ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

GROUP CLASSIFICATION FOR A HIGHER-ORDER BOUSSINESQ EQUATION

M.Sc. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Program

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

YÜKSEK MERTEBELİ BOUSSINESQ DENKLEMİNİN GRUP SINIFLANDIRMASI

YÜKSEK LİSANS TEZİ

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Matematik Mühendisliği Anabilim Dalı

Matematik Mühendisliği Programı

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TEMMUZ 2020

Yasin HASANOĞLU, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509171255 successfully defended the thesis entitled "GROUP CLASSIFICATION FOR A HIGHER-ORDER BOUSSINESQ EQUATION", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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Date of Submission : 15 June 2020
Date of Defense : 13 July 2020

To my family,

FOREWORD

First and foremost I would like to thank my thesis advisor Assoc.Prof.Dr. Cihangir ÖZEMİR for his tireless efforts and information sharing during the thesis work. Without his enlightening lectures, sincerity and patience I would not be able to complete this thesis work. I am thankful to him for introducing me Lie symmetry analysis of PDEs.
I am grateful to the Professors in the Department of Mathematics who taught me during my scientific preparation and graduate education for their efforts.
I would like to thank my friend Furkan KAYA for his valuable mathematical suggestions and discussions.
Finally, I owe special thanks to my parents Cevdet, Kerime and my brothers Rafet, Kadir and Ersin for encouraging me to continue on Mathematics.

July 2020 Yasin HASANOĞLU
(M.Sc. Student)

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ABBREVIATIONS

ODE	**:** Ordinary Differential Equation
PDE	**:** Partial Differential Equation
HBq	**:** Higher-Order Boussinesq Equation
KDD	**:** Kısmi Diferansiyel Denklem

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GROUP CLASSIFICATION FOR A HIGHER-ORDER BOUSSINESQ EQUATION

SUMMARY

Lie symmetry analysis of partial differential equations (PDE) is a connection for many mathematical fields, including Lie algebras, Lie groups, differential geometry, ordinary differential equations, partial differential equations and mathematical physics. This list can be extended according to the research topic, type of the PDE and so on. Finding analytical solution of a PDE is not easy in general. A powerful tool which is used by both mathematicians and physicists to find analytical solution of a PDE is transformation groups. Transformation groups, simply, can be defined as groups of which action leave the solution space of an equation invariant. One can reduce the number of independent variables of a PDE by using Lie groups and Lie algebras. The Lie algorithm to find symmetry generators can be summarized as follows: First, one generates the determining equations for the symmetries of the system. Second, these equations are solved manually or with a computer package to determine the explicit forms of the vector fields of which flows generate the transformation groups. By using Lie series and commutation relations, one can compute adjoint representations, determine the structure of the Lie algebra of the equation. From the Lie algebras, symmetry groups are obtained and actions of these symmetry groups leave the solution space of the PDE invariant.

One can use Lie theory to classify differential equations. The procedure for the classification of symmetry algebras can be summarized as follows: First, find equivalence transformation of the equation. Second, find non-equivalent forms of the symmetry generator. Last, determine the invariance algebra of the equation from two and higher dimensional Lie algebras (the well-known structural results on the classification of low dimensional Lie algebras make this procedure possible). The result of this procedure is a list of representative equations with canonical invariance algebras, classified up to equivalence transformations.

Symmetry classification of PDEs are studied by both mathematicians and physicists. Some mathematicians focus on Lie symmetry classification itself since it can be useful for finding integrable systems of PDEs.

This thesis can be seen as an application of Lie symmetry analysis which is described above. In this thesis, a family of higher-order Boussinesq (HBq) equations of the form

$$u_{tt} = \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + (f(u))_{xx}$$

where $f(u)$ is an arbitary function, is considered to be classified according to the Lie symmetry algebras the equation admits depending on the formulation of the nonlinearity $f(u)$.

