koneksiyonuna ait rekürant kovaryant vektör alanı μ_k'nün gradyant olması ve

$$\lambda_{ij} + \alpha_{ji} = 0,$$

eşitliğinin sağlanmasıdır, burada λ_{ij}

$$\lambda_{ij} = \nabla_i \lambda_j - \lambda_i \lambda_j + \frac{1}{2} g_{ij} \lambda_t \lambda^t$$

ile verilir.
Ayrıca tüm bunlara ek, 3-boyutlu WS_n üzerinde κ keyfi bir sabit olmak üzere ve $1 - \kappa r^2 > 0$ koşulu ile verilen

$$ds^2 = \frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (r > 0\ ,\ 0 \leq \theta < \pi,\ 0 \leq \phi < 2\pi)$$

metriği ile tanımlı bir örnek oluşturularak; hangi durum altında WS_n uzayının skaler eğriliğinin, Riemann uzayının skaler eğriliği ile aynı olduğu çözümlenmiştir.
Teorem. n-boyutlu Einstein Weyl manifoldunun EW_n, yarı-simetrik rekürant metrik koneksiyonlu Einstein Weyl manifoldu EWS_n olması için gerek ve yeter koşul,

$$\nabla_j \lambda_i + \nabla_i \lambda_j + 2(w_i \lambda_j + w_j \lambda_i + \lambda_j \lambda_i) = \beta g_{ij}$$

ifadesi ile verilir, ve burada β, WS_n üzerinde tanımlı skaler bir fonksiyondur.
Teorem. Bir EWS_n uzayında, β katsayısı her zaman sabit değildir.
Teorem. WS_n uzayı için tanımlı genelleştirilmiş Einstein tensörü genelleştirilmiş kovaryant türeve göre sabittir.
Theorem. Herhangi bir izotropik WS_n, lokal anlamda EWS_n uzayına konformal olarak dönüştürülebilir.
Teorem. WS_n uzayına ait $\mathbb{C}_{kjim}$ konformal eğrilik tensörü,

$$\begin{aligned}\mathbb{C}_{kjim} = {}& \overline{R}_{kjim} - \overline{R}_{mj}\left(\frac{(3n+2)g_{ik}}{2n(1+n)}\right) - \overline{R}_{ik}\left(\frac{(3n+2)g_{jm}}{2n(1+n)}\right) - \overline{R}_{jm}\left(\frac{(3n+2)g_{ki}}{2n(1+n)}\right)\\ &+ \overline{R}_{jk}\left(\frac{1-g_{im}}{2(1+n)}\right) + \overline{R}_{kj}\left(\frac{g_{im}}{2(1+n)}\right) + \overline{R}_{km}\left(\frac{1-g_{ji}}{2(1+n)}\right) + \overline{R}_{mk}\left(\frac{g_{ji}}{2(1+n)}\right) - \overline{R}_{ki}\left(\frac{g_{mj}}{2(1+n)}\right)\\ &+ \overline{R}_{mi}\left(\frac{1-g_{kj}}{2(1+n)}\right) + \overline{R}_{im}\left(\frac{g_{kj}}{2(1+n)}\right) + \overline{R}_{ij}\left(\frac{1-g_{mk}}{2(1+n)}\right) - \overline{R}_{ji}\left(\frac{g_{mk}}{2(1+n)}\right)\\ &+ \overline{R}\left(\frac{1}{(n-1)(n+1)}\right)[2g_{jm}g_{ki} - g_{ji}g_{km} - g_{kj}g_{mi}]\end{aligned}$$

formundadır.
Teorem. WS_n uzayına ait $\mathbb{P}_{ikh}^{\ \ \ l}$ projektif eğrilik tensörü,

$$\begin{aligned}\mathbb{P}_{ikh}^{\ \ \ l} = {}& \overline{R}_{ikh}^{\ \ \ l} + \mathbb{Q}_{ikh}^{\ \ \ l} - \mathbb{Q}_{kih}^{\ \ \ l}\\ &+ \frac{1}{n^2-1}[(n(\overline{R}_{ih} + \mathbb{Q}_{ih} - \mathbb{Q}_{hi}) + \overline{R}_{hi} + \mathbb{Q}_{hi} - \mathbb{Q}_{ih})\delta_k^l - (n(\overline{R}_{ik} + \mathbb{Q}_{ik} - \mathbb{Q}_{ki}) + \overline{R}_{ki} + \mathbb{Q}_{ki} - \mathbb{Q}_{ik})\delta_h^l]\\ &+ \frac{1}{n+1}(\overline{R}_{kh} - \overline{R}_{hk} + 2(\mathbb{Q}_{kh} - \mathbb{Q}_{hk}))\delta_i^l\end{aligned}$$

formundadır.

Teorem. Riemann ve WS_n uzaylarının aynı jeodezik denklemlere sahip olması için yeter koşul, $\mu_k + w_k = 0$ ifadesi ile verilir. Burada w ve μ, 1-formlardır.
Ayrıca, WS_n üzerinde tanımlı $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$, $y > 0$ metriği ile verilen hiperbolik düzleme ait jeodezik denklemleri örnek olarak incelenmiştir.

1. INTRODUCTION

Differential geometry studies the geometrical concepts by using differential calculus. One of the most studied problems in differential geometry is classification of spaces with respect to the structures defined on them. This allows one to classify spaces and to present new interesting properties of the geometrical quantities. In Riemannian geometry, Riemannian space admits a bilinear symmetric form; metric and covariant derivative of the metric with respect to this linear connection is zero, [1], [2].

Besides its geometrical attention, Riemannian geometry finds a lot of applicable area in physics, especially theoretical and mathematical physics field. Theory of Einstein's General Relativity can be given, individually, as almost full success of differential geometry. In the beginning, theory is stated in differential geometric language first and then the Einstein-Hilbert action followed later, [3], [4], [5], [6], [7].

A generalization of Riemannian spaces can be provided by defining new connections on them. For example, symmetric but non-metric connections (Weyl spaces), semi-symmetric connections, non-symmetric connections (torsion spaces) etc. These spaces are also attractive for physical sciences especially particle physics and gravitational physics for vectorial and spinorial materials, [8], [9].

In literature, many authors have been studied Weyl spaces [10], [11], [12], [13]. The starting point of Weyl was to state a unified theory for the electromagnetic and gravitational interactions. But the electric charge conservation is violated in this unified theory [14], [15]. In spite of unacceptable theory for physics, it presents some special properties for differential geometry. It relates spaces by conformal transformations and display some conserved quantities during transformations. The change in curvature and curvature related geometric quantities by introducing some structures on these type of spaces have been widely studied, [2], [16].

On the other hand, technological improvements made huge contributions in the astrophysical science in a couple of decades and observations show that the theory of gravity is beyond the Einstein's theory in the large scale. This is a great and new practice area of differential geometry. Some of theoretical physicists are insterested in modified gravity theories to explain the system, and therefore, they may need spaces having some special connections or having some characteristic curvature properties or dimensions, [17], [18].

When we look at Einstein's geometrization of gravity, we see the usage of the metric tensor g_{ij} and the symmetric linear connection $\Gamma_{jk}^{\ i}$ instead of the Newtonian gravitational potential and the Newtonian gravitational force. In essence, the general theory of relativity the gravitational field is represented by the metric tensor and indirectly by the curvature of spacetime.

In Einstein's theory of relativity non-vacuum solutions admit conserved energy-momentum tensors. Then, electromagnetic field as an example of conserved source is included in Einstein's theory of relativity, [10].

1.1 Purpose of Thesis

Main purpose of this thesis is to examine certain geometrical structures on Weyl manifolds having some special connections. Especially, by considering spaces with semi-symmetric recurrent metric connection the geometrical structures of the spaces are studied, the differences and similarities between Riemannian and Weyl spaces would be presented.
Within this context, the following problems which are, to investigate the semi-symmetric recurrent metric connection on Weyl manifolds and obtain some properties of curvatures, and curvature related quantities; to give a necessary and sufficient condition for an Einstein Weyl manifold to be an Einstein Weyl manifold with semi symmetric recurrent metric connection; to obtain geometric structures of semi-symmetric recurrent metric connection on Weyl manifolds under conformal and projective transformations and to construct some examples on this manifold having zero and constant scalar curvatures studied. In addition to all of above, comparing changes of geodesic equations with Riemannian, Weyl, and Weyl space with semi-symmetric recurrent metric, and to solve these equations for the specific examples are given.

1.2 Literature Review

In this chapter, the geometric quantities in Riemannian space, such as the curvature, curvature related properties and theorems are given primarily to examine Weyl Space with similar analogy.

1.2.1 Riemannian Spaces

In an n-dimensional Riemannian space, for any given coordinates x^i $(i = 1, \cdots, n)$, the infinite small distance between very close points x^i and $x^i + dx^i$ is defined by

$$ds^2 = g_{ij} dx^i dx^j, \quad (i, j = 1, 2, \cdots, n) . \tag{1.1}$$

In the equation (1.1), the coefficient g_{ij} is coordinate function of x^i and said to be Riemannian metric tensor. In addition, the space and the geometry which are characterized by such a metric, are called Riemannian space and the Riemannian geometry respectively, [2], [19], [20].

If a vector A is parallel transported along a curve between very close points such as x^i and $x^i + dx^i$, the components of the vector change as follows;

$$dA^i = -\Gamma^i_{jk} A^j dx^k , \tag{1.2}$$

where Γ^i_{jk} represents the connection coefficients, [21].

Definition. Let M be a Hausdorff space. Each point p of M has an open neighbourhood U for which there is a homeomorphism such that $\varphi : U \to V$ where V is an open subset of R^n, then we call M as an n-dimensional topological manifold and the pair of (U, φ) is said to be local chart around p.

Let M be a topological manifold. Suppose that A be an index set and U_α be the union of open sets which is defined by A. If there is a collection such that $S = \{(U_\alpha, \varphi_\alpha)_{\alpha \in A}\}$ on M which satisfies the following conditions, then we say that the collection construct an n-dimensional differentiable structure on M_n, [1].

(i) $\bigcup_{\alpha \in A} U_\alpha = M$

(ii) For any $\alpha, \beta \in A$, the functions $f_{\beta\alpha}$ and $f_{\alpha\beta}$ as,

$$f_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset R^n \to \phi_\beta(U_\alpha \cap U_\beta) \subset R^n$$

,

$$f_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset R^n \to \phi_\alpha(U_\alpha \cap U_\beta) \subset R^n$$

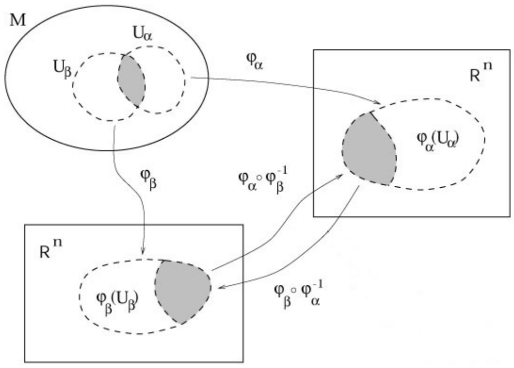

Figure 1.1 : Manifold

(iii) The family set of $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is maximal by the conditions **(i)** and **(ii)**.

Definition. Let M_n be a Riemannian space defined by the metric tensor g_{ij}, there exists only one connection which is compatible with the metric tensor g_{ij} of M_n.

Assume $\Gamma_{jk}^{\ i}$ and $\Gamma_{\alpha\beta}^{'\ \gamma}$ be the coordinate functions of x and x', respectively, so the following statement is provided [22];

$$\Gamma_{\alpha\beta}^{'\ \gamma} \frac{\partial x^i}{\partial x'^\gamma} = \frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} + \Gamma_{jk}^{\ i} \frac{\partial x^j}{\partial x'^\alpha} \frac{\partial x^k}{\partial x'^\beta} \,, \tag{1.3}$$

which means $\Gamma_{jk}^{\ i}$ is not a tensor. [21]

Furthermore, if the connection $\Gamma_{jk}^{\ i}$ is Riemannian (i.e. Levi-Civita), then we have

$$g^{ih} g_{jh} = \delta^i_j \quad , \qquad \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.4}$$

$$\Gamma_{jk}^{\ i} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = g^{ij}[jk, h] \quad , \qquad [jk, h] = \frac{1}{2}\left(\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right) . \tag{1.5}$$

Here, we call $[jk, h]$ and $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ as Christoffel symbols of the first kind and the second kind, respectively. It is seen that Christoffel symbols are symmetric respect to the lower two indices.

In general, if connection is different from Levi-Civita, the connection coefficient $\Gamma_{jk}^{\ i}$ is composed of its symmetric and anti-symmetric parts on n-dimensional Riemannnian space M_n. The symmetric and anti-symmetric part of the connection coefficients $\Gamma_{jk}^{\ i}$ are defined by, respectively,

$$\Gamma_{(jk)}^{\ \ i} = \frac{1}{2}(\Gamma_{jk}^{\ i} + \Gamma_{kj}^{\ i}) \,, \tag{1.6}$$

$$\Gamma_{[jk]}^{\ \ i} = \frac{1}{2}(\Gamma_{jk}^{\ i} - \Gamma_{kj}^{\ i}) \,, \tag{1.7}$$

where $\Gamma_{(jk)}^{\ \ i}$ is a connection coefficient and $\Gamma_{[jk]}^{\ \ i}$ is a tensor which is called the torsion tensor of the connection. Recall that, since $\Gamma_{(jk)}^{\ \ i}$ in (1.3) is not a tensor, $\Gamma_{[jk]}^{\ \ i}$ is a tensor.

By (1.6) and (1.7), we write

$$\Gamma_{jk}^{\ i} = \Gamma_{(jk)}^{\ i} + \Gamma_{[jk]}^{\ i} . \tag{1.8}$$

In Riemannian space, $\Gamma_{jk}^{\ i}$ is symmetric and the torsion tensor $\Gamma_{[jk]}^{\ i}$ equals to zero.
Let v^i, v_i and $T_{ji}^{\ \ h}$ be the components of a contravariant vector field, a covariant vector field and a tensor field, in order. The covariant derivative of these quantities with respect to the Riemannian connection ∇, are defined by, respectively [22],

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + v^h \Gamma_{hj}^{\ i} , \tag{1.9}$$

$$\nabla_j v_i = \frac{\partial v_i}{\partial x^j} - v_k \Gamma_{ij}^{\ k} , \tag{1.10}$$

$$\nabla_k T_{ji}^{\ \ h} = \frac{\partial T_{ji}^{\ \ a}}{\partial x^k} + T_{ij}^{\ \ a} \Gamma_{ka}^{\ h} - T_{ai}^{\ \ h} \Gamma_{kj}^{\ a} - T_{ja}^{\ \ h} \Gamma_{ki}^{\ a} . \tag{1.11}$$

The covariant derivative of the metric tensor g_{ij} respect to Levi-Civita connection is

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \left\{ {a \atop kj} \right\} g_{ai} - \left\{ {a \atop ki} \right\} g_{aj} = 0 . \tag{1.12}$$

In general, the curvature tensor of a space M_n is defined by

$$R_{kji}^{\ \ \ h} = \partial_k \Gamma_{ji}^{\ h} - \partial_j \Gamma_{ki}^{\ h} + \Gamma_{kt}^{\ h} \Gamma_{ji}^{\ t} - \Gamma_{jt}^{\ h} \Gamma_{ki}^{\ t}, \quad (\partial_k = \frac{\partial}{\partial x^k}) . \tag{1.13}$$

If we take the coefficients $\Gamma_{ji}^{\ h}$ as Christoffel symbols of second kind, then the curvature tensor of the Riemannian space admitting the metric tensor g_{ij} turns into

$$R_{kji}^{\ \ \ h} = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop ka} \right\} \left\{ {a \atop ji} \right\} - \left\{ {h \atop ja} \right\} \left\{ {a \atop ki} \right\} . \tag{1.14}$$

Let v^i, v_i, f, $T_{li}^{\ \ h}$ and $S_{kj}^{\ \ t}$ be the components of a contravariant vector field, a covariant vector field, a scalar function, a tensor field of type $(1,2)$ and a torsion tensor, on M_n respectively. The covariant tensor $R_{kji}^{\ \ \ h}$ satisfies the Ricci identities

$$\nabla_k \nabla_j v^i - \nabla_j \nabla_k v^i = R_{kjh}^{\ \ \ i} v^h - 2 S_{kj}^{\ \ t} \nabla_t v^i , \tag{1.15}$$

$$\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i = -R_{kji}^{\ \ \ h} v_h - 2 S_{kj}^{\ \ t} \nabla_t v_i , \tag{1.16}$$

$$\nabla_k \nabla_j f - \nabla_j \nabla_k f = -2 S_{kj}^{\ \ t} \nabla_t f , \tag{1.17}$$

$$\nabla_k \nabla_j T_{li}^{\ \ h} - \nabla_j \nabla_k T_{li}^{\ \ h} = R_{kjt}^{\ \ \ h} T_{li}^{\ \ t} - R_{kjl}^{\ \ \ t} T_{ti}^{\ \ h} - R_{kji}^{\ \ \ t} T_{lt}^{\ \ h} - 2 S_{kj}^{\ \ t} \nabla_t T_{li}^{\ \ h} . \tag{1.18}$$

In particular, if M_n is the Riemannian space, then torsion tensor $S_{kj}^{\ \ t} = 0$. Thus, we reach the Ricci identities for Riemannian case, [23].
The covariant curvature tensor of R_{kjih} is defined as

$$R_{kjih} = R_{kji}^{\ \ \ a} g_{ah} . \tag{1.19}$$

From (1.14), it can be seen that the Riemannian curvature satisfies the following properties

$$R_{kjih} + R_{jikh} + R_{ikjh} = 0 \,, \tag{1.20}$$

$$R_{kjih} = -R_{jkih} \,, \tag{1.21}$$

$$R_{kjih} = -R_{kjhi} \,, \tag{1.22}$$

$$R_{kjih} = R_{ihkj} \,, \tag{1.23}$$

$$R_{kkih} = -R_{kjhh} = 0 \,. \tag{1.24}$$

The first property is called the first Bianchi identity.
Also, the covariant derivative of the curvature tensor satisfies

$$\nabla_l R_{kji}{}^h + \nabla_k R_{jli}{}^h + \nabla_j R_{lki}{}^h = 0 \,, \tag{1.25}$$

which is called the second Bianchi identity.
In the equation of (1.14), contracting indices h and k, then we reach the Ricci curvature tensor

$$R_{ji} = R_{aji}{}^a \,. \tag{1.26}$$

Also, we have

$$R_{ji} = R_{aji}{}^a = g^{ab} R_{ajib} = g^{ba} R_{ibaj} = g^{ba} R_{bija} = R_{ij} \,, \tag{1.27}$$

which shows that the Ricci curvature tensor is symmetric. By using Ricci curvature tensor, the scalar curvature of the Riemannian space is defined by, [21],

$$R = g^{ji} R_{ji} \,. \tag{1.28}$$

1.2.2 Weyl Spaces

In this section, we give some preliminary concepts related to Weyl spaces. First, we define the Weyl space and obtain the curvature, the covariant curvature, the Ricci curvature tensor and scalar curvature of Weyl space in terms of Riemannian tensor and the metric tensor. Also, we mention about the Bianchi identities and give some symmetry properties of the curvature tensors of Weyl space. And then, curvature similarities between Riemannian and Weyl space are emphasized, [21], [22], [24], [25].
Two Riemannian metrics g and $\bar{g}$ are conformal if they coincide up to a factor which is positive function, i.e. $\bar{g} = e^{2\lambda}g$. This is an equivalence relation, each class $\mathbb{G}$ being called a conformal structure. A Weyl structure is a map $w : \mathbb{G} \to \Lambda^1(W)$ satisfying $w(e^{2\lambda}g) = w(g) + 2d\lambda$ where $\Lambda^1(W)$ is space of 1-forms defined on Weyl manifold. A n-dimensional manifold with a Weyl structure is called a Weyl manifold denoted by $W_n(g,w)$, or W_n briefly.
In [26], it is proved that for a Weyl manifold W_n, there exists an unique torsion-free connection ∇ that preserves the conformal class $\mathbb{G}$. Preserving the conformal class means that for any $g \in \mathbb{G}$ there exists 1-form w such that

$$\nabla g = 2w \otimes g\,. \tag{1.29}$$

The equation (1.29) can be expressed in local coordinates as

$$\nabla_k g_{ij} = 2\,w_k\,g_{ij}\,. \tag{1.30}$$

Here, w is a 1-form named complementary co-vector field.
Under the re-normalization of the metric tensor g

$$\overline{g} = \Omega^2 g,\ \ (\Omega > 0) \tag{1.31}$$

the 1-form w is transformed by the law

$$\overline{w} = w + d\ln\Omega, \tag{1.32}$$

so that

$$\nabla_k \overline{g}_{ij} = 2\overline{w}_k \overline{g}_{ij}. \tag{1.33}$$

Here, Ω is a positive scalar differentiable function defined on W_n, [27], [28].

In local coordinates, we examine the transformation of the complementary vector $\bar{w}$ to get the covariant derivative of $\bar{g}$ by using (1.33). If we use (1.31) in (1.33), we find

$$\nabla_k(\Omega^2 g) = 2\,\bar{w}_k\,\Omega^2 g\,. \tag{1.34}$$

If we take the derivative of (1.34) with respect to ∇_k, then we obtain

$$2\Omega\Omega_k\, g + \Omega^2\, 2w_k\, g - 2\bar{w}_k\,\Omega^2\, g = 0\,, \tag{1.35}$$

and re-arranging (1.35), then we get by bracketing above equation with respect to g,

$$g\,(2\Omega\Omega_k + \Omega^2\, 2w_k - 2\bar{w}_k\,\Omega^2) = 0\,. \tag{1.36}$$

As the metric tensor g is different than the zero, then,

$$2\Omega\Omega_k + \Omega^2\, 2w_k - 2\bar{w}_k\,\Omega^2 = 0\,. \tag{1.37}$$

Rearranging (1.37), we get

$$\bar{w}_k = w_k + \frac{\Omega_k}{\Omega}\,. \tag{1.38}$$

So, we reach (1.32).

The relation between Weyl connection ∇, and Riemannian connection ∇^g is

$$\nabla_X Y = \nabla^g_X Y - w(X)Y - w(Y)X + g(X,Y)\psi, \tag{1.39}$$

where X, Y are vector fields on W_n and ψ is the dual vector field to w such that $w(X) = g(X,\psi)$.

In local coordinates, (1.39) can be given by

$$\Gamma_{ji}^{\;\,l} = \left\{ \begin{matrix} l \\ ji \end{matrix} \right\} - (w_j\delta^l_i + w_i\delta^l_j - w^l g_{ji}), \tag{1.40}$$

where $\Gamma^{\,l}_{ji}$ are the coefficients of the Weyl connection in which

$$\left\{ \begin{matrix} l \\ ji \end{matrix} \right\} = \frac{1}{2}\, g^{lm}\,(\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ji}) \tag{1.41}$$

are the coefficients of the Levi-Civita connection ∇^g , [22], [28].

The curvature tensor W of ∇ is given by

$$W(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,. \tag{1.42}$$

Using (1.39) in (1.42), the curvature tensor of W_n is obtained

$$W(X,Y)Z = R(X,Y)Z - s(X,Z)Y + s(Y,Z)X + s(Y,X)Z - s(Y,X)Z + g(Y,Z)\overline{S}X - g(X,Z)\overline{S}Y \tag{1.43}$$

for any vector fields $X,\ Y,\ Z$ where s is the tensor field of type $(0,2)$ defined by

$$s(X,Y) = (\nabla_X w)(Y) + w(X)w(Y) - \frac{1}{2}w(\psi)g(X,Y)\ , \tag{1.44}$$

and $\overline{S}$ is the tensor field of type $(1,1)$ defined by

$$g(\overline{S}X,Y) = s(X,Y). \tag{1.45}$$

In local coordinates, using

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\,v_k = -W_{kji}{}^t\, v_t\ , \tag{1.46}$$

the curvature tensor of W_n is obtained

$$W_{kji}{}^l = \partial_k\Gamma_{ji}{}^l - \partial_j\Gamma_{ki}{}^l - \Gamma_{ki}{}^t\Gamma_{jt}{}^l + \Gamma_{ji}{}^t\Gamma_{kt}{}^l\ , \tag{1.47}$$

where $\Gamma_{ji}{}^l$ are the coefficients of the Weyl connection as in (1.40), substituting (1.40) in (1.47), the curvature tensor of W_n is obtained

$$\begin{aligned} W_{kji}{}^l = {} & \partial_k[\left\{ {l \atop ji} \right\} - (w_j\delta_i^l + w_i\delta_j^l - g_{ji}w^l)] - \partial_j[\left\{ {l \atop ki} \right\} - (w_k\delta_i^l + w_i\delta_k^l - g_{ki}w^l)] \\ & + [\left\{ {t \atop ji} \right\} - (w_i\delta_j^t + w_j\delta_i^t - g_{ji}w^t)][\left\{ {l \atop kt} \right\} - (w_k\delta_t^l + w_t\delta_k^l - g_{kt}w^l)] \\ & - [\left\{ {t \atop ki} \right\} - (w_k\delta_i^t + w_i\delta_k^t - g_{ki}w^t)][\left\{ {l \atop jt} \right\} - (w_j\delta_t^l + w_t\delta_j^l - g_{jt}w^l)]\ , \end{aligned} \tag{1.48}$$

and using Riemannian curvature (1.14),

$$\begin{aligned} W_{kji}{}^l = {} & R_{kji}{}^l - \partial_k(w_i\delta_j^l + w_j\delta_i^l - g_{ji}w^l) + \partial_j(w_k\delta_i^l + w_i\delta_k^l - g_{ki}w^l) \\ & - \left\{ {t \atop ji} \right\}(w_k\delta_t^l + w_t\delta_k^l - g_{kt}w^l) - \left\{ {l \atop kt} \right\}(w_i\delta_j^t + w_j\delta_i^t - g_{ji}w^t) \\ & + (w_k\delta_t^l + w_t\delta_k^l - g_{kt}w^l)(w_i\delta_j^t + w_j\delta_i^t - g_{ji}w^t) \\ & + \left\{ {t \atop ki} \right\}(w_j\delta_t^l + w_i\delta_t^j - g_{jt}w^l) + \left\{ {l \atop jt} \right\}(w_k\delta_i^t + w_i\delta_k^t - g_{ki}w^t) \\ & - (w_j\delta_t^l + w_i\delta_t^j - g_{jt}w^l)(w_k\delta_i^t + w_i\delta_k^t - g_{ki}w^t)\ . \end{aligned} \tag{1.49}$$

Then, using the properties of the Riemannian curvature tensor and simplying the equation (1.49), we reach

$$W_{kji}{}^l = R_{kji}{}^l - \delta_j^l w_{ki} + \delta_k^l w_{ji} + \delta_i^l(w_{jk} - w_{kj}) + g^{ls}(g_{ji}w_{ks} - g_{ki}w_{js})\ , \tag{1.50}$$

where

$$w_{jk} = \nabla_j w_k + w_j w_k - \frac{1}{2} g_{jk} w_t w^t \,, \tag{1.51}$$

and ${R_{kji}}^l$ represents the Riemannian curvature tensor.

The curvature tensor and covariant curvature tensor, the Ricci tensor, and the scalar curvature of Weyl space are defined through parallel transportation of vector fields v, respectively by,

$${W_{kji}}^h = W_{kjil}\, g^{lh} \,, \tag{1.52}$$

$$W_{kjil} = {W_{kji}}^m g_{ml} \,, \tag{1.53}$$

$$W_{ji} = g^{kl}\, W_{kjil} = {W_{kji}}^k \,, \tag{1.54}$$

$$W = g^{ji}\, W_{ji} \,. \tag{1.55}$$

Multiplying (1.50) by the metric tensor g_{lm} and using (1.53), we find

$$W_{kjim} = {R_{kji}}^l g_{lm} - \delta^l_j w_{ki} g_{lm} + \delta^l_k w_{ji} g_{lm} + \delta^l_i (w_{jk} - w_{kj}) g_{lm} + g^{ls}(g_{ji} w_{ks} - g_{ki} w_{js}) g_{lm} \,. \tag{1.56}$$

Using the property of the curvature tensor of Riemannian space, [22]

$${R_{kji}}^h g_{hm} = R_{kjim} \tag{1.57}$$

from (1.19), (1.4) and obtain

$$W_{kjim} = R_{kjim} - g_{jm} w_{ki} + g_{km} w_{ji} + g_{im}(w_{jk} - w_{kj}) + g^{ls} g_{lm}(g_{ji} w_{ks} - g_{ki} w_{js}) \,. \tag{1.58}$$

Also, the covariant curvature tensor of W_n is found as

$$W_{kjim} = R_{kjim} - g_{jm} w_{ki} + g_{mk} w_{ji} + g_{im}(w_{jk} - w_{kj}) + (g_{ji} w_{km} - g_{ki} w_{jm}) \,. \tag{1.59}$$

It is observed that the covariant curvature tensor of W_n satisfies the following symmetry properties:

$$W_{kjim} = -W_{jkim} \,, \tag{1.60}$$

$$W_{kjim} + W_{kjmi} = 2 g_{im}(\nabla_j w_k - \nabla_k w_j) = 4 g_{im} \nabla_{[j} w_{k]} \,. \tag{1.61}$$

In (1.59), by interchanging the indices k and j, the covariant curvature tensor W_{jkim} is obtained

$$W_{jkim} = R_{jkim} - g_{km} w_{ji} + g_{mj} w_{ki} + g_{im}(w_{kj} - w_{jk}) + (g_{ki} w_{jm} - g_{ji} w_{km}) \,. \tag{1.62}$$

Since the metric tensor g_{ij} is symmetric and using (1.24), we get

$$W_{jkim} = -[-R_{jkim} + g_{mk}w_{ji} - g_{jm}w_{ki} + g_{im}(-w_{kj} + w_{jk}) + (-g_{ki}w_{jm} + g_{ji}w_{km})] \ . \quad (1.63)$$

Using the symmetry property of covariant curvature of Riemannian tensor (1.24) in (1.63) and by rearranging the (1.63), we find

$$W_{jkim} = -[R_{kjim} - g_{jm}w_{ki} + g_{mk}w_{ji} + g_{im}(w_{jk} - w_{kj}) + (g_{ji}w_{km} - g_{ki}w_{jm})] \ , \quad (1.64)$$

so, we see that

$$W_{kjim} = -W_{jkim} \ . \quad (1.65)$$

Now, using (1.64) we calculate the sum of the covariant tensors W_{kjim} and W_{kjmi}

$$W_{kjim} = R_{kjim} - g_{jm}w_{ki} + g_{mk}w_{ji} + g_{im}(w_{jk} - w_{kj}) + (g_{ji}w_{km} - g_{ki}w_{jm}) \ , \quad (1.66)$$

$$W_{kjmi} = R_{kjmi} - g_{ji}w_{km} + g_{ik}w_{jm} + g_{mi}(w_{jk} - w_{kj}) + (g_{jm}w_{ki} - g_{km}w_{ji}) \ , \quad (1.67)$$

as,

$$W_{kjim} + W_{kjmi} = R_{kjim} + R_{kjmi} + 2g_{im}(w_{jk} - w_{kj}) \ . \quad (1.68)$$

Using the symmetry property of covariant curvature tensor of Riemannian space (1.24), (1.68) turns into,

$$W_{kjim} + W_{kjmi} = 2g_{im}(w_{jk} - w_{kj}) \ , \quad (1.69)$$

where w_{jk} is defined in (1.51).
If we substituting (1.51) in (1.69), then (1.69) reduces to

$$W_{kjim} + W_{kjmi} = 2g_{im}(\nabla_j w_k - \nabla_k w_j) = 4g_{im}\nabla_{[j}w_{k]} \ . \quad (1.70)$$

Let us now consider the covariant curvature tensor W_{kjim} on Weyl space. Multiplying (1.59) by the metric tensor g^{km}, we obtain

$$W_{ji} = g^{km}[R_{kjim} - g_{jm}w_{ki} + g_{mk}w_{ji} + g_{im}(w_{jk} - w_{kj}) + (g_{ji}w_{km} - g_{ki}w_{jm})] \quad (1.71)$$

by the rule of the (1.54).
Using the property (1.27) in (1.71), we get

$$W_{ji} = R_{ji} - g^{km}g_{mj}w_{ki} + g^{km}g_{mk}w_{ji} + g^{km}g_{mi}(w_{jk} - w_{kj}) + g^{km}g_{ji}w_{km} - g^{mk}g_{ki}w_{jm} \ . \quad (1.72)$$

From (1.4), (1.72) can be written as

$$W_{ji} = R_{ji} - \delta^k_j w_{ki} + \delta^m_m w_{ji} + \delta^k_i (w_{jk} - w_{kj}) + g^{km} g_{ji} w_{km} - \delta^m_i w_{jm} \ . \tag{1.73}$$

Rearranging (1.73), we find that the Ricci curvature tensor of W_n is calculated in terms of the Ricci curvature of the Riemannian space R_{ji} as

$$W_{ji} = R_{ji} + (n-2) w_{ji} + (w_{ji} - w_{ij}) + g_{ji} g^{km} w_{km} \ . \tag{1.74}$$

If we interchange the indices appropriately, we find

$$W_{kj} = R_{kj} + (n-2) w_{kj} + (w_{kj} - w_{jk}) + g_{kj} g^{st} w_{st} \ , \tag{1.75}$$

where w_{ji} is defined in (1.51).

The Ricci curvature tensor in (1.73) is not necessarily symmetric on W_n since the Weyl connection ∇ is non-metric. The Ricci curvature tensor W_{ji} of W_n can be decomposed symmetric and anti-symmetric parts as

$$W_{ji} = W_{(ji)} + W_{[ji]} \ . \tag{1.76}$$

Here, $W_{(ji)}$ represents the symmetric part of W_{ji} and $W_{[ji]}$ represents the anti-symmetric part of W_{ji}

$$W_{(ji)} = \frac{1}{2}(W_{ji} + W_{ij}) \ , \tag{1.77}$$

$$W_{[ji]} = \frac{1}{2}(W_{ji} - W_{ij}) = n \nabla_{[j} w_{i]} \ . \tag{1.78}$$

By using (1.73) the symmetric and anti-symmetric parts of the Ricci curvature tensor of W_{ji} are given as follows

$$W_{(ji)} = R_{ji} + \frac{(n-2)}{2}(w_{ji} + w_{ij}) + g_{ji} g^{kt} w_{kt} \ , \tag{1.79}$$

or

$$W_{(ji)} = R_{ji} + (n-2)[\nabla_j w_i + \nabla_i w_j + 2 w_j w_i] - g_{ji}(2 w^t w_t - g^{kt} \nabla_k w_t) \ , \tag{1.80}$$

and

$$W_{[ji]} = n \nabla_{[j} w_{i]} \ . \tag{1.81}$$

Also, multiplying (1.74) by g^{ji} and using (1.28), we find the scalar curvature of W_n

$$W = R + 2(n-1) \nabla_j w^j - (n-1)(n-2) w_j w^j \tag{1.82}$$

in terms of the Riemannian scalar curvature R and the complementary co-vector w.

The following identities are known as the first and the second Bianchi identities for W_n, respectively, [22], [29], [30]

$$W_{kji}{}^{l} + W_{jik}{}^{l} + W_{ikj}{}^{l} = 0\,, \tag{1.83}$$

$$\nabla_m W_{kji}{}^{l} + \nabla_k W_{jmi}{}^{l} + \nabla_j W_{mki}{}^{l} = 0\,. \tag{1.84}$$

The first and the second Bianchi identities can be also written for the covariant curvature tensors, as

$$W_{kjim} + W_{jikm} + W_{ikjm} = 0\,, \tag{1.85}$$

$$\nabla_m W_{kjin} + \nabla_k W_{jmin} + \nabla_j W_{mkin} = 0\,. \tag{1.86}$$

By using (1.59), and interchanging the indices, we find the difference of the covariant tensors W_{kjim} and W_{imkj} as

$$\begin{aligned} W_{kjim} - W_{imkj} &= g_{mk}(w_{ji} - w_{ij}) + g_{mj}(w_{ik} - w_{ki}) + g_{ji}(w_{km} - w_{mk}) \\ &+ g_{ik}(w_{mj} - w_{jm}) + g_{im}(w_{jk} - w_{kj}) + g_{kj}(w_{mi} - w_{im})\,. \end{aligned} \tag{1.87}$$

Substituting (1.51) in (1.87),and after some calculations, we find

$$\begin{aligned} W_{kjim} - W_{imkj} &= g_{mk}(\nabla_j w_i - \nabla_i w_j) + g_{mj}(\nabla_i w_k - \nabla_k w_i) + g_{ji}(\nabla_k w_m - \nabla_m w_k) \\ &+ g_{ik}(\nabla_m w_j - \nabla_j w_m) + g_{im}(\nabla_j w_k - \nabla_k w_j) + g_{kj}(\nabla_m w_i - \nabla_i w_m)\,. \end{aligned} \tag{1.88}$$

Thus, the covariant curvature tensor W_{kjim} satisfies

$$\begin{aligned} W_{kjim} - W_{imkj} &= g_{mk}(w_{i,j} - w_{j,i}) + g_{mj}(w_{k,i} - w_{i,k}) + g_{ji}(w_{m,k} - w_{k,m}) \\ &+ g_{ik}(w_{j,m} - w_{m,j}) + g_{im}(w_{k,j} - w_{j,k}) + g_{kj}(w_{i,m} - w_{m,i})\,, \end{aligned} \tag{1.89}$$

where $\nabla_m w_j = w_{j,m}$.

1.2.2.1 Semi-Symmetric Spaces

Friedmann and Schouten in 1924, [31], introduced the concept of the semi-symmetric linear connection on differentiable manifolds. Afterwards, Hayden in 1932, [32], considered a metric connection but non-torsion on Riemannian manifolds.

Definition 1.2.1. *Let π_j be the component of any vector field and $\Gamma_{jk}^{\;i}$ be the connection coefficient of Riemannian manifold whose anti-symmetric part $\Omega_{jk}^{\;i}$ represents torsion* *tensor satisfying the equation*

$$\Omega_{jk}^{\ i} = \pi_j \delta_k^i - \pi^k g_{ij} \ , \tag{1.90}$$

and this connection is called as semi-symmetric connection, [33].

Definition 1.2.2. *Let M_n be n-dimensional manifold with the Riemannian metric tensor g, and ∇ be Levi-Civita, $\overline{\nabla}$ be any linear connection on M_n. For any 1-form π on M_n, if the torsion tensor of $\overline{\nabla}$ connection which is defined as*

$$T(X,Y) = \pi(Y)X - \pi(X)Y \ , \tag{1.91}$$

it is called semi-symmetric connection on manifold, [34].