In Chapter 1, the literature about HBq is reviewed and main results of the thesis are given. In Chapter 2, some fundamental definitions, theorems and notations regarding Lie group analysis of differential equations is introduced. In Chapter 3, the main result of the thesis is proved, and three possible canonical forms of $f(u)$ is obtained so that the equation admits finite-dimensional Lie algebras. In Chapter 4, some exact solutions to HBq is found by focusing on traveling wave solutions which is widely concerned in literature.

Through this thesis, we believe that we contribute to the current literature on symmetry algebras of Boussinesq-type equations and also on the solutions of this PDE.

YÜKSEK MERTEBELİ BOUSSINESQ DENKLEMİNİN GRUP SINIFLANDIRMASI

ÖZET

Norveçli matematikçi Sophus Lie'nin dönüşüm grupları konusunda yaptığı çalışmaların üzerinden yaklaşık yüz elli yıl geçmesine rağmen birçok matematikçi ve fizikçi için bu çalışmalar güncelliğini yitirmemiştir.

Geometrik bir nesnenin simetrisi, kabaca, bu geometrik nesneyi değiştirmeyen dönüşüm olarak tanımlanır. Fizikçi ve matematikçiler genel olarak yapı koruyan simetriler ile ilgilenmişlerdir. Katı bir geometrik nesnenin yapısını koruyan sonlu sayıda dönüşüm bulabiliriz. Örneğin herhangi bir üçgeni gözönünde bulunduralım. Üçgeni katı halde bırakan dönüşümler sadece yansıma, öteleme ve döndürme simetrileridir. Bu simetriler sonucunda üçgen üzerindeki herhangi iki nokta arasındaki mesafe değişmez. Ancak, eğer üçgen silgi malzemesinden yapılırsa, yapı koruyan (yani yapıldığı malzeme değişmeyen) dönüşümlerin grubu daha büyük olur. Dolayısıyla, bu üçgen için yeni simetriler elde edebiliriz.

Dönüşümlerin simetri kabul edilme şartları: 1. Yapıyı korumalıdır, 2. Difeomorfizma olmalıdır. Yani, eğer x herhangi bir nesne üzerindeki konumu gösteren rastgele bir nokta ve $\Gamma : x \to \tilde{x}(x)$ herhangi bir simetri ise, $\tilde{x}$ x'e göre sonsuz türevlenebilir olmalıdır ve aynı şekilde ters fonksiyon Γ^{-1} de sonsuz türevlenebilir olmalıdır. 3. Geometrik nesneyi kendisine dönüştürmelidir. Yani, (x,y) düzlemindeki düzlemsel bir nesne ile $(\tilde{x},\tilde{y})$ düzlemindeki görüntüsü birbirinin aynısı olmalıdır. Bu, aynı zamanda simetri şartı olarak da bilinir.

Kısmi diferansiyel denklemlerin Lie simetri analizi, Lie Cebiri, Lie Grupları, Diferansiyel Geometri, Adi Diferansiyel Denklemler, Kısmi Diferansiyel Denklemler, Matematiksel Fizik gibi birçok alanı kapsayan bir bağlantı noktasıdır. Bu liste araştırma konusu, KDD'nin türü v.s. göre daha da uzatılabilir. KDD'lerin analitik çözümlerini bulmak genel olarak kolay değildir. Fizikçilerin ve matematikçilerin KDDlerin analitik çözümlerini bulmak maksadıyla kullandıkları en güçlü araçlardan biri dönüşüm gruplarıdır. Dönüşüm grupları, basitçe, etkileri bir denklemin çözüm uzayını değişmez bırakan gruplardır. Lie grupları ve Lie cebirleri kullanılarak bir KDD'nin bağımsız değişken sayısı düşürülebilir. Simetri üretecinin bulunması için kullanılan Lie'nin algoritması müteakip şekilde özetlenebilir: Öncelikle sistemin simetrilerine ait belirleyici denklemleri bulunur. Daha sonra bu denklemleri elle veya bilgisayar paket programları vasıtasıyla çözerek integral eğrileri diferansiyel denklemin simetri dönüşümünü veren vektör alanları elde edilir. Komütasyon bağıntıları ile Lie cebirinin yapısı, Lie serileri ile adjoint temsil elde edilir . Lie cebirlerinden Lie grupları elde edilir ve bu grupların etkileri çözüm uzayını değişmez bırakır.