In addition to this, if $\overline{\nabla}$ connection satisfies the condition $\overline{\nabla}g = 0$, then $\overline{\nabla}$ is called as metric connection. Furthermore, the connection coefficient of a manifold with semi-symmetric connection is expressed as,

$$\Gamma_{jk}^{\ i} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \pi_j \delta_k^i - \pi^i g_{jk} \tag{1.92}$$

in general. Here, the first term, $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ represents Riemannian connection coefficient which is symmetric, and the rest of the equation

$$\Omega_{jk}^{\ i} = \pi_j \delta_k^i - \pi^i g_{jk} \tag{1.93}$$

represents the torsion tensor.
The curvature tensor of a space with semi-symmetric metric connection is expressed in terms of covariant derivative with respect to metric tensor as follows, [35]

$$L_{ijk}^{\ \ \ h} = R_{ijk}^{\ \ \ h} - \delta_j^h(\nabla_k \pi_i - g_{ik}\pi_l \pi^l) + \delta_k^h(\nabla_j \pi_i - g_{ij}\pi_l \pi^l) - g_{ik}\nabla_j \pi^h + g_{ij}\nabla_k \pi^h \tag{1.94}$$

by re-arranging, it is seen that

$$L_{ijk}^{\ \ \ h} = R_{ijk}^{\ \ \ h} + \delta_j^h \alpha_{ik} - \delta_i^h \alpha_{jk} + \alpha_{jl} g^{lh} g_{ik} - \alpha_{il} g^{lh} g_{jk} \ , \tag{1.95}$$

where $R_{ijk}^{\ \ \ h}$ is the curvature tensor of Riemannian manifold, and

$$\alpha_{ki} = \nabla_k \pi_i - \pi_i \pi_k + \frac{1}{2} g_{ki} \pi_t \pi^t \ . \tag{1.96}$$

Remark. Yano [35], proved that the necessary and sufficient condition for a curvature tensor of a manifold with semi-symmetric connection is zero if and only if its conformal curvature tensor is zero with respect to Riemannian connection.

Imai-Yano, Kim, Liang [36], [37], [38], [39] re-investigated the connection which is studied above, by using recurrency condition of metric tensor instead of the metricity on spaces with semi-symmetric metric connection.

Let (M_n, g), $(n > 2)$ be an n-dimensional differentiable manifold with metric tensor g, and let ∇ be Riemannian connection. A linear connection $\overset{\circ}{\nabla}$ on M_n, whose coefficients are $\overset{\circ}{\Gamma}{}_{jk}^{\,i}$, is said to be semi-symmetric if the torsion tensor S of $\overset{\circ}{\nabla}$ satisfies

$$S_{jk}^{\ i} = \overset{\circ}{\Gamma}{}_{jk}^{\,i} - \overset{\circ}{\Gamma}{}_{kj}^{\,i} = \pi_k \delta_j^i - \pi_j \delta_k^i , \tag{1.97}$$

where $\pi_j = \lambda_j + \mu_j$ is defined by arbitrary 1-forms λ_j and μ_j.

In addition, if a semi-symmetric metric connection has recurrency condition

$$\overset{\circ}{\nabla}_k g_{ij} = 2\mu_k g_{ij} , \tag{1.98}$$

then the connection $\overset{\circ}{\nabla}$ is said to be semi-symmetric recurrent-metric connection and μ is called the recurrent covariant vector field. And, the connection coefficient of the linear connection $\overset{\circ}{\nabla}$ is written in terms of 1-forms such as, [36], [37], [38], [40],

$$\overset{\circ}{\Gamma}{}_{ji}^{\,h} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} + (\delta_j^h \lambda_i - \mu_j \delta_i^h - g_{ji} \lambda^h) . \tag{1.99}$$

Definition 1.2.3. *Generalized connection on Weyl manifold is given by Murgescu [41] as*

$$L_{jk}^{\ i} = \Gamma_{jk}^{\ i} + a_{jkh} g^{hi} , \tag{1.100}$$

where

$$a_{jkh} = g_{jl} \Omega_{kh}^{\ l} + g_{lk} \Omega_{jh}^{\ l} + g_{lh} \Omega_{jk}^{\ l} . \tag{1.101}$$

Ünal-Uysal [42] constrained the concept of the generalized connection on Weyl manifold by choosing $\Omega_{jk}^{\ i}$ as

$$\Omega_{jk}^{\ i} = \delta_j^{\,i} a_k - \delta_k^{\,i} a_j , \tag{1.102}$$

so, the semi-symmetric connection on Weyl manifold is obtained. Besides, the curvature tensor of the manifold with given ∇ connection which satisfies the compatibility condition is given as below,

$$\begin{aligned} \overline{R}_{ijk}^{\ \ h} &= R_{ijk}^{\ \ h} + \delta_k^h S_{ij} - \delta_j^h S_{ik} + g_{ij} g^{hr} S_{rk} - g_{ik} g^{hr} S_{rj} , \\ S_{ij} &= \nabla_j S_i - S_i S_j + \frac{1}{2} g_{ij} S_s S^s . \end{aligned} \tag{1.103}$$

In the following section, we will investigate the existence and the geometry of Weyl manifolds having the connection which is defined by [36], [37], [38].

2. WEYL SPACES WITH A SPECIAL CONNECTION

In literature, the idea of semi-symmetric connection is introduced by [35], [26], [43], [44], [45], [46], [47], and curvature related properties are studied widely. Let M_n be an n-dimensional, $(n > 2)$ differentiable manifold. A linear connection ∇^* on M_n, whose coefficients are $\Gamma^*{}_{jk}^{\,i}$, is said to be semi-symmetric if the torsion tensor T of ∇^* satisfies the relation

$$T(X,Y) = \pi(Y)X - \pi(X)Y \,, \tag{2.1}$$

where π is a 1-form, and X,Y are smooth vector fields on M_n. In local coordinates, (2.1) can be written as

$$T_{jk}^{\,i} = \Gamma^*{}_{jk}^{\,i} - \Gamma^*{}_{kj}^{\,i} = \pi_k \delta^i_j - \pi_j \delta^i_k \,. \tag{2.2}$$

In addition, if a semi-symmetric connection has recurrency condition

$$\nabla^*_X g = 2\,\mu(X) g \,. \tag{2.3}$$

In local coordinates, (2.3) can be written as

$$\nabla^*{}_k\, g_{ij} = 2\mu_k\, g_{ij} \,, \tag{2.4}$$

then the connection ∇^* is said to be semi-symmetric recurrent-metric connection and μ is called the recurrent covariant vector field, [36], [38].

2.1 Weyl Spaces With Semi-Symmetric Recurrent Metric Connection

In [48], we use the notion of a semi-symmetric recurrent metric connection for Weyl manifolds. Let $\overline{\nabla}$ be a linear connection with coefficients $\overline{\Gamma}_{jk}^{\,i}$ on a Weyl manifold W_n satisfying (2.2). If, also the following relation holds on W_n

$$\overline{\nabla}_X g(Y,Z) = 2(w+\mu)(X) g(Y,Z) \,. \tag{2.5}$$

In local coordinates, (2.5) is represented by

$$\overline{\nabla}_k g_{ij} = \nabla_k g_{ij} + 2\mu_k g_{ij} = 2(w_k + \mu_k) g_{ij}, \tag{2.6}$$

then, W_n is called a Weyl manifold with a semi-symmetric recurrent-metric connection denoted by $WS_n(g,w,\pi,\mu)$, or WS_n briefly.
From (1.30), we have

$$\nabla_k g_{ij} = \partial_k g_{ij} - g_{hj}\Gamma_{ki}^{\ h} - g_{ih}\Gamma_{kj}^{\ h} \quad (2.7)$$
$$= 2w_k g_{ij},$$

and from (2.6) more explicitly,

$$\overline{\nabla}_k g_{ij} = \partial_k g_{ij} - g_{hj}\overline{\Gamma}_{ki}^{\ h} - g_{ih}\overline{\Gamma}_{kj}^{\ h} \quad (2.8)$$
$$= 2(w_k + \mu_k)g_{ij}.$$

By using (2.8), we have

$$\overline{\nabla}_k g^{ij} = -2(w_k + \mu_k)g^{ij}. \quad (2.9)$$

Here, we will examine the existence and uniqueness of the semi-symmetric recurrent-metric connection $\overline{\nabla}$ on a Weyl manifold, and will prove the following theorem, [48].

Theorem 2.1.1. *Let $WS_n(g,w,\pi,\mu)$ be an n-dimensional Weyl manifold equipped with the semi-symmetric recurrent-metric connection $\overline{\nabla}$ associated with 1-forms, w, π, and μ satisfying (1.30), (2.2), (2.4) respectively. Then, there exists a unique connection $\overline{\nabla}$ on $WS_n(g,w,\pi,\mu)$ given by*

$$\overline{\nabla}_X Y = \nabla_X Y - \mu(X)Y - \mu(Y)X + g(X,Y)\xi + \pi(Y)X - g(X,Y)\eta, \quad (2.10)$$

where ξ ,η are dual vector fields such that

$$\mu(X) = g(X,\xi), \qquad \pi(X) = g(X,\eta). \quad (2.11)$$

Proof. Let $\overline{\nabla}$ be semi-symmetric recurrent metric connection, and ∇ be Weyl connection. We have

$$(\overline{\nabla}_X g)(Y,Z) = \overline{\nabla}_X g(Y,Z) - g(\overline{\nabla}_X Y,Z) - g(Y,\overline{\nabla}_X Z), \quad (2.12)$$

and

$$(\nabla_X g)(Y,Z) = \nabla_X g(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z), \quad (2.13)$$

for any vector fields X, Y, and Z.

We put

$$\overline{\nabla}_X Y = \nabla_X Y + U(X,Y) , \tag{2.14}$$

where U is a tensor field of type $(1,2)$ defined as the difference of the connections. Using (2.1), and (2.14) it is obtained that

$$\begin{aligned} T(X,Y) =& \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] , \\ =& U(X,Y) - U(Y,X) . \end{aligned} \tag{2.15}$$

From (2.12), (2.13), (2.14), and (2.15) we get

$$g(U(X,Y),Z) + g(U(X,Z),Y) = -2\mu(X)g(Y,Z) . \tag{2.16}$$

By using from (2.15), and permuting vector fields X, Y, and Z for T, we get

$$g(T(X,Y),Z) = g(U(X,Y),Z) - g(U(Y,X),Z) , \tag{2.17}$$

$$g(T(Z,X),Y) = g(U(Z,X),Y) - g(U(X,Z),Y) , \tag{2.18}$$

$$g(T(Z,Y),X) = g(U(Z,Y),X) - g(U(Y,Z),X) . \tag{2.19}$$

From (2.16), and (2.17) we obtain

$$\begin{aligned} g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X) = 2g(U(X,Y),Z) + 2\mu(X)g(Y,Z) \\ + 2\mu(Y)g(Z,X) - 2\mu(Z)g(X,Y) . \end{aligned} \tag{2.20}$$

Defining the tensor $\acute{T}$ of type $(1,2)$ as

$$g(T(Z,X),Y) = g(\acute{T}(X,Y),Z) . \tag{2.21}$$

The equation (2.20) can be written

$$\begin{aligned} g(U(X,Y),Z) =& \frac{1}{2}[g(T(X,Y),Z) + g(\acute{T}(X,Y),Z) + g(\acute{T}(Y,X),Z)] \\ & - \mu(X)g(Y,Z) - \mu(Y)g(Z,X) + \mu(Z)g(X,Y) . \end{aligned} \tag{2.22}$$

Thus, we find

$$U(X,Y) = \frac{1}{2}[T(X,Y) + \acute{T}(X,Y) + \acute{T}(Y,X)] - \mu(X)Y - \mu(Y)X + g(X,Y)\xi , \tag{2.23}$$

where $\mu(X) = g(X,\xi)$.

From (2.1), and (2.21) we have

$$\begin{aligned} g(T(Z,X),Y) =& g(\pi(X)Z,Y) - g(\pi(Z)X,Y) , \\ =& g(\acute{T}(X,Y),Z) . \end{aligned} \tag{2.24}$$

From (2.21), and (2.24), we reach

$$g(\acute{T}(X,Y),Z) = \pi(X)g(Z,Y) - g(Z,\eta)g(X,Y)\,, \tag{2.25}$$

which implies

$$\acute{T}(X,Y) = \pi(X)Y - g(X,Y)\eta\,, \tag{2.26}$$

where $\pi(X) = g(X,\eta)$.

Hence, (2.23) turns into

$$U(X,Y) = \pi(Y)X - \mu(X)Y - \mu(Y)X + g(X,Y)\xi - g(X,Y)\eta\,. \tag{2.27}$$

Then (2.14) becomes

$$\overline{\nabla}_X Y = \nabla_X Y - \mu(X)Y - \mu(Y)X + g(X,Y)\xi + \pi(Y)X - g(X,Y)\eta\,, \tag{2.28}$$

from which concludes the proof.

Also, the equation (2.10) is obtained in local coordinates as

$$\overline{\Gamma}_{ik}^{\ \ l} = \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} - (w_i\delta_k^l + w_k\delta_i^l - w^l g_{ik}) + (\lambda_k\delta_i^l - \mu_i\delta_k^l - \lambda^l g_{ik})\,, \tag{2.29}$$

where $\lambda_k = \pi_k - \mu_k$.

The following sub-section is devoted to presentation of curvature tensors of Weyl manifolds with the semi-symmetric recurrent-metric connection, WS_n, in local coordinates in detail. The covariant curvature tensor, the Ricci tensor and the scalar curvature of WS_n will be denoted by $\overline{R}_{kjim}$, $\overline{R}_{ji}$ and $\overline{R}$, respectively, [48].

2.1.1 The Curvature Tensor of $WS_n(g,w,\pi,\mu)$

Theorem 2.1.2. *The curvature tensor of Weyl manifold with semi-symmetric recurrent-metric connection WS_n is obtained as*

$$\overline{R}_{kjim} = W_{kjim} + Q_{kjim} - g_{mk}\alpha_{ij} + g_{mj}\alpha_{ik} - g_{ij}\alpha_{mk} + g_{ik}\alpha_{mj}\,, \tag{2.30}$$

where

$$\alpha_{ij} = (\lambda_i w_j + \lambda_j w_i - g_{ij} w_l \lambda^l)\,. \tag{2.31}$$

Proof. The curvature tensor of WS_n can be computed by using the Ricci identity for a covariant vector field v_i:

$$(\overline{\nabla}_k\overline{\nabla}_j - \overline{\nabla}_j\overline{\nabla}_k)v_i = -\overline{R}_{kji}^{\ \ \ t}v_t - T_{kj}^{\ \ t}\overline{\nabla}_t v_i\,, \tag{2.32}$$

where $\overline{R}_{kji}{}^{h}$ is the curvature tensor of WS_n

$$\overline{R}_{kji}{}^{h} = \partial_k \overline{\Gamma}_{ji}{}^{h} - \partial_j \overline{\Gamma}_{ki}{}^{h} - \overline{\Gamma}_{ki}{}^{t} \overline{\Gamma}_{jt}{}^{h} + \overline{\Gamma}_{ji}{}^{t} \overline{\Gamma}_{kt}{}^{h}, \tag{2.33}$$

and $T_{ij}{}^{h}$ is the torsion tensor of WS_n

$$T_{ij}{}^{h} = \overline{\Gamma}_{ij}{}^{h} - \overline{\Gamma}_{ji}{}^{h}. \tag{2.34}$$

Multiplying (2.33) by g_{hm},and using (2.29), we obtain

$$\begin{aligned}
\overline{R}_{kjim} = W_{kjim} &- (\left\{ {h \atop kt} \right\} - g^{hl}(g_{lk}w_t + g_{lt}w_k - g_{kt}w_l))g_{hm}g_{ij}\lambda^t \\
&+ (\left\{ {h \atop jt} \right\} - g^{hl}(g_{lj}w_t + g_{lt}w_j - g_{jt}w_l))g_{hm}g_{ki}\lambda^t \\
&+ (\left\{ {t \atop ji} \right\} - g^{tl}(g_{lj}w_i + g_{li}w_j - g_{ji}w_l))g_{hm}\delta_k^h\lambda_t \\
&- (\left\{ {t \atop ki} \right\} - g^{tl}(g_{lk}w_i + g_{li}w_k - g_{ki}w_l))g_{jm}\lambda_t \\
&- (\left\{ {t \atop ji} \right\} - g^{tl}(g_{lj}w_i + g_{li}w_j - g_{ji}w_l))g_{kt}\lambda_m \\
&+ (\left\{ {t \atop ki} \right\} - g^{tl}(g_{lk}w_i + g_{li}w_k - g_{ki}w_l))g_{jt}\lambda_m \\
&+ g_{km}(\lambda_i\lambda_j - g_{ij}\lambda_t\lambda^t - \lambda_{i,j}) \\
&- g_{mj}(\lambda_k\lambda_i - g_{ki}\lambda_t\lambda^t - \lambda_{i,k}) - g_{mi}(\mu_{k,j} - \mu_{j,k}) \\
&+ \lambda_m\lambda_k g_{ij} - \lambda_m\lambda_j g_{ik} + \lambda_m(g_{ki,j} - g_{ij,k}) \\
&- g_{ij}g_{hm}\partial_k\lambda^h + g_{ki}g_{hm}\partial_j\lambda^h .
\end{aligned} \tag{2.35}$$

Rearranging terms in (2.35), we have

$$\begin{aligned}
\overline{R}_{kjim} = W_{kjim} &- (\left\{ {h \atop kt} \right\} g_{hm}g_{ij}\lambda^t + g_{ij}\lambda^t(g_{mk}w_t + g_{mt}w_k - g_{kt}w_m)) \\
&+ (\left\{ {h \atop jt} \right\} g_{hm}g_{ki}\lambda^t - g_{ki}\lambda^t(g_{mj}w_t + g_{mt}w_j - g_{jt}w_m)) \\
&+ (\left\{ {t \atop ji} \right\} g_{km}\lambda_t - g_{km}\lambda^l(g_{lj}w_i + g_{li}w_j - g_{ji}w_l)) \\
&- (\left\{ {t \atop ki} \right\} g_{jm}\lambda_t + g_{jm}\lambda^l(g_{lk}w_i + g_{li}w_k - g_{ki}w_l)) \\
&- (\left\{ {t \atop ji} \right\} \lambda_m g_{kt} + \lambda_m(g_{kj}w_i + g_{ki}w_j - g_{ji}w_k)) \\
&+ (\left\{ {t \atop ki} \right\} \lambda_m g_{jt} - \lambda_m(g_{jk}w_i + g_{ji}w_k - g_{ki}w_j)) \\
&+ g_{km}(\lambda_i\lambda_j - g_{ij}\lambda_t\lambda^t - \lambda_{i,j}) - g_{mj}(\lambda_k\lambda_i - g_{ki}\lambda_t\lambda^t - \lambda_{i,k}) - g_{mi}(\mu_{k,j} - \mu_{j,k}) \\
&+ \lambda_m\lambda_k g_{ij} - \lambda_m\lambda_j g_{ik} + \lambda_m(g_{ki,j} - g_{ij,k}) - g_{ij}g_{hm}\partial_k\lambda^h + g_{ki}g_{hm}\partial_j\lambda^h ,
\end{aligned} \tag{2.36}$$

after some calculations finally we obtain the curvature tensor of WS_n as

$$\overline{R}_{kji}{}^{h} = W_{kji}{}^{h} + Q_{kji}{}^{h} - \delta_k^h(\lambda_j w_i + \lambda_i w_j - g_{ij} w_l \lambda^l) + \delta_j^h(\lambda_k w_i + \lambda_i w_k - g_{ik} w_l \lambda^l) \tag{2.37}$$
$$- g_{ij}(\lambda_k w^h + \lambda^h w_k - \delta_k^h w_l \lambda^l) + g_{ik}(\lambda_j w^h + \lambda^h w_j - \delta_j^h w_l \lambda^l),$$

where $W_{kji}{}^{h}$ represents curvature tensor of Weyl space defined in (1.50).
To simplify our calculations we defined the tensor $Q_{kji}{}^{h}$ in (2.37) as

$$Q_{kji}{}^{h} = \delta_j^h \lambda_{ki} - \delta_k^h \lambda_{ji} + \lambda_{jl} g^{lh} g_{ki} - \lambda_{kl} g^{lh} g_{ji} + \delta_i^h (\nabla_j \mu_k - \nabla_k \mu_j), \tag{2.38}$$

where

$$\lambda_{ki} = \nabla_k \lambda_i - \lambda_k \lambda_i + \frac{1}{2} g_{ki} \lambda_t \lambda^t. \tag{2.39}$$

Multiplying (2.38) by the metric tensor, we obtain

$$Q_{kjim} = (g_{mj}\lambda_{ki} - g_{mk}\lambda_{ji} + \lambda_{jm} g_{ki} - \lambda_{km} g_{ji}) + 2 g_{mi} \nabla_{[j}\mu_{k]}, \tag{2.40}$$

also, we get

$$Q_{ji} = Q_{kjim} g^{km} = (n-2)[-\nabla_j \lambda_i + \lambda_i \lambda_j - g_{ji} \lambda_t \lambda^t] - g_{ji} \nabla_t \lambda^t + 2\nabla_{[j}\mu_{i]}. \tag{2.41}$$

From (2.40) we see the following anti-symmetry property

$$Q_{kjil} = -Q_{jkil}. \tag{2.42}$$

Multiplying (2.37) by metric tensor and using (2.40) and (1.59), we reach

$$\overline{R}_{kjim} = W_{kjim} + Q_{kjim} - g_{mk}\alpha_{ij} + g_{mj}\alpha_{ik} - g_{ij}\alpha_{mk} + g_{ik}\alpha_{mj}, \tag{2.43}$$

where

$$\alpha_{ij} = (\lambda_i w_j + \lambda_j w_i - g_{ij} w_l \lambda^l). \tag{2.44}$$

□

Now, we examine the properties of covariant curvature tensor of Weyl space with semi-symmetric recurrent-metric connection WS_n. Using the properties of R_{kjil} and W_{kjil}, it can be seen that the curvature tensor of WS_n satisfies the following symmetry relations:

Theorem 2.1.3. *The curvature tensor of WS_n satisfies the following symmetry relations:*

$$\overline{R}_{kjil} + \overline{R}_{jkil} = 0\,, \tag{2.45}$$

$$\begin{aligned}\overline{R}_{kjil} + \overline{R}_{kjli} &= W_{kjil} + W_{kjli} + Q_{kjil} + Q_{kjli} \\ &= 4g_{il}(\nabla_{[j}\mu_{k]} + \nabla_{[j}w_{k]})\,.\end{aligned} \tag{2.46}$$

Note that if w_k and μ_k are gradient or if w_k and μ_k have opposite signs, then $\overline{R}_{kjil} = -\overline{R}_{kjli}$.

Proof. **(i)** Interchanging the indices k and j in the equation (2.30), we get

$$\overline{R}_{kjim} + \overline{R}_{jkim} = W_{kjim} + W_{jkim} + Q_{kjim} + Q_{jkim}\,, \tag{2.47}$$

which proves (2.45).
(ii) Using (1.60), (2.40), and (2.31) in the equation of (2.30), we obtain (2.46).

$$\begin{aligned}\overline{R}_{kjim} + \overline{R}_{kjmi} &= W_{kjim} + W_{kjmi} + 2g_{im}(\mu_{jk} - \mu_{kj}) \\ &= 2g_{im}(\nabla_{[j}w_{k]} - \nabla_{[k}w_{j]}) + 2g_{im}(\mu_{jk} - \mu_{kj}) \\ &= 4g_{im}(\nabla_{[j}\mu_{k]} + \nabla_{[j}w_{k]})\,.\end{aligned} \tag{2.48}$$

Next, in the following theorem, we firstly calculate extended (generalized) first Bianchi identity and then the generalized second Bianchi identity for WS_n.

Theorem 2.1.4. *The curvature tensor of WS_n satisfies the following the first and the second Bianchi identities for WS_n, respectively,*

$$\overline{R}_{kji}{}^{l} + \overline{R}_{jik}{}^{l} + \overline{R}_{ikj}{}^{l} = 2(\delta^l_j\nabla_{[k}\pi_{i]} + \delta^l_i\nabla_{[j}\pi_{k]} + \delta^l_k\nabla_{[i}\pi_{j]})\,, \tag{2.49}$$

$$\overline{\nabla}_l\overline{R}_{kji}{}^{t} + \overline{\nabla}_j\overline{R}_{lki}{}^{t} + \overline{\nabla}_k\overline{R}_{jli}{}^{t} = 2(\pi_l\overline{R}_{kji}{}^{t} + \pi_j\overline{R}_{lki}{}^{t} + \pi_k\overline{R}_{jli}{}^{t})\,. \tag{2.50}$$

Proof. **(i)** Using (2.37) and (2.39), we write,

$$\overline{R}_{kji}{}^{l} = W_{kji}{}^{l} + Q_{kji}{}^{l} - \delta^l_k\alpha_{ij} + \delta^l_j\alpha_{ik} - g_{ij}\alpha_{mk}g^{ml} + g_{ik}\alpha_{mj}g^{ml}\,, \tag{2.51}$$

by changing indices k,j,i,l cyclically, we get

$$\begin{aligned}\overline{R}_{kji}{}^{l} + \overline{R}_{jik}{}^{l} + \overline{R}_{ikj}{}^{l} &= W_{kji}{}^{l} + W_{jik}{}^{l} + W_{ikj}{}^{l} + Q_{kji}{}^{l} + Q_{jik}{}^{l} + Q_{ikj}{}^{l} \\ &\quad + \delta^l_k(\alpha_{ji} - \alpha_{ij}) + \delta^l_j(\alpha_{ik} - \alpha_{ki}) + \delta^l_i(\alpha_{kj} - \alpha_{jk})\,,\end{aligned} \tag{2.52}$$

from which is obtained

$$\overline{R}_{kji}{}^{l} + \overline{R}_{jik}{}^{l} + \overline{R}_{ikj}{}^{l} = Q_{kji}{}^{l} + Q_{jik}{}^{l} + Q_{ikj}{}^{l} . \tag{2.53}$$

On the other hand, using (2.39), we calculate $Q_{kji}{}^{l} + Q_{jik}{}^{l} + Q_{ikj}{}^{l}$ as

$$\begin{aligned} Q_{kji}{}^{l} + Q_{jik}{}^{l} + Q_{ikj}{}^{l} &= \delta^l_j(\lambda_{ki} - \lambda_{ik}) + \delta^l_k(\lambda_{ij} - \lambda_{ji}) + \delta^l_i(\lambda_{jk} - \lambda_{kj}) \\ &\quad + \delta^l_i(\mu_{jk} - \mu_{kj}) + \delta^l_k(\mu_{ij} - \mu_{ji}) + \delta^l_j(\mu_{ki} - \mu_{ik}) , \end{aligned} \tag{2.54}$$

where

$$\mu_{kj} = \nabla_k \mu_j - \mu_j \mu_k + \frac{1}{2} g_{jk} \mu_t \mu^t . \tag{2.55}$$

Using (2.39), we have

$$\begin{aligned} Q_{kji}{}^{l} + Q_{jik}{}^{l} + Q_{ikj}{}^{l} &= \delta^l_j[\nabla_k(\lambda_i + \mu_i) - \nabla_i(\lambda_k + \mu_k)] + \delta^l_k[\nabla_i(\lambda_j + \mu_j) - \nabla_j(\lambda_i + \mu_i)] \\ &\quad + \delta^l_i[\nabla_j(\lambda_k + \mu_k) - \nabla_k(\lambda_j + \mu_j)]. \end{aligned} \tag{2.56}$$

Arranging (2.56), we find

$$Q_{kji}{}^{l} + Q_{jik}{}^{l} + Q_{ikj}{}^{l} = \delta^l_j[\nabla_k(\pi_i - \nabla_i \pi_k)] + \delta^l_k[\nabla_i(\pi_j - \nabla_j \pi_i)] + \delta^l_i[\nabla_j(\pi_k - \nabla_k \pi_j)] . \tag{2.57}$$

Using (2.57) in (2.53), we get generalized first Bianchi identity,

$$\overline{R}_{kji}{}^{l} + \overline{R}_{jik}{}^{l} + \overline{R}_{ikj}{}^{l} = 2(\delta^l_j \nabla_{[k}\pi_{i]} + \delta^l_i \nabla_{[j}\pi_{k]} + \delta^l_k \nabla_{[i}\pi_{j]}) . \tag{2.58}$$

Also, we obtain,

$$\overline{R}_{kjim} + \overline{R}_{jikm} + \overline{R}_{ikjm} = 2(g_{jm} \nabla_{[k}\pi_{i]} + g_{im} \nabla_{[j}\pi_{k]} + g_{km} \nabla_{[i}\pi_{j]}) . \tag{2.59}$$

□

(ii) We use the Ricci identity (2.32) and we differentiate covariantly the both sides of (2.32). This gives us

$$-\overline{\nabla}_l \overline{\nabla}_k \overline{\nabla}_j v_i + \overline{\nabla}_l \overline{\nabla}_j \overline{\nabla}_k v_i = \overline{\nabla}_l(\overline{R}_{kji}{}^{t}) v_t + \overline{R}_{kji}{}^{t} \overline{\nabla}_l(v_t) + \overline{\nabla}_l(T_{kj}{}^{t}) \overline{\nabla}_t v_i + T_{kj}{}^{t}(\overline{\nabla}_l \overline{\nabla}_t v_i) , \tag{2.60}$$

which is written in terms of covariant derivative of curvature tensor and torsion tensor. Interchanging the indices l, k and j in (2.60) we get

$$-\overline{\nabla}_k \overline{\nabla}_j \overline{\nabla}_l v_i + \overline{\nabla}_k \overline{\nabla}_l \overline{\nabla}_j v_i = \overline{\nabla}_k(\overline{R}_{jli}{}^{t}) v_t + \overline{R}_{jli}{}^{t} \overline{\nabla}_k(v_t) + \overline{\nabla}_l(T_{kj}{}^{t}) \overline{\nabla}_t v_i + T_{kj}{}^{t}(\overline{\nabla}_l \overline{\nabla}_t v_i) , \tag{2.61}$$

and

$$-\overline{\nabla}_j\overline{\nabla}_l\overline{\nabla}_k v_i+\overline{\nabla}_j\overline{\nabla}_k\overline{\nabla}_l v_i=\overline{\nabla}_j(\overline{R}_{lki}{}^{t})v_t+\overline{R}_{lki}{}^{t}\overline{\nabla}_j(v_t)+\overline{\nabla}_j(T_{lk}{}^{t})\overline{\nabla}_t v_i+T_{lk}{}^{t}(\overline{\nabla}_j\overline{\nabla}_t v_i)\,. \tag{2.62}$$

Adding the equations in (2.60)-(2.62) we find

$$\begin{aligned}(\overline{\nabla}_l\overline{R}_{kji}{}^{t}+\overline{\nabla}_j\overline{R}_{lki}{}^{t}+\overline{\nabla}_k\overline{R}_{lki}{}^{t})v_t=(\overline{R}_{lkj}{}^{t}+\overline{R}_{kjl}{}^{t}+\overline{R}_{jlk}{}^{t})\overline{\nabla}_t v_i+T_{lk}{}^{t}(\overline{\nabla}_t\overline{\nabla}_j-\overline{\nabla}_j\overline{\nabla}_t)v_i\\+T_{jl}{}^{t}(\overline{\nabla}_t\overline{\nabla}_k-\overline{\nabla}_k\overline{\nabla}_t)v_i+T_{kj}{}^{t}(\overline{\nabla}_t\overline{\nabla}_l-\overline{\nabla}_l\overline{\nabla}_t)v_i\\-\overline{\nabla}_t v_i(\overline{\nabla}_jT_{lk}{}^{t}+\overline{\nabla}_kT_{jl}{}^{t}+\overline{\nabla}_lT_{kj}{}^{t})\,.\end{aligned} \tag{2.63}$$

Using the components of torsion tensor

$$T_{lk}{}^{t}=(\pi_k\delta_l^t-\pi_l\delta_k^t)\,, \tag{2.64}$$

in (2.63), the terms in the second row can be written as

$$\begin{aligned}&T_{lk}{}^{t}(\overline{\nabla}_t\overline{\nabla}_j-\overline{\nabla}_j\overline{\nabla}_t)v_i+T_{jl}{}^{t}(\overline{\nabla}_t\overline{\nabla}_k-\overline{\nabla}_k\overline{\nabla}_t)v_i+T_{kj}{}^{t}(\overline{\nabla}_t\overline{\nabla}_l-\overline{\nabla}_l\overline{\nabla}_t)v_i\\&=2\pi_k(\overline{\nabla}_l\overline{\nabla}_j-\overline{\nabla}_j\overline{\nabla}_l)v_i+2\pi_l(\overline{\nabla}_j\overline{\nabla}_k-\overline{\nabla}_k\overline{\nabla}_j)v_i+2\pi_j(\overline{\nabla}_k\overline{\nabla}_l-\overline{\nabla}_l\overline{\nabla}_k)v_i\,.\end{aligned} \tag{2.65}$$

Thus, the Ricci identity (2.32) reduces to

$$\begin{aligned}&T_{lk}{}^{t}(\overline{\nabla}_t\overline{\nabla}_j-\overline{\nabla}_j\overline{\nabla}_t)v_i+T_{jl}{}^{t}(\overline{\nabla}_t\overline{\nabla}_k-\overline{\nabla}_k\overline{\nabla}_t)v_i+T_{kj}{}^{t}(\overline{\nabla}_t\overline{\nabla}_l-\overline{\nabla}_l\overline{\nabla}_t)v_i\\&=-2\pi_k(\overline{R}_{lji}{}^{t}v_t+T_{lj}{}^{t}\overline{\nabla}_t v_i)-2\pi_l(\overline{R}_{jki}{}^{t}v_t+T_{jk}{}^{t}\overline{\nabla}_t v_i)-2\pi_j(\overline{R}_{kli}{}^{t}v_t+T_{kl}{}^{t}\overline{\nabla}_t v_i)\,.\end{aligned} \tag{2.66}$$

We now simplify (2.63) as

$$\begin{aligned}(\overline{\nabla}_l\overline{R}_{kji}{}^{t}+\overline{\nabla}_j\overline{R}_{lki}{}^{t}+\overline{\nabla}_k\overline{R}_{jli}{}^{t})v_t=(\overline{R}_{lkj}{}^{t}+\overline{R}_{kjl}{}^{t}+\overline{R}_{jlk}{}^{t})\overline{\nabla}_t v_i\\-2(\pi_k\overline{R}_{lji}{}^{t}+\pi_l\overline{R}_{jki}{}^{t}+\pi_j\overline{R}_{kli}{}^{t})v_t\\-(\overline{\nabla}_jT_{lk}{}^{t}+\overline{\nabla}_kT_{jl}{}^{t}+\overline{\nabla}_lT_{kj}{}^{t})(\overline{\nabla}_t v_i)\,.\end{aligned} \tag{2.67}$$

Using (2.64), we reach

$$(\overline{\nabla}_l\overline{R}_{kji}{}^{t}+\overline{\nabla}_j\overline{R}_{lki}{}^{t}+\overline{\nabla}_k\overline{R}_{jli}{}^{t})v_t=2(\pi_k\overline{R}_{jli}{}^{t}+\pi_l\overline{R}_{kji}{}^{t}+\pi_j\overline{R}_{lki}{}^{t})v_t\,, \tag{2.68}$$

and then we find

$$(\overline{\nabla}_l\overline{R}_{kji}{}^{t}+\overline{\nabla}_j\overline{R}_{lki}{}^{t}+\overline{\nabla}_k\overline{R}_{jli}{}^{t})=2(\pi_l\overline{R}_{kji}{}^{t}+\pi_j\overline{R}_{lki}{}^{t}+\pi_k\overline{R}_{jli}{}^{t})\,, \tag{2.69}$$

which is called the generalized second Bianchi identity for WS_n. □

Next, we examine the Ricci curvature and its symmetric properties for WS_n. Multiplying (2.30) by g^{mk}, we get the Ricci tensor of WS_n as

$$\overline{R}_{ji} = W_{ji} + Q_{ji} - (n-2)(\lambda_j w_i + \lambda_i w_j) + 2(n-2)g_{ji}w_l\lambda^l\,, \tag{2.70}$$

where W_{ji} and Q_{ji} are given by (1.75) and (2.41), respectively.
It follows that the Ricci tensor of WS_n is, in general, not symmetric. So, the symmetric and anti-symmetric parts of $\overline{R}_{ji}$ can be calculated, as

$$\overline{R}_{(ji)} = W_{(ji)} - \frac{(n-2)}{2}[(\nabla_j\lambda_i + \nabla_i\lambda_j) + 2\lambda_i\lambda_j + 2(\lambda_j w_i + \lambda_i w_j)] - g_{ji}[2(n-2)\lambda_t\lambda^t + \nabla_t\lambda^t]\,, \tag{2.71}$$

and

$$\overline{R}_{[ji]} = n\nabla_{[j}w_{i]} - (n-2)\nabla_{[j}\lambda_{i]} + 2\nabla_{[j}\mu_{i]}\,. \tag{2.72}$$

We also have

$$\overline{R}_j^{\ k} = \overline{R}_{ji}\,g^{ik}, \quad \overline{R} = \overline{R}_{ji}\,g^{ji}\,. \tag{2.73}$$

Finally, the scalar curvature of WS_n is

$$\overline{R} = R + 2(n-1)(\nabla_t w^t - \nabla_t\lambda^t) - (n-1)(n-2)(w_t - \lambda_t)(w^t - \lambda^t)\,. \tag{2.74}$$

Theorem 2.1.5. *WS_n and W_n have same curvature tensors if and only if, the recurrent covariant vector field μ_k of $\overline{\nabla}$ defined by (2.4), is gradient vector, and the following equation holds*

$$\lambda_{ij} + \alpha_{ji} = 0, \tag{2.75}$$

where α_{ij}, λ_{ij} in (2.31), and (2.39), respectively, [48].