Diferansiyel denklemlerin sınıflandırılmasında Lie teorisi kullanılabilir. Simetri cebirlerinin sınıflandırılmasını müteakip şekilde özetlenebilir: Öncelikle denklemin

denklik dönüşümlerini bulunur. İkinci olarak, simetri üreteçlerinin denk-olmayan formlarını formları bulunur. Son olarak iki veya daha yüksek boyutlu Lie cebirlerini kulanarak denklemin değişmez cebirleri elde edilir. (Düşük boyutlu Lie cebirlerinin sınıflandırılması konusundaki iyi bilinen yapısal sonuçlar bunu kolaylaştırır.) Bu işlemlerin sonucunda; denklik dönüşümlerine göre sınıflandırılmış, kurallı değişmez cebirlere sahip bir temsilci denklemler listesi elde edilir.

KDDlerin simetri sınıflandırılması konusunda hem matematikçiler hem de fizikçiler çalışmaktadır. Bazı matematikçiler KDDlerin integrallenebilir sistemlerini bulmak maksadıyla sadece Lie simetri sınıflandırmalarına odaklanmışlardır.

Bu tez yukarıda ifade edilen Lie simetri analizinin bir uygulaması olarak görülebilir. Bu tezde yüksek mertebeli Boussinesq denklemi

$$u_{tt} = \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + (f(u))_{xx}$$

doğrusal olmayan $f(u)$ fonksiyonunun formülüne bağlı olarak Lie simetri cebirlerine göre sınıflandırılmaya çalışılmıştır. Bu, daha önce çalışılmamış olan orijinal açık bir problemdir.

Bölüm 1'de yüksek mertebeli Boussinesq denklemi ile ilgili literatür gözden geçirilmiş ve tezin ana sonuçları verilmiştir. Burada da belirtildiği gibi , bu denklem matematikçi Rosenau tarafından elde edilmiştir. Rosenau bu denklemi, daha önce geliştirdiği bir metodu (quasi-continuous formalism) yoğun ayrık sistemlerin (dense discrete systems) dinamiğine uygulayarak elde etmiştir.

Bölüm 2'de diferansiyel denklemlerin Lie grubu analizinin temel tanımları, teoremleri ve gösterimleri verilmiştir.

Bölüm 3'te, tezin ana sonucu kanıtlanmıştır ve $f(u)$ fonksiyonlarının kanonik formları belirtilen şekilde elde edilmiştir:

$$\begin{array}{lll} \text{(A)} \quad f(u) = \alpha e^u, & \alpha = \mp 1, & \\ \text{(B)} \quad f(u) = \alpha \ln(u), & \alpha = \mp 1, & \\ \text{(C)} \quad f(u) = \alpha u^n, & \alpha = \mp 1, & \mathbb{R} \ni n \neq 0, 1. \end{array}$$

Ayrıca, özel bir $f(u)$ fonksiyonu için dört boyutlu Lie simetri cebiri elde edilmiştir ki bu durum $\eta_2 = 0$ olduğu zaman da geçerlidir. $\eta_2 = 0$ durumu için simetri sınıflandırması literatürde mevcuttur, ancak bunun incelendiği çalışmada simetri cebirinin maksimal boyutu 3 olarak gösterilmektedir. Dolayısıyla analizimiz, $\eta_2 = 0$ durumunda literatürde bulunan sonuçlara da bir düzeltme getirmekte, $\eta_2 \neq 0$ durumunu inceleyerek de orijinal bir çalışma olarak literatürde yerini almaktadır. Durum (A) için sonuçlar açık bir şekilde ifade edilmiştir. Durum (B) ve Durum (C) için elde edilen sonuçlar bir tabloda özetlenmiştir.