Proof. Let WS_n and W_n have same curvature tensors

$$\overline{R}_{kjim} = W_{kjim}. \tag{2.76}$$

Using (2.30), we have

$$Q_{kjim} = g_{mk}\alpha_{ij} - g_{mj}\alpha_{ik} + g_{ij}\alpha_{mk} - g_{ik}\alpha_{mj}. \tag{2.77}$$

Also, from (2.31), (2.39), and (2.40), we obtain the relation

$$g_{mj}(\alpha_{ik} + \lambda_{ki}) - g_{mk}(\alpha_{ij} + \lambda_{ji}) + g_{ik}(\alpha_{mj} + \lambda_{jm}) - g_{ij}(\alpha_{mk} + \lambda_{km}) + 2g_{mi}\nabla_{[j}\mu_{k]} = 0. \tag{2.78}$$

From (2.78), it follows that $\alpha_{ik} + \lambda_{ki} = 0$, and $\nabla_{[j}\mu_{k]} = 0$ simultaneously.
Conversely, using (2.30),

$$\overline{R}_{kjim} = W_{kjim} + Q_{kjim} - \alpha_{ij}g_{mk} + \alpha_{ik}g_{mj} - \alpha_{mk}g_{ij} + \alpha_{mj}g_{ik}, \tag{2.79}$$

and substituting (2.40) in (2.30), we get

$$\begin{aligned}\overline{R}_{kjim} =& W_{kjim} + g_{mj}(\alpha_{ik} + \lambda_{ki}) - g_{mk}(\alpha_{ij} + \lambda_{ji}) + g_{ik}(\alpha_{mj} + \lambda_{jm}) \\ & - g_{ij}(\alpha_{mk} + \lambda_{km}) + 2g_{mi}\nabla_{[j}\mu_{k]}.\end{aligned} \tag{2.80}$$

By using given assumptions that are $\alpha_{ik} + \lambda_{ki} = 0$, and $\nabla_{[j}\mu_{k]} = 0$, therefore we conclude that

$$\overline{R}_{kjim} = W_{kjim}. \tag{2.81}$$

Next, we give an example of 3-dimensional WS_n with a constant curvature in which components of torsion tensor, complementary, and recurrency co-vector fields are chosen specially, [48].

Example: Let us consider three dimensional metric as

$$ds^2 = \frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (r > 0\,,\ 0 \le \theta < \pi,\ 0 \le \phi < 2\pi) \tag{2.82}$$

where $1 - \kappa r^2 > 0$, and κ is an arbitrary constant.

The scalar curvature of (2.82) is obtained as $R = 6\kappa$. For $\kappa = 1, 0, -1$, space is called spherical, planar, and hyperbolic, respectively.

Here, all 1-forms w, π, and μ are represented with three components in spherical directions r, θ, ϕ, i.e. $w = (w_r, w_\theta, w_\phi)$, $\pi = (\pi_r, \pi_\theta, \pi_\phi)$. For this example, we choose the complementary vector w as $w = (0, w_\theta, 0)$; co-vector fields π as $\pi = (0, \pi_\theta, 0)$; and recurrency form μ as $\mu = (0, \mu_\theta, 0)$ which are defined in equations (1.30), (2.2), and (2.4) respectively. Thus, for the metric (2.82), we find the connection coefficients, the Ricci curvature, and the scalar curvature of WS_n as follows:

$$\begin{aligned}
&\overline{\Gamma}_{rr}^{\ r} = \frac{\kappa r}{1-\kappa r^2}, && \overline{\Gamma}_{r\theta}^{\ r} = -w_\theta + \mu_\theta, \\
&\overline{\Gamma}_{\theta r}^{\ r} = -(w_\theta + \mu_\theta), && \overline{\Gamma}_{\theta\theta}^{\ r} = -r(1-\kappa r^2), \\
&\overline{\Gamma}_{\phi\phi}^{\ r} = -r(1-\kappa r^2)\sin^2\theta, && \overline{\Gamma}_{rr}^{\ \theta} = \frac{w_\theta - \mu_\theta}{r^2(1-\kappa r^2)}, \\
&\overline{\Gamma}_{\phi r}^{\ \phi} = \overline{\Gamma}_{\theta r}^{\ \theta} = \overline{\Gamma}_{r\phi}^{\ \phi} = \overline{\Gamma}_{r\theta}^{\ \theta} = \frac{1}{r}, && \overline{\Gamma}_{\phi\theta}^{\ \phi} = \cot\theta - w_\theta + \mu_\theta, \\
&\overline{\Gamma}_{\theta\theta}^{\ \theta} = -(w_\theta + \mu_\theta) = -\pi_\theta, && \overline{\Gamma}_{\phi\phi}^{\ \theta} = -\sin\theta(\cos\theta + (\mu_\theta - w_\theta)sin\theta), \\
&\overline{\Gamma}_{\theta\phi}^{\ \phi} = \cot\theta - \pi_\theta,\ .
\end{aligned} \tag{2.83}$$

Here, if we choose w_θ, and μ_θ to be functions of radial coordinate r, and also, π_θ to satisfy the relation $\pi_\theta = w_\theta(r) + \mu_\theta(r)$, then from (2.15), we can compute components of the torsion tensor for (2.84).

$$T_{r\theta}{}^{r} = T_{\phi\theta}{}^{\phi} = 2w_\theta\,, \qquad T_{\theta r}{}^{r} = T_{\theta\phi}{}^{\phi} = -2w_\theta\,, \tag{2.84}$$

and the components of the Ricci tensor of WS_n are

$$\begin{aligned}
\overline{R}_{rr} &= \frac{2\kappa r^2 + (w_\theta - \mu_\theta)(\cot\theta - w_\theta + \mu_\theta)}{r^2(1-\kappa r^2)}\,,\\
\overline{R}_{r\theta} &= \frac{2rw'_\theta - w_\theta + \mu_\theta}{r}\,,\\
\overline{R}_{\theta r} &= -\frac{r(w'_\theta + \mu'_\theta) + w_\theta - \mu_\theta}{r}\,,\\
\overline{R}_{\theta\theta} &= 2\kappa r^2 + \cot\theta(w_\theta - \mu_\theta)\,,\\
\overline{R}_{\phi\phi} &= \sin\theta(2\kappa r^2 \sin\theta + (w_\theta - \mu_\theta)[2\cos\theta + (\mu_\theta - w_\theta)\sin\theta])\,,
\end{aligned} \tag{2.85}$$

where prime ($'$) denotes derivative with respect to r, and the scalar curvature of WS_n is obtained as

$$\overline{R} = \frac{1}{r^2}\Big(6\kappa r^2 + 2(w_\theta - \mu_\theta)(2\cot\theta - w_\theta + \mu_\theta)\Big)\,. \tag{2.86}$$

Particularly, in (2.86), by taking

$$w_\theta = \lambda_\theta = \mu_\theta = c_1\sqrt{1-\kappa r^2} + c_2\left[-1 + \operatorname{artanh}\left(\frac{1}{\sqrt{1-\kappa r^2}}\right)\right], \quad 1-\kappa r^2 > 0 \tag{2.87}$$

where c_1, c_2 are any real constant, we obtain the scalar curvature of WS_n is $\overline{R} = 6\kappa$. Thus, the scalar curvature of WS_n becomes the same as the scalar curvature of Riemannian space.

3. EINSTEIN WEYL SPACES

We call a Weyl manifold WS_n as an Einstein manifold with semi-symmetric recurrent-metric connection $EWS_n(g,w,\pi,\mu)$, or EWS_n briefly, if the symmetric part of the Ricci tensor is proportional to the metric, that is

$$\overline{R}_{(ji)} = \theta g_{ij}, \tag{3.1}$$

for a scalar function θ defined on WS_n.
It follows from the symmetric part of Ricci tensor of WS_n that

Corollary 3.0.1. *Every 2-dimensional manifold WS_n is an Einstein manifold.*

3.1 Einstein Weyl Spaces With Semi-Symmetric Recurrent Metric Connection

Theorem 3.1.1. *For an Einstein Weyl manifold EW_n to be an EWS_n it is the necessary and sufficient condition that*

$$\nabla_j\lambda_i + \nabla_i\lambda_j + 2(w_i\lambda_j + w_j\lambda_i + \lambda_j\lambda_i) = \beta g_{ij}. \tag{3.2}$$

Proof. Let us consider an Einstein Weyl manifold EW. Then we have $W_{(ji)} = \gamma g_{ji}$. Thus, the symmetric part reduces to

$$\overline{R}_{(ji)} = \gamma g_{ji} - \beta g_{ji} - g_{ji}[2(n-2)\lambda_t\lambda^t + \nabla_t\lambda^t], \tag{3.3}$$

from which it follows that the symmetric part of the Ricci curvature tensor is proportional to the metric

$$\overline{R}_{(ji)} = \theta g_{ji}. \tag{3.4}$$

That is, the manifold is an EWS_n.
We now consider an EWS_n, that is $\overline{R}_{(ji)} = \theta g_{ji}$. Using the symmetric part of Ricci tensor of WS_n, we get

$$\theta g_{ji} = W_{(ji)} - \beta g_{ji} - g_{ji}[2(n-2)\lambda_t\lambda^t + \nabla_t\lambda^t]. \tag{3.5}$$

This equation shows that EWS_n is an EW_n, [49]. □

Theorem 3.1.2. *In an EWS_n, coefficient θ in (3.1) is not constant in general.*

Proof. By equating indices k and t in the second Bianchi identity of WS_n, we get

$$(\overline{\nabla}_l\overline{R}_{ji} + \overline{\nabla}_j\overline{R}_{lti}{}^{t} + \overline{\nabla}_t\overline{R}_{jli}{}^{t}) = 2(\pi_l\overline{R}_{ji} + \pi_j\overline{R}_{lti}{}^{t} + \pi_t\overline{R}_{jli}{}^{t}). \tag{3.6}$$

By using the symmetry properties of WS_n, we find that

$$(\overline{\nabla}_l\overline{R}_{ji} - \overline{\nabla}_j\overline{R}_{li} + \overline{\nabla}_t\overline{R}_{jli}{}^{t}) = 2(\pi_l\overline{R}_{ji} - \pi_j\overline{R}_{li} + \pi_t\overline{R}_{jli}{}^{t}). \tag{3.7}$$

Multiplying (3.7) by g^{ji} and using the compability condition of WS_n, we get

$$\overline{\nabla}_l\overline{R} + 2(w_l+\mu_l)\overline{R} - 2\overline{\nabla}_t\overline{R}_l{}^{t} + 2(w_t+\mu_t)\overline{R}_l{}^{t} + g^{ji}\overline{\nabla}_t\overline{R}_{jli}{}^{t} = 2(\pi_l\overline{R} - \pi_j\overline{R}_l{}^{j} + \pi_t g^{ji}\overline{R}_{jli}{}^{t}). \tag{3.8}$$

If we use $\pi_j = w_j + \mu_j$ and the symmetry properties of WS_n, we can write

$$\overline{\nabla}_t\overline{R}_{jli}{}^{t} = \overline{\nabla}_t(\overline{R}_{jlim}g^{mt}) = \overline{\nabla}_t(-g^{mt}\overline{R}_{jlmi} + 4\delta_i^t\nabla_{[l}\pi_{j]}), \tag{3.9}$$

and then

$$g^{ij}(\overline{\nabla}_t\overline{R}_{jli}{}^{t}) = \overline{\nabla}_t(-\overline{R}_l{}^{t}) - 2\pi_t\overline{R}_l{}^{t} + 4g^{ji}\overline{\nabla}_i\nabla_{[l}\pi_{j]}. \tag{3.10}$$

After some calculations (3.10) reduces to

$$\overline{\nabla}_l\overline{R} = 2\overline{\nabla}_t\overline{R}_l{}^{t} - 4g^{ji}\overline{\nabla}_t(\nabla_{[l}\pi_{j]}) + 8\pi^j(\nabla_{[l}\pi_{j]}). \tag{3.11}$$

On the other hand, scalar curvature $\overline{R}$ of EWS_n can be written as

$$\overline{R} = \overline{R}_{ji}g^{ji} = [\overline{R}_{(ji)} + \overline{R}_{[ji]}]g^{ji} = \overline{R}_{(ji)}g^{ji} = \theta g_{ji}g^{ji} = n\theta, \tag{3.12}$$

from which we find that

$$\theta = \frac{\overline{R}}{n}. \tag{3.13}$$

By using (3.13), (3.11) can be written as

$$\overline{\nabla}_l\overline{R} = 2\overline{\nabla}_l(\frac{\overline{R}}{n}) + 2\overline{\nabla}_t(\overline{R}_{[lm]}g^{mt}) + 4\left(2\pi^j - g^{ji}\overline{\nabla}_i\right)\nabla_{[l}\pi_{j]}. \tag{3.14}$$

We may simplify (3.14) as

$$\overline{\nabla}_l\overline{R} = \frac{2n}{(n-2)}\left[\overline{\nabla}_t\overline{\nabla}_t(\overline{R}_{[lm]}g^{mt}) + 2\left(2\pi^j - g^{ji}\overline{\nabla}_i\right)\nabla_{[l}\pi_{j]}\right]. \tag{3.15}$$

We see that covariant derivative of $\overline{R}$ is not zero, in general. □

3.2 Generalized Einstein Tensor of $WS_n(g,w,\pi,\mu)$

In this section, we define generalized Einstein tensor for WS_n in sense of [50]. Using the generalized the second Bianchi of WS_n identity we have

$$(\overline{\nabla}_l - 2\pi_l)\overline{R}_{kji}{}^{t} + (\overline{\nabla}_j - 2\pi_l)\overline{R}_{lki}{}^{t} + (\overline{\nabla}_k - 2\pi_k)\overline{R}_{jli}{}^{t} = 0. \tag{3.16}$$

Contracting t and k in (3.16) we obtain

$$(\overline{\nabla}_l - 2\pi_l)\overline{R}_{ji} + (\overline{\nabla}_j - 2\pi_l)\overline{R}_{lki}{}^{k} + (\overline{\nabla}_k - 2\pi_k)\overline{R}_{jli}{}^{k} = 0. \tag{3.17}$$

Using the symmetry relation of WS_n we get

$$(\overline{\nabla}_l - 2\pi_l)\overline{R}_{ji} - (\overline{\nabla}_j - 2\pi_l)\overline{R}_{li}{}^{k} + (\overline{\nabla}_k - 2\pi_k)\overline{R}_{jli}{}^{k} = 0\,. \tag{3.18}$$

Multiplying (3.18) by g^{ij} and using the compability condition of WS_n, we get

$$[\overline{\nabla}_l + 2(w_l+\mu_l)]\overline{R} - 2[\overline{\nabla}_k + 2(w_k+\mu_k)]\overline{R}_l{}^{k} + 2[\overline{\nabla}_k + 2(w_l+\mu_l)][g^{jk}(\nabla_{[l}w_{j]} + \nabla_{[l}\mu_{j]})] = 0. \tag{3.19}$$

Arranging (3.19), we obtain

$$[\overline{\nabla}_k + 2(w_k+\mu_k)]\left(\delta_l^k\overline{R} - 2\overline{R}_l{}^{k} + 2[g^{jk}(\nabla_{[l}w_{j]} + \nabla_{[l}\mu_{j]})]\right) = 0. \tag{3.20}$$

In (3.20) the term

$$\left(\delta_l^k\overline{R} - 2\overline{R}_l{}^{k} + 2[g^{jk}(\nabla_{[l}w_{j]} + \nabla_{[l}\mu_{j]})]\right) \tag{3.21}$$

is called generalized Einstein tensor of WS_n and is denoted

$$\overline{G}_l{}^{k} = \left(\delta_l^k\overline{R} - 2\overline{R}_l{}^{k} + 2[g^{jk}(\nabla_{[l}w_{j]} + \nabla_{[l}\mu_{j]})]\right), \tag{3.22}$$

and

$$\dot{\overline{\nabla}}_k = \overline{\nabla}_k + 2(w_k+\mu_k) \tag{3.23}$$

is said to be prolonged covariant derivative.

If we use (3.23) in (3.22) , we get

$$\dot{\overline{\nabla}}_k\overline{G}_l{}^{k} = 0\,, \tag{3.24}$$

that is $\overline{G}_l{}^{k}$ covariantly conserved or covariantly constant.

We state the following theorem:

Theorem 3.2.1. *Generalized Einstein tensor for WS_n is prolonged covariant constant.*

3.3 Sectional Curvatures on $WS_n(g,w,\pi,\mu)$

Let $\overline{\nabla}$ be a semi-symmetric metric connection, and $P(x^k)$ be a point on WS_n. Also, here, $X^i$and Y^i be two linearly independent vector which span the section Π.

The sectional curvature K for Π at P is defined as

$$K(\Pi,P) = \frac{\overline{R}_{ijkl} X^i Y^j X^k Y^l}{(g_{ik}g_{jl} - g_{il}g_{jk}) X^i Y^j X^k Y^l} \, . \tag{3.25}$$

The point P on WS_n is said to be isotropic point of $K(\Pi,P)$ if is the same for every section at P. And, if every point on WS_n is isotropic point then, the manifold is called isotropic.

Now, we recall a Weyl manifold WS_n to be an Einstein manifold with the semi-symmetric recurrent-metric connection EWS_n, if the symmetric part of the Ricci tensor is proportional to the metric, that is

$$\overline{R}_{(ji)} = \theta g_{ij} \, , \tag{3.26}$$

for a scalar function θ defined on WS_n in [48].

In [51], [52], it is shown that every 2-dimensional Weyl manifold is an Einstein manifold, and it is also given a sufficient condition for a Weyl manifold to be locally conformal to an Einstein manifold by using sectional curvature.

In the following theorem, we will generalize above mentioned sufficient condition to an isotropic Weyl manifold with the semi-symmetric recurrent metric connection.

We now quote the following lemma from [53] which will be used in our proof.

Lemma 3.3.1. *Suppose that S is any 4-covariant tensor, and that X and Y are two arbitrary linearly independent vectors. If for all X and Y*

$$S_{ijkl} X^i Y^j X^k Y^l = 0 \, , \tag{3.27}$$

then we have

$$S_{ijkl} + S_{klij} + S_{ilkj} + S_{kjil} = 0 \, . \tag{3.28}$$

Theorem 3.3.2. *Any isotropic Weyl manifold with the semi-symmetric recurrent-metric connection can be locally mapped conformally to an Einstein manifold with the semi-symmetric recurrent-metric connection, EWS_n.*

Proof. Assume that WS_n is an isotropic manifold. In the Lemma 3.3.1, by taking

$$S_{ijkl} = \overline{R}_{ijkl} - K(x)(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{3.29}$$

and using (3.28), we get

$$\overline{R}_{ijkl} + \overline{R}_{klij} + \overline{R}_{ilkj} + \overline{R}_{kjil} = 4Kg_{ik}g_{jl} - 2K(g_{lk}g_{ij} + g_{li}g_{kj}). \tag{3.30}$$

Transvecting (3.30) by g^{lh}

$$\overline{R}_{ijk}{}^{h} + \overline{R}_{kji}{}^{h} + (\overline{R}_{kjil} + \overline{R}_{kjil})g^{lh} = [4Kg_{ik}g_{jl} - 2K(g_{lk}g_{ij} + g_{li}g_{kj})]g^{lh}, \tag{3.31}$$

and using symmetry properties (2.45), (2.46) and the first Bianchi identity for EWS_n,

$$\overline{R}_{kji}{}^{l} + \overline{R}_{jik}{}^{l} + \overline{R}_{ikj}{}^{l} = 2(\delta^l_j\nabla_{[k}\pi_{i]} + \delta^l_i\nabla_{[j}\pi_{k]} + \delta^l_k\nabla_{[i}\pi_{j]}), \tag{3.32}$$

we find that

$$\overline{R}_{ijk}{}^{h} + \overline{R}_{kji}{}^{h} + \overline{R}_{ikj}{}^{h} + 2\overline{R}_{ilkj}g^{lh} = 2K(2g_{ik}\delta^h_j - g_{kj}\delta^h_i - g_{ij}\delta^h_k) - g^{lh}A_{lkji} + 4\delta^h_j(\nabla_{[k}w_{i]} + \nabla_{[k}\mu_{i]}), \tag{3.33}$$

where

$$A_{lkji} := 2(g_{ij}\nabla_{[l}\pi_{k]} + g_{kj}\nabla_{[i}\pi_{l]} + g_{lj}\nabla_{[k}\pi_{i]}). \tag{3.34}$$

Contracting (3.33) with h and i and using the equation (3.32),

$$\overline{R}_{jk} + \overline{R}_{kji}{}^{i} + \overline{R}_{kj} + 2\overline{R}_{ilkj}g^{il} = 2K(2g_{ik}\delta^i_j - g_{kj}\delta^i_i - g_{ij}\delta^i_k) - g^{li}A_{lkji} + 4\delta^i_j(\nabla_{[k}w_{i]} + \nabla_{[k}\mu_{i]}). \tag{3.35}$$

For (3.35), by using (2.30), (2.38) and (1.51), let us calculate $\overline{R}_{kji}{}^{i}$ and $g^{li}A_{lkji}$

$$\overline{R}_{kji}{}^{i} = Q_{kji}{}^{i} + W_{kji}{}^{i}, \tag{3.36}$$

where

$$\begin{aligned} Q_{kji}{}^{i} &= \delta^i_j\lambda_{ki} - \delta^i_k\lambda_{ji} + \lambda_{jl}g^{li}g_{ki} - \lambda_{km}g^{mi}g_{ji} + \delta^i_i(\partial_j\mu_k - \partial_k\mu_j) \\ &= 2n\nabla_{[j}\mu_{k]}. \end{aligned} \tag{3.37}$$

Thus,

$$Q_{kji}{}^{i} = 2n\nabla_{[j}\mu_{k]}, \tag{3.38}$$

$$W_{kji}{}^{i} = n(w_{jk} - w_{kj}) = 2n\nabla_{[j}w_{k]}\ , \tag{3.39}$$

and from (3.34)

$$A_{lkji}g^{li} = 2g^{li}g_{kj}\nabla_{[i}\pi_{l]}\ . \tag{3.40}$$

Hence, we reach

$$\overline{R}_{jk} + \overline{R}_{kj} + 2n(\nabla_{[j}\mu_{k]} + \nabla_{[j}w_{k]}) = 2K(2g_{ik}\delta_i^j - ng_{kj} - g_{kj}) + 4(\nabla_{[k}w_{j]} + \nabla_{[k}\mu_{j]}) - 2g^{li}g_{kj}\nabla_{[i}\pi_{l]}\ . \tag{3.41}$$

From (3.41), we observe that the symmetric part of the Ricci tensor is

$$\overline{R}_{(jk)} = (1-n)Kg_{kj} - (n+2)(\nabla_{[j}w_{k]} + \nabla_{[j}\mu_{k]})\,. \tag{3.42}$$

Since $\overline{R}_{(jk)}$ is symmetric, second term of (3.42) must satisfy the relation

$$\nabla_{[j}w_{k]} + \nabla_{[j}\mu_{k]} = 0\,. \tag{3.43}$$

Thus, the symmetric Ricci tensor of EWS_n is

$$\overline{R}_{(jk)} = (1-n)Kg_{kj}\,, \tag{3.44}$$

and equation (3.43) implies that w_k and μ_k are gradient, [48]. □

4. TRANSFORMATIONS UNDER SPECIAL CONNECTION

4.1 Conformal Transformation

Norden [28], extended the idea of conformal transformation from Riemanniann space into Weyl spaces, between W_n, and W_n^* which satisfy $\tau : W_n(g_{ij}, w_k) \rightarrow W_n^*(g_{ij}^*, w_k^*)$ where using proper conformal scaling on these space at corresponding points which holds for

$$g_{ij}^* = g_{ij} \,, \tag{4.1}$$

and the covector field transform as $\rho = w_k - w_k^*$.
Therefore, the connection coefficients of W_n^* is obtained

$$\Gamma_{kl}^{\;\;i*} = \Gamma_{ij}^{\;\;h} + \delta_k^i \rho_l + \delta_l^i \rho_k - g_{kl} g^{im} \rho_m \,, \tag{4.2}$$

[28], [29], [54].
The conformal curvature tensor of W_n^* is

$$W_{ijk}^{\;\;\;h} = R_{ijk}^{\;\;\;h} - \frac{1}{(n-2)} [\delta_i^h (L_{jk} - L_{kj}) + \delta_j^h L_{ik} - \delta_k^h L_{ij} - g^{hm} (g_{ij} L_{mk} - g_{ik} L_{mj})] \,. \tag{4.3}$$

Here

$$L_{ij} = \frac{1}{n} [(n-1) R_{ij} + R_{ji}] - \frac{1}{(n-1)} g_{ij} R \,, \tag{4.4}$$

[55], [56], [57], [58], [59].

4.1.1 Conformal Transformation on $WS_n(g, w, \pi, \mu)$

Now, we examine the conformal transformation for Weyl Spaces with a semi-symmetric recurrent metric connection, [60]. WS_n, and WS_n^* represent two Weyl spaces with semi-symmetric recurrent metric connection and $\sigma : WS_n \rightarrow WS_n^*$ be a conformal transformation such that
$\mathbb{W}_i = w_i - w_i^*$, $\gamma_k = \lambda_k - \lambda_k^*$, $\aleph_i = \mu_i - \mu_i^*$ and $g_{ij} = g_{ij}^*$. Thus, the equation (2.29) can be rewritten as,

$$\overline{\Gamma}_{ik}^{\;\;l*} = \left\{ \begin{matrix} l \\ ik \end{matrix} \right\}^* - (\delta_k^{l*} w_i^* + \delta_i^{l*} w_k^* - g_{ik}^* w^{l*}) + (\lambda_k^* \delta_i^{l*} - \mu_i^* \delta_k^{l*} - g_{ik}^* \lambda^{l*}) \,, \tag{4.5}$$

by applying new form relations into the equation of (4.5), we reach

$$\overline{\Gamma}_{ik}^{\ \ l*} = \overline{\Gamma}_{ik}^{\ \ l} - \delta_k^l(\mathbb{W}_i + \aleph_i) + \delta_i^l(\mathbb{W}_k - \gamma_k) - g_{ik}(\mathbb{W}^l - \gamma^l)\ . \tag{4.6}$$

To calculate the conformal curvature tensor, we use the equation (4.6) into

$$\overline{R}_{kji}^{\ \ \ h*} = \partial_k\overline{\Gamma}_{ji}^{\ \ h*} - \partial_j\overline{\Gamma}_{ki}^{\ \ h*} - \overline{\Gamma}_{ki}^{\ \ t*}\overline{\Gamma}_{jt}^{\ \ h*} + \overline{\Gamma}_{ji}^{\ \ t*}\overline{\Gamma}_{kt}^{\ \ h*}\ , \tag{4.7}$$

by differentiating indices with respect to k, j, and i.

Hence, the equation of (4.7) becomes

$$\begin{aligned}
\overline{R}_{kji}^{\ \ \ h*} =&\overline{R}_{kji}^{\ \ \ h} + \delta_i^h\partial_k(\mathbb{W}_j + \aleph_j) + \delta_j^h\partial_k(\mathbb{W}_i - \gamma_i) - \partial_k g_{ji}(\mathbb{W}^h - \gamma^h) - g_{ji}\partial_k(\mathbb{W}^h - \gamma^h)\\
&- \delta_i^h\partial_j(\mathbb{W}_k + \aleph_k) - \delta_k^h\partial_j(\mathbb{W}_i - \gamma_i) + \partial_j g_{ki}(\mathbb{W}^h - \gamma^h) + g_{ki}\partial_j(\mathbb{W}^h - \gamma^h)\\
&+ \overline{\Gamma}_{ji}^{\ \ h}(\mathbb{W}_k + \aleph_k) + \overline{\Gamma}_{ji}^{\ \ t}\delta_k^h(\mathbb{W}_t - \gamma_t) - \overline{\Gamma}_{ji}^{\ \ t} g_{kt}(\mathbb{W}^h - \gamma^h) + \overline{\Gamma}_{ki}^{\ \ h}(\mathbb{W}_j + \aleph_j)\\
&+ \delta_i^h(\mathbb{W}_j + \aleph_j)(\mathbb{W}_k + \aleph_k) + \delta_i^t\delta_k^h(\mathbb{W}_j + \aleph_j)(\mathbb{W}_t - \gamma_t) - g_{ki}(\mathbb{W}_j + \aleph_j)(\mathbb{W}^h - \gamma^h)\\
&+ \overline{\Gamma}_{kj}^{\ \ h}(\mathbb{W}_i - \gamma_i) + \delta_j^h(\mathbb{W}_i - \gamma_i)(\mathbb{W}_k + \aleph_k) + \delta_j^t\delta_k^h(\mathbb{W}_i - \gamma_i)(\mathbb{W}_t - \gamma_t)\\
&- g_{kj}(\mathbb{W}_i - \gamma_i)(\mathbb{W}^h - \gamma^h) - \overline{\Gamma}_{kt}^{\ \ h} g_{ji}(\mathbb{W}^t - \gamma^t) - \delta_t^h g_{ji}(\mathbb{W}^t - \gamma^t)(\mathbb{W}_k + \aleph_k)\\
&- \delta_k^h g_{ji}(\mathbb{W}^t - \gamma^t)(\mathbb{W}_t - \gamma_t) + g_{ji}g_{kt}(\mathbb{W}^t - \gamma^t)(\mathbb{W}^h - \gamma^h) - \overline{\Gamma}_{ki}^{\ \ h}(\mathbb{W}_j + \aleph_j)\\
&- \overline{\Gamma}_{ki}^{\ \ t}\delta_j^h(\mathbb{W}_t - \gamma_t) + \overline{\Gamma}_{ki}^{\ \ t} g_{jt}(\mathbb{W}^h - \gamma^h) - \overline{\Gamma}_{ji}^{\ \ h}(\mathbb{W}_k + \aleph_k) - \delta_i^h(\mathbb{W}_k + \aleph_k)(\mathbb{W}_j + \aleph_j)\\
&- \delta_i^t\delta_j^h(\mathbb{W}_k + \aleph_k)(\mathbb{W}_t - \gamma_t) + g_{ji}(\mathbb{W}_k + \aleph_k)(\mathbb{W}^h - \gamma^h) - \overline{\Gamma}_{jk}^{\ \ h}(\mathbb{W}_i - \gamma_i)\\
&- \delta_k^h(\mathbb{W}_i - \gamma_i)(\mathbb{W}_j + \aleph_j) - \delta_k^t\delta_j^h(\mathbb{W}_i - \gamma_i)(\mathbb{W}_t - \gamma_t) + g_{jk}(\mathbb{W}_i - \gamma_i)(\mathbb{W}^h - \gamma^h)\\
&+ \overline{\Gamma}_{jt}^{\ \ h} g_{ki}(\mathbb{W}^t - \gamma^t) + \delta_t^h g_{ki}(\mathbb{W}^t - \gamma^t)(\mathbb{W}_j + \aleph_j) + \delta_j^h g_{ki}(\mathbb{W}^t - \gamma^t)(\mathbb{W}_t - \gamma_t)\\
&- g_{ki}g_{jt}(\mathbb{W}^t - \gamma^t)(\mathbb{W}^h - \gamma^h)\ .
\end{aligned} \tag{4.8}$$

By using the definition of derivative for a metric tensor that is

$$\partial_k g_{ij} = \begin{Bmatrix} s \\ kj \end{Bmatrix} g_{si} + \begin{Bmatrix} s \\ ki \end{Bmatrix} g_{sj}\ , \tag{4.9}$$

into (4.8) and arranging terms, we get

$$
\begin{aligned}
\overline{R}_{kji}^{\ \ \ h*} =& \overline{R}_{kji}^{\ \ \ h} + \delta_i^h[\partial_k(\mathbb{W}_j + \aleph_j) - \partial_j(\mathbb{W}_k + \aleph_k)] \\
&+ \delta_j^h[\partial_k(\mathbb{W}_i - \gamma_i) + w_k(\mathbb{W}_i - \gamma_i) + w_i(\mathbb{W}_k - \gamma_k) - g_{ki}w^t(\mathbb{W}_t - \gamma_t) - \lambda_i(\mathbb{W}_k - \gamma_k) \\
&+ \lambda^t g_{ki}(\mathbb{W}_t - \gamma_t) - \lambda_k(\mathbb{W}_i - \gamma_i) - (\mathbb{W}_i - \gamma_i)(\mathbb{W}_k - \gamma_k) - g_{ki}w^a(\mathbb{W}_a - \gamma_a) \\
&+ g_{ki}\lambda^a(\mathbb{W}_a - \gamma_a) + g_{ki}g^{at}(\mathbb{W}_a - \gamma_a)(\mathbb{W}_t - \gamma_t) - \left\{ {t \atop ki} \right\}(\mathbb{W}_t - \gamma_t)] \\
&- \delta_k^h[\partial_j(\mathbb{W}_i - \gamma_i) - \left\{ {t \atop ji} \right\}(\mathbb{W}_t - \gamma_t) + w_i(\mathbb{W}_j - \gamma_j) - g_{ji}w^t(\mathbb{W}_t - \gamma_t) - \lambda_i(\mathbb{W}_j - \gamma_j) \\
&+ g_{ji}\lambda^t(\mathbb{W}_t - \gamma_t) + w_j(\mathbb{W}_i - \gamma_i) - \lambda_j(\mathbb{W}_i - \gamma_i) - (\mathbb{W}_i - \gamma_i)(\mathbb{W}_j - \gamma_j) - g_{ji}w^a(\mathbb{W}_a - \gamma_a) \\
&+ g_{ji}\lambda^a(\mathbb{W}_a - \gamma_a) + g_{ji}g^{at}(\mathbb{W}_a - \gamma_a)(\mathbb{W}_t - \gamma_t)] \\
&- g_{ji}g^{ha}[\partial_k(\mathbb{W}_a - \gamma_a) - \lambda_k(\mathbb{W}_a - \gamma_a) + w_a(\mathbb{W}_k - \gamma_k) - \lambda_a(\mathbb{W}_k - \gamma_k) \\
&- (\mathbb{W}_k - \gamma_k)(\mathbb{W}_a - \gamma_a) + w_k(\mathbb{W}_a - \gamma_a)] \\
&+ g_{ki}g^{ha}[\partial_j(\mathbb{W}_a - \gamma_a) + w_j(\mathbb{W}_a - \gamma_a) - \lambda_j(\mathbb{W}_a - \gamma_a) - \lambda_a(\mathbb{W}_j - \gamma_j) \\
&- (\mathbb{W}_j - \gamma_j)(\mathbb{W}_a - \gamma_a) + w_a(\mathbb{W}_j - \gamma_j)] \, . \qquad (4.10)
\end{aligned}
$$

In the equation of (4.10), the definition of covariant derivative of a form, $\nabla_k \mathbb{W}_i$ which is

$$\overline{\nabla}_k(\mathbb{W}_i) = \partial_k \mathbb{W}_i - \mathbb{W}_j \overline{\Gamma}_{ik}^{\ \ t} \, , \qquad (4.11)$$

is used as the coefficient terms for δ_j^h and δ_k^h, respectively.

Hence, the equation (4.10) can be rewritten as,

$$
\begin{aligned}
\overline{R}_{kji}^{\ \ \ h*} =& \overline{R}_{kji}^{\ \ \ h} + \delta_i^h[\partial_k(\mathbb{W}_j + \aleph_j) - \partial_j(\mathbb{W}_k + \aleph_k)] \\
&+ \delta_j^h[\nabla_k(\mathbb{W}_i - \gamma_i) - \mu_i(\mathbb{W}_k - \gamma_k) - \lambda_i(\mathbb{W}_k - \gamma_k) - (\mathbb{W}_i - \gamma_i)(\mathbb{W}_k - \gamma_k) \\
&- g_{ki}w^a(\mathbb{W}_a - \gamma_a) + g_{ki}(\mathbb{W}^t - \gamma^t)(\mathbb{W}_t - \gamma_t) + \lambda^a g_{ki}(\mathbb{W}_a - \gamma_a)] \\
&- \delta_k^h[\nabla_j(\mathbb{W}_i - \gamma_i) - \mu_i(\mathbb{W}_j - \gamma_j) - \lambda_i(\mathbb{W}_j - \gamma_j) - (\mathbb{W}_i - \gamma_i)(\mathbb{W}_j - \gamma_j) \\
&- g_{ji}w^a(\mathbb{W}_a - \gamma_a) + g_{ji}(\mathbb{W}^t - \gamma^t)(\mathbb{W}_t - \gamma_t) + \lambda^a g_{ji}(\mathbb{W}_a - \gamma_a)] \\
&- g_{ji}g^{ha}[\partial_k(\mathbb{W}_a - \gamma_a) - \lambda_k(\mathbb{W}_a - \gamma_a) + w_a(\mathbb{W}_k - \gamma_k) - \lambda_a(\mathbb{W}_k - \gamma_k) \\
&- (\mathbb{W}_k - \gamma_k)(\mathbb{W}_a - \gamma_a) + w_k(\mathbb{W}_a - \gamma_a)] \\
&+ g_{ki}g^{ha}[\partial_j(\mathbb{W}_a - \gamma_a) + w_j(\mathbb{W}_a - \gamma_a) - \lambda_j(\mathbb{W}_a - \gamma_a) - \lambda_a(\mathbb{W}_j - \gamma_j) \\
&- (\mathbb{W}_j - \gamma_j)(\mathbb{W}_a - \gamma_a) + w_a(\mathbb{W}_j - \gamma_j)] \, . \qquad (4.12)
\end{aligned}
$$

To simplify the equation (4.12), we define new expressions such as

$$\begin{aligned}\mathbb{K}_{ij} :=&\nabla_j(\mathbb{W}_i-\gamma_i)-\mu_i(\mathbb{W}_j-\gamma_j)-\lambda_i(\mathbb{W}_j-\gamma_j)-(\mathbb{W}_i-\gamma_i)(\mathbb{W}_j-\gamma_j)\,,\\ \mathbb{L}_{ij} :=&\mathbb{K}_{ij}-g_{ji}w^a(\mathbb{W}_a-\gamma_a)+g_{ji}(\mathbb{W}^t-\gamma^t)(\mathbb{W}_t-\gamma_t)+\lambda^a g_{ji}(\mathbb{W}_a-\gamma_a)\,,\end{aligned}\tag{4.13}$$

and,

$$\begin{aligned}\mathbb{H}_{ak} :=&\partial_k(\mathbb{W}_a-\gamma_a)-\lambda_k(\mathbb{W}_a-\gamma_a)+w_a(\mathbb{W}_k-\gamma_k)-\lambda_a(\mathbb{W}_k-\gamma_k)-(\mathbb{W}_k-\gamma_k)(\mathbb{W}_a-\gamma_a)\\ &+w_k(\mathbb{W}_a-\gamma_a)\,.\end{aligned}\tag{4.14}$$

If we differentiate the equations (4.13) and (4.14) with respect to indices k and j, then the equation (4.12) turns into,

$$\overline{R}_{kji}^{\;\;\;h*}=\overline{R}_{kji}^{\;\;\;h}+\delta_i^h\nabla_{[k}(\mathbb{W}+\aleph)_{j]}+\delta_j^h\mathbb{L}_{ik}-\delta_k^h\mathbb{L}_{ij}+g_{ki}g^{ha}\mathbb{H}_{aj}-g_{ji}g^{ha}\mathbb{H}_{ak}\,.\tag{4.15}$$

By transvecting with g_{hm}, we reach

$$\overline{R}_{kjim}^{\;*}=\overline{R}_{kjim}+2g_{im}\nabla_{[k}(\mathbb{W}+\aleph)_{j]}+g_{mj}\mathbb{L}_{ik}-g_{mk}\mathbb{L}_{ij}+g_{ki}\mathbb{H}_{mj}-g_{ji}\mathbb{H}_{mk}\,.\tag{4.16}$$

Also, by multiplying with g^{km}, we have

$$\overline{R}_{ji}^{\;*}=\overline{R}_{ji}+2\nabla_{[i}(\mathbb{W}+\aleph)_{j]}+\mathbb{L}_{ij}-n\mathbb{L}_{ij}+\mathbb{H}_{ij}-g_{ji}\mathbb{H}\,,\tag{4.17}$$

then contracting with g^{ji}, we obtain

$$\overline{R}^*=\overline{R}-(n-1)(\mathbb{L}+\mathbb{H})\,,\tag{4.18}$$

where n denotes n-dimensional space of WS_n.