Bölüm 4'de literatürde yaygın olarak ilgilenilen hareketli dalga çözümleri kullanılarak, yüksek mertebeli Boussinesq denklemi için bazı tam çözümler bulunmuştur. Bunun için öncelikle yüksek mertebeli Boussinesq denklemi $f(u) = u + \alpha u^2$ için hareketli dalga çözümlerinin araştırılmasıyla aşağıda belirtilen forma dönüştürülmüştür:

$$\eta_2 k^4 c^2 \left[F''' F' - \frac{1}{2}(F'')^2 \right] - \frac{\eta_1 k^2 c^2}{2}(F')^2 - \frac{\alpha k^2}{3} F^3 + \frac{c^2 - k^2}{2} F^2 = K_0.$$

Daha sonra katsayılar üzerinde çeşitli tahminlerde bulunularak yüksek mertebeli Boussinesq denklemlerinin tam çözümleri bulunmaya çalışılmıştır. Bu şekilde literatürde mevcut olmayan bazı trigonometrik, hiperbolik ve elliptik çözümler elde edilmiştir.

Bu tez sayesinde, Boussinesq sınıfı KDDlerin simetri cebirleri ve bu KDDlerin çözümleriyle ilgili olarak literatüre katkıda bulunduğumuza inanmaktayız.

1. INTRODUCTION

1.1 Purpose of Thesis

The aim of the thesis is to classify higher-order Boussinesq (HBq) equations of the form

$$u_{tt} = \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + (f(u))_{xx} \tag{1.1.1}$$

according to the Lie symmetry algebras the equation admits depending on the formulation of the nonlinearity $f(u)$ and to study possible reductions of this equation to find exact solutions. Here we assume η_1, η_2 are nonzero constants and $f_{uu} \neq 0$. More explicitly, the purpose is to determine the classes of functions $f(u)$ for which the equation has finite-dimensional Lie symmetry algebras. Among these classes, we will concentrate on a specific family, which is widely concerned in literature, to find exact traveling wave solutions.

1.2 Literature Review

The derivation of (1.1.1) appears in [1]. In that paper Rosenau uses quasi-contionuous formalism which he introduced in his previous works to treat the dynamics of dense discrete systems. After introducing the method he gives three physical example which necessitates the use of higher order effects of discreteness: Two neighbors interaction, non-linear 3-D motion of a string and transmission line modelled by a long chain of L-C circuits. The approximation of the equations of motion of a 1-dimensional lattice to the continuum requires considering higher order effects. Eq. (1.1.1) is also derived in [2] for the propagation of longitudinal waves in an infinite elastic medium within the context of nonlinear non-local elasticity. In that paper the authors also investigate the well-posedness of the Cauchy problem. As a recent literature, Eq. (1.1.1) is seen in [3] where the authors study the local and global existence and blow-up of solutions to the initial and boundary value problem of the equation. In that literature, η_1 and η_2 are positive constants and $f(u)$ is considered to be an arbitrary nonlinearity. HBq equations of [2] and [3] are obtained from (1.1.1) when we replace $f(u) \to u + f(u)$. The Lie symmetry algebra of the Boussinesq equation

$$u_{tt} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0 \tag{1.2.1}$$

is the Lie algebra of the vector fields

$$D = x\partial_x + 2t\partial_t - 2u\partial_u, \quad P_1 = \partial_x, \quad P_0 = \partial_t, \tag{1.2.2}$$

which generate translations and dilations, see Refs. [4–7]. Classical and non-classical similarity reductions of the Boussinesq equation

$$u_{tt} + au_{xx} + b(u^2)_{xx} + cu_{xxxx} = 0 \tag{1.2.3}$$

are obtained in [5] and these nonclassical reductions are given a group-theoretical framework in the context of conditional symmetries in [4].

In connection with classification problem in Lie theory, [8] performs the symmetry classification of the generalized Boussinesq equation

$$u_{tt} = u_{xxxx} + (f(u))_{xx}. \tag{1.2.4}$$

In [9], the authors perform Lie symmetry analysis of the equation

$$u_{tt} - u_{xx} + u_{xxxx} + (f(u))_{xx} = 0. \tag{1.2.5}$$

Ref. [10] handles the double-dispersion equation

$$u_{tt} = u_{xx} + au_{xxtt} - bu_{xxxx} + du_{xxt} + (f(u))_{xx} \tag{1.2.6}$$

and exhibits the functional forms of $f(u)$ so that the equation has Lie symmetry algebras.