Now, we derive the invariant part of the curvature tensor under the conformal transformation σ which is going to be the conformal curvature tensor. Hence, we rewrite $\nabla_{[k}(\mathbb{W}+\aleph)_{j]}$, $\mathbb{L}_{ik}$, $\mathbb{H}_{mj}$ of the equation (4.16) in terms of curvature tensor $\overline{R}_{kjim}$, Ricci tensor $\overline{R}_{ji}$, and the scalar curvature tensor $\overline{R}$ of WS_n.

By using the equation (4.17), we have

$$\mathbb{H}_{ij}=\overline{R}_{ji}^{\;*}-\overline{R}_{ji}-2\nabla_{[i}(\mathbb{W}+\aleph)_{j]}+(n-1)\mathbb{L}_{ij}+g_{ji}\mathbb{H}\,.\tag{4.19}$$

Also, from the equation (4.18), we have

$$\mathbb{H}=\frac{\overline{R}^*-\overline{R}}{n-1}-\mathbb{L}\,.\tag{4.20}$$

Putting the equations (4.19) and (4.20) into the equation of (4.16), we have

$$\begin{aligned}\overline{R}^{*}_{kjim} =&\overline{R}_{kjim} + 2g_{im}\nabla_{[k}(\mathbb{W}+\aleph)_{j]} - 2g_{ki}\nabla_{[m}(\mathbb{W}+\aleph)_{j]} + 2g_{ji}\nabla_{[m}(\mathbb{W}+\aleph)_{k]} \\ &+ g_{ki}(\overline{R}_{jm}^{\ *} - \overline{R}_{jm}) - g_{ji}(\overline{R}_{km}^{\ *} - \overline{R}_{km}) + \frac{g_{ki}g_{jm}}{1-n}(\overline{R}^{*} - \overline{R}) - \frac{g_{ji}g_{km}}{1-n}(\overline{R}^{*} - \overline{R}) \\ &+ g_{mj}\mathbb{L}_{ik} - g_{mk}\mathbb{L}_{ij} + (n-1)g_{ki}\mathbb{L}_{mj} - (n-1)g_{ji}\mathbb{L}_{mk} - g_{ki}g_{jm}\mathbb{L} + g_{ji}g_{km}\mathbb{L}\ . \end{aligned} \tag{4.21}$$

By transvecting with g^{mj}, we have

$$\overline{R}^{*}_{kjim}g^{mj} = \overline{R}_{kjim}g^{mj} - (\overline{R}_{ki}^{\ *} - \overline{R}_{ki})\ . \tag{4.22}$$

Also, by multiplying the equation (4.21) with the metric tensor g^{kj}, we get

$$\overline{R}^{*}_{kjim}g^{kj} = \overline{R}_{kjim}g^{kj}\ . \tag{4.23}$$

Multiplying the equation (4.21) with g^{ij}, we obtain

$$\begin{aligned}\overline{R}^{*}_{kjim}g^{ij} =&\overline{R}_{kjim}g^{ij} + (2n-4)\nabla_{[m}(\mathbb{W}+\aleph)_{k]} + (1-n)(\overline{R}_{km}^{\ *} - \overline{R}_{km}) \\ &+ g_{km}(\overline{R}^{*} - \overline{R}) + (2n-n^2)\mathbb{L}_{mk} + (n-2)g_{km}\mathbb{L}\ , \end{aligned} \tag{4.24}$$

which concludes

$$\begin{aligned}\nabla_{[m}(\mathbb{W}+\aleph)_{k]} =&\frac{1}{2n-4}[\overline{R}^{*}_{kjim}g^{ij} - \overline{R}_{kjim}g^{ij} + (n-1)(\overline{R}_{km}^{\ *} - \overline{R}_{km}) - g_{km}(\overline{R}^{*} - \overline{R}) \\ &+ (n^2-2n)\mathbb{L}_{mk} - (n-2)g_{km}\mathbb{L}]\ . \end{aligned} \tag{4.25}$$

Next, by transvecting the equation (4.21) with g^{ki}, we find

$$\begin{aligned}\nabla_{[m}(\mathbb{W}+\aleph)_{j]} =&[\overline{R}^{*}_{kjim}g^{ki} - \overline{R}_{kjim}g^{ki} - (n-1)(\overline{R}_{jm}^{\ *} - \overline{R}_{jm}) + g_{jm}(\overline{R}^{*} - \overline{R}) \\ &- (n^2-2n)\mathbb{L}_{mj} + (n-2)g_{mj}\mathbb{L}]\frac{1}{4-2n}\ . \end{aligned} \tag{4.26}$$

Furthermore, by taking the equation (4.21) into action with g^{im}, we finally find out

$$\begin{aligned}\nabla_{[k}(\mathbb{W}+\aleph)_{j]} =&\frac{1}{2n-4}[\overline{R}^{*}_{kjim}g^{im} - \overline{R}_{kjim}g^{im} - (\overline{R}_{jk}^{\ *} - \overline{R}_{jk}) + (\overline{R}^{*}_{kj} - \overline{R}_{kj}) \\ &- (n-2)(\mathbb{L}_{kj} + \mathbb{L}_{jk})]\ . \end{aligned} \tag{4.27}$$

By using the equations (4.25), (4.26), (4.27) into equation of (4.21), we have

$$\begin{aligned}(1-\frac{3n}{n-2})\overline{R}^{*}_{kjim} =&(1-\frac{3n}{n-2})\overline{R}_{kjim} + (\overline{R}_{jk}^{\ *} - \overline{R}_{jk})(-\frac{g_{im}}{n-2}) + (\overline{R}_{kj}^{\ *} - \overline{R}_{kj})(\frac{g_{im}}{n-2}) \\ &+ (\overline{R}_{jm}^{\ *} - \overline{R}_{jm})(\frac{g_{ki}}{2-n}) + (\overline{R}_{km}^{\ *} - \overline{R}_{km})(\frac{1}{n-2}) \\ &+ (\overline{R}^{*} - \overline{R})(\frac{g_{ki}g_{jm}}{(n-2)(n-1)} - \frac{g_{ji}g_{km}}{(n-2)(n-1)}) \\ &- (\mathbb{L}_{kj} - \mathbb{L}_{jk})g_{im} - \mathbb{L}_{mj}g_{ki} + \mathbb{L}_{mk}g_{ji} + \mathbb{L}_{ik}g_{mj} - \mathbb{L}_{ij}g_{mk}\ . \end{aligned} \tag{4.28}$$

From the equation (4.28), we write down all possible terms for $\overline{R}^{*}_{kjim}$ by changing the indices cyclically and then adding all of them, we obtain

$$\begin{aligned}
&\overline{R}^{*}_{kjim}+\overline{R}^{*}_{mkji}+\overline{R}^{*}_{imkj}+\overline{R}^{*}_{jimk}+\overline{R}_{jk}^{*}(\frac{1-g_{im}}{2(1+n)})+\overline{R}_{kj}^{*}(\frac{g_{im}}{2(1+n)})-\overline{R}_{jm}^{*}(\frac{g_{ki}}{2(1+n)})\\
&+\overline{R}_{km}^{*}(\frac{1-g_{ji}}{2(1+n)})+\overline{R}_{mk}^{*}(\frac{g_{ji}}{2(1+n)})-\overline{R}_{ki}^{*}(\frac{g_{mj}}{2(1+n)})+\overline{R}_{mi}^{*}(\frac{1-g_{kj}}{2(1+n)})+\overline{R}_{im}^{*}(\frac{g_{kj}}{2(1+n)})\\
&-\overline{R}_{mj}^{*}(\frac{g_{ik}}{2(1+n)})+\overline{R}_{ij}^{*}(\frac{1-g_{mk}}{2(1+n)})+\overline{R}_{ji}^{*}(\frac{g_{mk}}{2(1+n)})-\overline{R}_{ik}^{*}(\frac{g_{jm}}{2(1+n)})\\
&+\overline{R}^{*}(\frac{1}{(n-1)(n+1)})[2g_{jm}g_{ki}-g_{ji}g_{km}-g_{kj}g_{mi}]\\
&=\overline{R}_{kjim}+\overline{R}_{mkji}+\overline{R}_{imkj}+\overline{R}_{jimk}+\overline{R}_{jk}(\frac{1-g_{im}}{2(1+n)})+\overline{R}_{kj}(\frac{g_{im}}{2(1+n)})-\overline{R}_{jm}(\frac{g_{ki}}{2(1+n)})\\
&+\overline{R}_{km}(\frac{1-g_{ji}}{2(1+n)})+\overline{R}_{mk}(\frac{g_{ji}}{2(1+n)})-\overline{R}_{ki}(\frac{g_{mj}}{2(1+n)})+\overline{R}_{mi}(\frac{1-g_{kj}}{2(1+n)})+\overline{R}_{im}(\frac{g_{kj}}{2(1+n)})\\
&-\overline{R}_{mj}(\frac{g_{ik}}{2(1+n)})+\overline{R}_{ij}(\frac{1-g_{mk}}{2(1+n)})+\overline{R}_{ji}(\frac{g_{mk}}{2(1+n)})-\overline{R}_{ik}(\frac{g_{jm}}{2(1+n)})\\
&+\overline{R}(\frac{1}{(n-1)(n+1)})[2g_{jm}g_{ki}-g_{ji}g_{km}-g_{kj}g_{mi}]\,. \qquad (4.29)
\end{aligned}$$

Also, by using the equation (4.22), we can rewrite the $\overline{R}^{*}_{mkji}$ such as;

$$\overline{R}^{*}_{mkji}=\overline{R}_{mkji}-(\overline{R}_{mj}^{*}-\overline{R}_{mj})\frac{g_{ik}}{n}\,, \qquad (4.30)$$

and changing the indices cyclically, and then put all of them into the equation (4.29), we obtain

$$\begin{aligned}
&\overline{R}^{*}_{kjim}-\overline{R}_{mj}^{*}(\frac{g_{ik}}{2(1+n)}+\frac{g_{ik}}{n})-\overline{R}_{ik}^{*}(\frac{g_{jm}}{2(1+n)}+\frac{g_{jm}}{n})-\overline{R}_{jm}^{*}(\frac{g_{ki}}{2(1+n)}+\frac{g_{ki}}{n})\\
&+\overline{R}_{jk}^{*}(\frac{1-g_{im}}{2(1+n)})+\overline{R}_{kj}^{*}(\frac{g_{im}}{2(1+n)})+\overline{R}_{km}^{*}(\frac{1-g_{ji}}{2(1+n)})+\overline{R}_{mk}^{*}(\frac{g_{ji}}{2(1+n)})-\overline{R}_{ki}^{*}(\frac{g_{mj}}{2(1+n)})\\
&+\overline{R}_{mi}^{*}(\frac{1-g_{kj}}{2(1+n)})+\overline{R}_{im}^{*}(\frac{g_{kj}}{2(1+n)})+\overline{R}_{ij}^{*}(\frac{1-g_{mk}}{2(1+n)})-\overline{R}_{ji}^{*}(\frac{g_{mk}}{2(1+n)})\\
&+\overline{R}^{*}(\frac{1}{(n-1)(n+1)})[2g_{jm}g_{ki}-g_{ji}g_{km}-g_{kj}g_{mi}]\\
&=\overline{R}_{kjim}-\overline{R}_{mj}(\frac{g_{ik}}{2(1+n)}+\frac{g_{ik}}{n})-\overline{R}_{ik}(\frac{g_{jm}}{2(1+n)}+\frac{g_{jm}}{n})-\overline{R}_{jm}(\frac{g_{ki}}{2(1+n)}+\frac{g_{ki}}{n})\\
&+\overline{R}_{jk}(\frac{1-g_{im}}{2(1+n)})+\overline{R}_{kj}(\frac{g_{im}}{2(1+n)})+\overline{R}_{km}(\frac{1-g_{ji}}{2(1+n)})+\overline{R}_{mk}(\frac{g_{ji}}{2(1+n)})-\overline{R}_{ki}(\frac{g_{mj}}{2(1+n)})\\
&+\overline{R}_{mi}(\frac{1-g_{kj}}{2(1+n)})+\overline{R}_{im}(\frac{g_{kj}}{2(1+n)})+\overline{R}_{ij}(\frac{1-g_{mk}}{2(1+n)})-\overline{R}_{ji}(\frac{g_{mk}}{2(1+n)})\\
&+\overline{R}(\frac{1}{(n-1)(n+1)})[2g_{jm}g_{ki}-g_{ji}g_{km}-g_{kj}g_{mi}]\,. \qquad (4.31)
\end{aligned}$$

From the equation (4.31), the invariant part

$$
\begin{aligned}
\mathbb{C}_{kjim} &= \overline{R}_{kjim} - \overline{R}_{mj}(\frac{(3n+2)g_{ik}}{2n(1+n)}) - \overline{R}_{ik}(\frac{(3n+2)g_{jm}}{2n(1+n)}) - \overline{R}_{jm}(\frac{(3n+2)g_{ki}}{2n(1+n)}) \\
&+ \overline{R}_{jk}(\frac{1-g_{im}}{2(1+n)}) + \overline{R}_{kj}(\frac{g_{im}}{2(1+n)}) + \overline{R}_{km}(\frac{1-g_{ji}}{2(1+n)}) + \overline{R}_{mk}(\frac{g_{ji}}{2(1+n)}) - \overline{R}_{ki}(\frac{g_{mj}}{2(1+n)}) \\
&+ \overline{R}_{mi}(\frac{1-g_{kj}}{2(1+n)}) + \overline{R}_{im}(\frac{g_{kj}}{2(1+n)}) + \overline{R}_{ij}(\frac{1-g_{mk}}{2(1+n)}) - \overline{R}_{ji}(\frac{g_{mk}}{2(1+n)}) \\
&+ \overline{R}(\frac{1}{(n-1)(n+1)})[2g_{jm}g_{ki} - g_{ji}g_{km} - g_{kj}g_{mi}] \,, \qquad (4.32)
\end{aligned}
$$

and is called conformal curvature tensor of WS_n.

Thus, we have proved the following theorem:

Theorem 4.1.1. *The conformal curvature tensor $\mathbb{C}_{kjim}$ of WS_n is obtained as in the form (4.32).*

4.2 Projective Transformation

Let ∇ and $\bar{\nabla}$ be two connection on W_n and $\bar{W}_n$ spaces, respectively. If the diffeomorphisma f which is $f : W_n(g,w) \to \bar{W}_n(\bar{g},\bar{w})$ preserves the geodesics, then this mapping is called as projective transformation. The connection coefficient is given by [61], [62], [63], [64],

$$\bar{\Gamma}_{ij}^{\ \ h} = \Gamma_{ij}^{\ \ h} + \delta_i^h \psi_j + \delta_j^h \psi_i \,, \qquad (4.33)$$

where the curvature tensor and the Ricci tensor are

$$\bar{R}_{ijk}^{\ \ \ h} = R_{ijk}^{\ \ \ h} + \delta_i^h(\psi_{kj} - \psi_{jk}) + \delta_k^h \psi_{ij} - \delta_j^h \psi_{ik} \,, \qquad (4.34)$$

$$\bar{R}_{ij} = R_{ij} + (n-1)\psi_{ij} + (\psi_{kj} - \psi_{jk}) \,. \qquad (4.35)$$

Here, $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$.

4.2.1 Projective Transformation on $WS_n(g,w,\pi,\mu)$

From [22], it is known that projective curvature for non-Riemannian geometry is given

$$\mathbb{P}_{ikh}^{\ \ \ l} = \overline{P_{ikh}^{\ \ \ l}} + \frac{1}{n^2-1}[(n\overline{P_{ih}} + \overline{P_{hi}})\delta_k^l - (n\overline{P_{ik}} + \overline{P_{ki}})\delta_h^l] + \frac{1}{n+1}(\overline{P_{kh}} - \overline{P_{hk}})\delta_i^l \,. \qquad (4.36)$$

Here, $\overline{P_{ikh}^{\ \ \ l}}$ is

$$\overline{P_{ikh}^{\ \ \ l}} = \partial_i \Gamma_{(kh)}^{\ \ \ l} - \partial_k \Gamma_{(ih)}^{\ \ \ l} + \Gamma_{(kh)}^{\ \ \ t}\Gamma_{(it)}^{\ \ \ l} - \Gamma_{(ih)}^{\ \ \ t}\Gamma_{(kt)}^{\ \ \ l} \,, \qquad (4.37)$$

where $\Gamma_{(kh)}^{\ \ \ l}$ represents the symmetric part of the connection coefficient in given space. Hence, we apply the similar analogy to our semi-symmetric recurrent metric Weyl space WS_n to derive the projective curvature tensor [60].