Ref. [11] studies the symmetry algebra and reductions of the equation

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + k\Delta u_t = \Delta f(u) \tag{1.2.7}$$

where $x \in \mathbb{R}^3$ and f is a power-type nonlinearity. [12] considers this equation for $n = 1$, in the form

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = (f(u))_{xx}, \tag{1.2.8}$$

and for $n = 2$, to derive conservation laws. Ref. [13] considers symmetry algebras of the equation

$$u_{tt} = cu_{xx} + bu_{xxxx} + au_{xxxxxx} + (f(u))_{xx} \tag{1.2.9}$$

and derives the conservation laws of this equation which admits a Hamiltonian form when written as a system.

Let us finally mention two references which consider the closest family of equations to the one considered. Ref. [14] considers

$$u_{tt} = u_{xx} + u_{xxtt} - u_{xxxxtt} - cu_{xxxx} + (f(u))_{xx} \tag{1.2.10}$$

in the case $c \neq 0$ and finds exact solutions to this equation in terms of trigonometric, hyperbolic and elliptic functions when $f(u)$ has some certain forms. Note that the case $c = 0$ is not considered in this article separately in the search of the Lie symmetry algebra, therefore they do not cover results in the thesis.

Classification of the family of equations (1.1.1) in the case $\eta_2 = 0$; explicitly, the family

$$u_{tt} = \delta u_{ttxx} + (f(u))_{xx} \tag{1.2.11}$$

according to symmetry algebras the equation admits is studied in [15]. Clearly,the main equation (1.1.1) is an extension of this family to sixth-order. According to the results of [15], the Lie symmetry algebra of an equation from the class (1.2.11) can be at most three-dimensional. However, it is shown in the thesis that for a specific form of $f(u)$, (1.1.1) has a four-dimensional symmetry algebra and this result is also valid when $\eta_2 = 0$, namely, for Eq. (1.2.11).

1.3 Main Results

As the original outcomes of this thesis, we have two main results.

The first one is the following Theorem.

Theorem 1.3.1 *The Lie symmetry algebra L of the higher order Boussinesq equation* (1.1.1) *can be 2-dimensional, 3-dimensional, or 4-dimensional.*

(i) The Abelian two-dimensional Lie algebra $2A_1$ is admitted as the invariance the algebra of Eq. (1.1.1) *for any $f(u)$, and is realized by the Lie algebra with basis* $\{X_1, X_2\} = \{\partial_t, \partial_x\}$.

(ii) *The three-dimensional Lie algebra* $A_2 \oplus A_1$ *(where* A_1 *is the one-dimensional Lie algebra and* A_2 *is the two-dimensional non-Abelian algebra) is admitted as the symmetry algebra of Eq.* (1.1.1) *if* $f(u)$ *respects one of the forms given in* (3.1.11), *or, equivalently,* (3.1.12). *The related generators of the Lie algebras for these cases are given in, respectively, for case A in* (3.2.2), *and in case B and case C.1 of the Table for the latter two. In all of these cases, the Lie algebra has the decomposition* $\{X_1, X_2\} \oplus X_3$.

(iii) *If* $f(u) = \alpha(\beta u + \delta)^{-3} + \gamma$, *or,equivalently, if* $f(u) = \alpha u^{-3}$, $\alpha = \mp 1$, *then L is 4-dimensional, which is denoted as case C.2 in the Table. The symmetry algebra has the structure*

$$L_{C.2} = \{X_1, X_2, X_3\} \oplus X_4 \simeq \mathrm{sl}(2, \mathbb{R}) \oplus \mathbb{R} \tag{1.3.1}$$

which contains the simple algebra $\mathrm{sl}(2, \mathbb{R})$ *as a subalgebra.*

(iv) *According to these results, maximal dimension of the Lie algebra of a higher-order Boussinesq equation belonging to the class* (1.1.1) *can be 4.*

In Chapter 3 the proof is given.