By making use of the equation (2.29), we get the symmetric part of connection coefficient of WS_n as

$$\frac{1}{2}(\overline{\Gamma}_{ik}{}^{l}+\overline{\Gamma}_{ki}{}^{l})=\overline{\Gamma}_{(ik)}^{l}$$
$$=\left\{ {l \atop ik} \right\}-(w_i\delta_k^l+w_k\delta_i^l-g_{ik}w^l)+\frac{1}{2}[\delta_i^l(\lambda_k-\mu_k)+\delta_k^l(\lambda_i-\mu_i)]-g_{ik}\lambda^l\,. \tag{4.38}$$

To write down the equation (4.38) in term of $\overline{\Gamma}_{ik}{}^{l}$, we need to add and substract some terms which are $\pm\frac{1}{2}\delta_i^l\lambda_k$, and $\pm\frac{1}{2}\delta_k^l\mu_i$, then we reach,

$$\overline{\Gamma}_{(ik)}^{l}=\overline{\Gamma}_{ik}{}^{l}+\frac{1}{2}[\delta_k^l(\lambda_i+\mu_i)-\delta_i^l(\lambda_k+\mu_k)]\,. \tag{4.39}$$

Now, we put this symmetric part $\overline{\Gamma}_{(ik)}^{l}$ of the connection coefficient into the equation (4.37) by differentiating with respect to indices k,h, and i, then we find,

$$\begin{aligned}\overline{P_{ikh}{}^{l}}=&\partial_i\overline{\Gamma}_{kh}{}^{l}+\frac{1}{2}\delta_h^l\partial_i(\lambda_k+\mu_k)-\frac{1}{2}\delta_k^l\partial_i(\lambda_h+\mu_h)-\partial_k\overline{\Gamma}_{ih}{}^{l}-\frac{1}{2}\delta_h^l\partial_k(\lambda_i+\mu_i)\\&+\frac{1}{2}\delta_i^l\partial_k(\lambda_h+\mu_h)+\overline{\Gamma}_{kh}{}^{t}\overline{\Gamma}_{it}{}^{l}+\frac{1}{2}\overline{\Gamma}_{kh}{}^{t}\delta_t^l(\lambda_i+\mu_i)-\frac{1}{2}\overline{\Gamma}_{kh}{}^{t}\delta_i^l(\lambda_t+\mu_t)\\&+\frac{1}{2}\overline{\Gamma}_{it}{}^{l}\delta_h^t(\lambda_k+\mu_k)+\frac{1}{4}\delta_h^t\delta_t^l(\lambda_k+\mu_k)(\lambda_i+\mu_i)-\frac{1}{4}\delta_h^t\delta_i^l(\lambda_k+\mu_k)(\lambda_t+\mu_t)\\&-\frac{1}{2}\overline{\Gamma}_{it}{}^{l}\delta_k^t(\lambda_h+\mu_h)-\frac{1}{4}\delta_k^t\delta_t^l(\lambda_h+\mu_h)(\lambda_i+\mu_i)+\frac{1}{4}\delta_k^t\delta_i^l(\lambda_h+\mu_h)(\lambda_t+\mu_t)\\&-\overline{\Gamma}_{ih}{}^{t}\overline{\Gamma}_{kt}{}^{l}-\frac{1}{2}\overline{\Gamma}_{ih}{}^{t}\delta_t^l(\lambda_k+\mu_k)+\frac{1}{2}\overline{\Gamma}_{ih}{}^{t}\delta_k^l(\lambda_t+\mu_t)-\frac{1}{2}\overline{\Gamma}_{kt}{}^{l}\delta_h^t(\lambda_i+\mu_i)\\&-\frac{1}{4}\delta_h^t\delta_t^l(\lambda_i+\mu_i)(\lambda_k+\mu_k)+\frac{1}{4}\delta_h^t\delta_k^l(\lambda_i+\mu_i)(\lambda_t+\mu_t)+\frac{1}{2}\overline{\Gamma}_{kt}{}^{l}\delta_i^t(\lambda_h+\mu_h)\\&+\frac{1}{4}\delta_i^t\delta_t^l(\lambda_h+\mu_h)(\lambda_k+\mu_k)-\frac{1}{4}\delta_i^t\delta_k^l(\lambda_h+\mu_h)(\lambda_t+\mu_t)\,.\end{aligned} \tag{4.40}$$

In the equation (4.40), some terms represent the definition of $\overline{R}_{ikh}{}^{l}$ in the equation of (2.33). So, by arranging some terms in (4.40), we have

$$\begin{aligned}\overline{P_{ikh}{}^{l}}=&\overline{R}_{ikh}{}^{l}-\frac{1}{2}\overline{\Gamma}_{kh}{}^{t}\delta_i^l(\lambda_t+\mu_t)-\frac{1}{2}\overline{\Gamma}_{ik}{}^{l}(\lambda_h+\mu_h)+\frac{1}{2}\overline{\Gamma}_{ih}{}^{t}\delta_k^l(\lambda_t+\mu_t)\\&+\frac{1}{2}\overline{\Gamma}_{ki}{}^{l}(\lambda_h+\mu_h)+\frac{1}{2}\delta_h^l\partial_i(\lambda_k+\mu_k)-\frac{1}{2}\delta_k^l\partial_i(\lambda_h+\mu_h)-\frac{1}{2}\delta_h^l\partial_k(\lambda_i+\mu_i)\\&\frac{1}{2}\delta_i^l\partial_k(\lambda_h+\mu_h)-\frac{1}{4}\delta_k^l(\lambda_h+\mu_h)(\lambda_i+\mu_i)+\frac{1}{4}\delta_i^l(\lambda_h+\mu_h)(\lambda_k+\mu_k)\,.\end{aligned} \tag{4.41}$$

It is easily seen that the equation (4.41) has terms such as $-\frac{1}{2}\overline{\Gamma}_{ik}{}^{l}(\lambda_h+\mu_h)$ and $+\frac{1}{2}\overline{\Gamma}_{ki}{}^{l}(\lambda_h+\mu_h)$ which are going to be taken into action to find the anti-symmetric part $\overline{\Gamma}_{[ik]}^{\;\;l}$ of the connection coefficient, that is

$$\frac{1}{2}(\overline{\Gamma}_{ki}^{\;\;l} - \overline{\Gamma}_{ik}^{\;\;l}) = \overline{\Gamma}_{[ki]}^{\;\;l}$$
$$= \frac{1}{2}[\delta_k^l(\lambda_i + \mu_i) - \delta_i^l(\lambda_k + \mu_k)] \,. \tag{4.42}$$

In addition to this, in the equation of (4.41), the definition of covariant derivative of an one-form, $\nabla_k(\lambda_i + \mu_i)$ which is

$$\nabla_k(\lambda_i + \mu_i) = \partial_k(\lambda_i + \mu_i) - (\lambda_t + \mu_t)\overline{\Gamma}_{ki}^{\;\;t} \,, \tag{4.43}$$

is used.

Hence, by using (4.42) and (4.43), we rewrite the equation of (4.41) arranged form such as

$$\begin{aligned}\overline{P_{ikh}^{\;\;\;l}} =& \overline{R}_{ikh}^{\;\;\;l} + \frac{1}{2}\delta_i^l \nabla_k(\lambda_h + \mu_h) - \frac{1}{2}\delta_k^l \nabla_i(\lambda_h + \mu_h) + \frac{1}{2}\delta_h^l \partial_i(\lambda_k + \mu_k) \\ & - \frac{1}{2}\delta_h^l \partial_k(\lambda_i + \mu_i) + \frac{1}{4}\delta_k^l(\lambda_h + \mu_h)(\lambda_i + \mu_i) - \frac{1}{4}\delta_i^l(\lambda_h + \mu_h)(\lambda_k + \mu_k) \,. \end{aligned} \tag{4.44}$$

To simplifying the equation of (4.44), we define new term as,

$$\mathbb{Q}_{ikh}^{\;\;\;l} := \frac{1}{2}\delta_i^l \nabla_k(\lambda_h + \mu_h) + \frac{1}{2}\delta_h^l \partial_i(\lambda_k + \mu_k) + \frac{1}{4}\delta_k^l(\lambda_h + \mu_h)(\lambda_i + \mu_i) \,. \tag{4.45}$$

Using the equation of (4.45) in (4.44), we get

$$\overline{P_{ikh}^{\;\;\;l}} = \overline{R}_{ikh}^{\;\;\;l} + \mathbb{Q}_{ikh}^{\;\;\;l} - \mathbb{Q}_{kih}^{\;\;\;l} \,. \tag{4.46}$$

By transvecting indices h, and l in the equation (4.46), we have

$$\overline{P_{ik}} = \overline{R}_{ik} + \mathbb{Q}_{ik} - \mathbb{Q}_{ki} \,. \tag{4.47}$$

Now, putting equations (4.46) and (4.47) into the equation of (4.36), we finally derive the projective curvature tensor $\mathbb{P}_{ikh}^{\;\;\;l}$ of WS_n as below,

$$\begin{aligned}\mathbb{P}_{ikh}^{\;\;\;l} =& \overline{R}_{ikh}^{\;\;\;l} + \mathbb{Q}_{ikh}^{\;\;\;l} - \mathbb{Q}_{kih}^{\;\;\;l} \\ & + \frac{1}{n^2-1}[(n(\overline{R}_{ih} + \mathbb{Q}_{ih} - \mathbb{Q}_{hi}) + \overline{R}_{hi} + \mathbb{Q}_{hi} - \mathbb{Q}_{ih})\delta_k^l - (n(\overline{R}_{ik} + \mathbb{Q}_{ik} - \mathbb{Q}_{ki}) + \overline{R}_{ki} + \mathbb{Q}_{ki} - \mathbb{Q}_{ik})\delta_h^l] \\ & + \frac{1}{n+1}(\overline{R}_{kh} - \overline{R}_{hk} + 2(\mathbb{Q}_{kh} - \mathbb{Q}_{hk}))\delta_i^l \,. \end{aligned} \tag{4.48}$$

Hence, we have proved the following theorem:

Theorem 4.2.1. *The projective curvature tensor* $\mathbb{P}_{ikh}^{\;\;\;l}$ *of* WS_n *is obtained as in the form (4.48).*

5. GEODESICS

Let τ be geodesic mapping which is $W_n \to \bar{W}_n$ for the curve $\zeta : x^i = x^i(t)$ is a geodesic of W_n if and only if the following equation

$$\frac{d^2x^l}{dt^2} + \Gamma_{ik}^{\ \ l} \frac{dx^i}{dt} \frac{dx^k}{dt} = \varrho(t)\frac{dx^i}{dt} \tag{5.1}$$

is satisfied. Here, $\varrho(t)$ is to be determinated function in [22], [65], [66], [67].

5.1 The Geodesic Equation on $WS_n(g, w, \pi, \mu)$

Now, we examine the geodesic equations on WS_n, and give an example for hyperbolic plane, [68]. Let us put the connection coefficient $\overline{\Gamma}_{ik}^{\ \ l}$ into (5.1), we get

$$\frac{d^2x^l}{dt^2} + \overline{\Gamma}_{ik}^{\ \ l} \frac{dx^i}{dt} \frac{dx^k}{dt} = 0\,. \tag{5.2}$$

By using the definition of connection coefficient, (2.29) in the equation (5.2), we obtain

$$\frac{d^2x^l}{dt^2} + [\left\{ \begin{matrix} l \\ ik \end{matrix} \right\} - (\delta^l_k w_i + \delta^l_i w_k - g_{ik} w^l) + (\lambda_k \delta^l_i - \mu_i \delta^l_k - g_{ik}\lambda^l)] \frac{dx^i}{dt} \frac{dx^k}{dt} = 0\,. \tag{5.3}$$

Also, using property of (1.4), and rearranging the indices with respect to i, and k, we reach

$$\frac{d^2x^l}{dt^2} + \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} \frac{dx^i}{dt} \frac{dx^k}{dt} - (w_k + \mu_k)\frac{dx^k}{dt} \frac{dx^l}{dt} + (\lambda_k - w_k)\frac{dx^l}{dt} \frac{dx^k}{dt} + (w^l - \lambda^l)g_{ik}\frac{dx^i}{dt} \frac{dx^k}{dt} = 0\,. \tag{5.4}$$

Since the first two terms in the equation (5.4) represent the geodesic equation of Riemannian Space which is also zero. Therefore the equation (5.4) becomes

$$(\lambda_k - 2w_k - \mu_k)\frac{dx^l}{dt} \frac{dx^k}{dt} + (w^l - \lambda^l)g_{ik}\frac{dx^i}{dt} \frac{dx^k}{dt} = 0\,. \tag{5.5}$$

Then,

$$[(\lambda_k - 2w_k - \mu_k)\frac{dx^l}{dt} + (w^l - \lambda^l)g_{ik}\frac{dx^i}{dt}] \frac{dx^k}{dt} = 0\,. \tag{5.6}$$

Hence, the term $\frac{dx^k}{dt}$ cannot be equal to zero, so

$$(\lambda_k - 2w_k - \mu_k)\frac{dx^l}{dt} + (w^l - \lambda^l)g_{ik}\frac{dx^i}{dt} = 0\,. \tag{5.7}$$

By defining $V^i = \frac{dx^i}{dt}$, then (5.7) becomes

$$(\lambda_k - 2w_k - \mu_k)V^l + (w^l - \lambda^l)V_k = 0\,. \tag{5.8}$$

Finally, to satisfy the equation (5.8), we have the relationship between forms, which is,

$$\mu_k + w_k = 0\,. \tag{5.9}$$

Thus, we have proved:

Theorem 5.1.1. *The sufficient condition for having the same geodesic equations of Riemmannian Space and WS_n is given by (5.9). Here, w, and μ are 1-forms satisfying (1.30) and (2.2).*

Now, we examine the geodesics of the *Poincaré* upper halfplane with given metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$ where $y > 0$ on WS_n.
The connection coefficients of given metric is calculated as below,

$${\Gamma_{12}}^1 = {\Gamma_{21}}^1 = -{\Gamma_{11}}^2 = {\Gamma_{22}}^2 = -\frac{1}{y}\,, \tag{5.10}$$

where

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \quad , \qquad g^{\mu\nu} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}. \tag{5.11}$$

Now, we study the geodesic equations by choosing the indices as $l = 1,2$, $k = 1,2$, $i = 1,2$ for the equation (5.3).
For the indice $l = 1$ and $i = 1,2$, $k = 1,2$, we get

$$\begin{aligned}
\frac{d^2x^1}{dt^2} &+ \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} \frac{dx^1}{dt}\frac{dx^1}{dt} + (\lambda_1 - 2w_1 - \mu_1)\frac{dx^1}{dt}\frac{dx^1}{dt} + g_{11}g^{1m}(w_m - \lambda_m)\frac{dx^1}{dt}\frac{dx^1}{dt} \\
&+ \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \frac{dx^2}{dt}\frac{dx^1}{dt} + (\lambda_1 - 2w_1 - \mu_1)\frac{dx^1}{dt}\frac{dx^1}{dt} + g_{21}g^{1m}(w_m - \lambda_m)\frac{dx^2}{dt}\frac{dx^1}{dt} \\
&+ \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} \frac{dx^1}{dt}\frac{dx^2}{dt} + (\lambda_2 - 2w_2 - \mu_2)\frac{dx^2}{dt}\frac{dx^1}{dt} + g_{12}g^{1m}(w_m - \lambda_m)\frac{dx^1}{dt}\frac{dx^2}{dt} \\
&+ \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \frac{dx^2}{dt}\frac{dx^2}{dt} + (\lambda_2 - 2w_2 - \mu_2)\frac{dx^2}{dt}\frac{dx^1}{dt} + g_{22}g^{1m}(w_m - \lambda_m)\frac{dx^2}{dt}\frac{dx^2}{dt} = 0\,.
\end{aligned} \tag{5.12}$$

By putting the values in (5.10) and (5.11) into (5.12), we get

$$\begin{aligned}
\frac{d^2x^1}{dt^2} &+ (\lambda_1 - 2w_1 - \mu_1)\frac{dx^1}{dt}\frac{dx^1}{dt} + \frac{1}{y^2}y^2(w_1 - \lambda_1)\frac{dx^1}{dt}\frac{dx^1}{dt} - \frac{1}{y}\frac{dx^2}{dt}\frac{dx^1}{dt} \\
&+ (\lambda_1 - 2w_1 - \mu_1)\frac{dx^1}{dt}\frac{dx^1}{dt} - \frac{1}{y}\frac{dx^1}{dt}\frac{dx^2}{dt} + (\lambda_2 - 2w_2 - \mu_2)\frac{dx^2}{dt}\frac{dx^1}{dt} \\
&+ (\lambda_2 - 2w_2 - \mu_2)\frac{dx^2}{dt}\frac{dx^1}{dt} + \frac{1}{y^2}y^2(w_1 - \lambda_1)\frac{dx^2}{dt}\frac{dx^2}{dt} = 0\,.
\end{aligned} \tag{5.13}$$

The equation (5.12) is calculated for the indice $m = 1$, since all the components of g^{lm} vanishes.

Finally, the equation (5.13) turns into

$$\ddot{x} + \dot{x}^2[\lambda_1 - 3w_1 - 2\mu_1] + \dot{y}\dot{x}[-\frac{2}{y} + 2\lambda_2 - 4w_2 - 2\mu_2] + \dot{y}^2[w_1 - \lambda_1] = 0\ . \tag{5.14}$$

Similarly, we calculate for the indices $l = 2$, and $i = 1,2,\ k = 1,2$, we reach

$$\ddot{y} + \dot{x}^2[\frac{1}{y} + w_2 - \lambda_2] + \dot{x}\dot{y}[2\lambda_1 - 4w_1 - 2\mu_1] + \dot{y}^2[\lambda_2 - 3w_2 - 2\mu_2 - \frac{1}{y}] = 0\ , \tag{5.15}$$

where

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y'\dot{x}\ . \tag{5.16}$$

By using the equations (5.14) and (5.15), one can classify the geodesics of the space with respect to indices. Here, we examine just one case which is by assuming the forms as $\lambda = w = \mu = 0$ in order to calculate the geodesic equations easily.

Thus, the equations becomes (5.14) and (5.15),

$$\ddot{x} + \dot{y}\dot{x}(-\frac{2}{y}) = 0\ , \tag{5.17}$$

$$\ddot{y} + \dot{x}^2(\frac{1}{y}) - \dot{y}^2(\frac{1}{y}) = 0\ . \tag{5.18}$$

Hence, from (5.16), we get

$$\begin{aligned}\ddot{y} = \frac{d}{dt}\dot{y} &= \frac{d}{dt}(y'\dot{x}) \\ &= y''\dot{x}^2 + y'\ddot{x}\ .\end{aligned} \tag{5.19}$$

By using the (5.19) into the equation (5.18), we rewrite as

$$y''\dot{x}^2 + y'\ddot{x} + (\frac{1}{y})[\dot{x}^2 - (y')^2\dot{x}^2] = 0\ . \tag{5.20}$$

From (5.17), we obtain

$$\dot{x}^2[y'' + (\frac{2}{y})(y')^2 + \frac{1}{y} - (\frac{1}{y})(y')^2] = 0\ . \tag{5.21}$$

By solving the equation (5.21) for x and y one by one, we get the solution for y such as

$$(x(t) - c_0)^2 + y(t)^2 = (c_0^2 + 2c_1)\ , \qquad c_0, c_1 \in \mathbb{R}\ , \tag{5.22}$$

which is the circle with radius $\sqrt{c_0^2 + 2c_1}$, center $(c_0, 0)$.

Another solution in the equation (5.21) for x is

$$\dot{x}^2 = 0\ , \tag{5.23}$$

that x is constant. This condition satisfies both equations (5.17) and (5.18). Hence, being constant of x gives another solution for (5.18) which becomes

$$\ddot{y} - \dot{y}^2(\frac{1}{y}) = 0\,, \tag{5.24}$$

therefore, the solution of (5.24) is

$$y(t) = be^{at} \quad a, b \in \mathbb{R}\ ,\ b \neq 0\,, \tag{5.25}$$

which represents curves.

6. CONCLUSIONS AND RECOMMENDATIONS

In differential geometry spaces can be classified with respect to structures defined on them. It is even just a problem how far we could generalized the connection in space. It is a necessary situation that generalize the semi-symmetric metric connection on Weyl manifolds to explain the properties under gauge invariant transformation for physical systems; either for differential geometry as well, under the special transformations such as projective, conformal, and geodesic.

Weyl spaces are similar to Riemannian space that admit symmetric connection but the covariant derivative of the metric tensor is not zero in Weyl spaces.

In this thesis, we introduce semi-symmetric recurrent metric connection on Weyl spaces WS_n to examine curvature properties of WS_n spaces having these structures. We also define Einstein Weyl space with semi-symmetric recurrent metric connection EWS_n, and we give a necessary and sufficient condition for an EWS_n space to be Einstein space EW_n. Moreover, we define the generalized Einstein tensor in WS_n space and express it in terms of semi-symmetric recurrent metric connection. In addition to this, we define sectional curvature, and isotropic WS_n.

Furthermore, we obtain conformal, projective curvature tensors on WS_n, and we examine geodesic equations of WS_n. Also, we study the example of hyperbolic plane for this space.

When we work over the connections on WS_n, we realized that it has been not studied Einstein equations - covariantly preserved - in general with respect to connections yet. Also, another lack of study is figuring out different covariantly preserved equation.One can examine these problems to make the space richer.

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