The second main result of the current thesis is the existence of trigonometric, hyperbolic and elliptic type solutions to HBq equation (1.1.1), many of which appear in the literature through this work the first time. In Chapter 4, we perform this analysis.

2. BASIC NOTIONS OF LIE SYMMETRY ANALYSIS

In Chapter 2, the fundamental concepts of the Lie groups and Lie algebras are given. All the information is collected from [16]. The proofs are not given.

2.1 Lie Groups of Transformations

Definition 2.1.1 *A transformation is a symmetry if it satisfies the following:*
(S1) The transformation preserves the structure,
(S2) The transformation is a diffeomorphism,
(S3) The transformation maps the object to itself. (For more information one can see the textbook [17])

Definition 2.1.2 *A group G is a set of elements with a law of composition φ between elements satisfying the following axioms:*
(i) Closure property: For any element a and b of G, $\varphi(a,b)$ is an element of G.
(ii) Associative property: For any elements a,b, and c of G,

$$\varphi(a,\varphi(b,c)) = \varphi(\varphi(a,b),c).$$

(iii) Identity element: There exists a unique identity element e of G such that for any element a of G, $\varphi(a,e) = \varphi(e,a) = a$.
(iv) Inverse element: For any element a of G there exists a unique inverse element a^{-1} in G such that $\varphi(a,a^{-1}) = \varphi(a^{-1},a)$.

Definition 2.1.3 *(Group of Transformations) Let $x = (x_1, x_2, ..., x_n)$ lie in region $D \subset$ R^{n}. The set of transformations*

$$\tilde{x} = X(x;\varepsilon) \tag{2.1.1}$$

defined for each x in D, depending on parameter ε lying in set $S \subset \mathrm{R}$, with $\varphi(\varepsilon,\delta)$ defining a law of composition of parameters ε and δ in S, forms a group of *transformations on D if:*

(i) For each parameter ε in S the transformations are one-to-one onto D, in particular $\tilde{x}$ lies in D.

(ii) S with the law of composition φ forms a group G.

(iii) $\tilde{x} = x$ when $\varepsilon = e$ (identity), i.e. $X(x;\varepsilon) = x$.

(iv) If $\tilde{x} = X(x;\varepsilon)$, $\tilde{\tilde{x}} = X(\tilde{x};\delta)$, then $\tilde{\tilde{x}} = X(x;\varphi(\varepsilon,\delta))$.

Definition 2.1.4 *(One-Parameter Lie Group of Transformation) A group of transformations defines a one-parameter Lie group of transformations if in addition to satisfying axioms (i)-(iv) of Definition 2.1.3:*

(v) ε is a continuous parameter, i.e. S is an interval in R. *Without loss of generality $\varepsilon = 0$ corresponds to the identity element e.*

(vi) X is infinitely differentiable with respect to x in D and an analytic function of ε in S.

(vii) $\varphi(\varepsilon,\delta)$ is an analytic function of ε and δ, $\varepsilon \in$ S, $\varepsilon \in$ S.

2.2 Infinitesimal Transformations

Definition 2.2.1 *(Infinitesimal Transformations) Consider a one-parameter (ε) Lie group of transformations*

$$\tilde{x} = X(x;\varepsilon) \tag{2.2.1}$$

with identity $\varepsilon = 0$ and law of composition φ. Expanding (2.2.1) *about $\varepsilon = 0$, we get (for some neighborhood of $\varepsilon = 0$)*

$$\begin{aligned} \tilde{x} &= x + \varepsilon\left(\frac{\partial X}{\partial \varepsilon}(x;\varepsilon)|_{\varepsilon=0}\right) + \frac{\varepsilon^2}{2}\left(\frac{\partial^2 X}{\partial \varepsilon^2}(x;\varepsilon)|_{\varepsilon=0}\right) + \ldots & \text{(2.2.2a)} \\ &= x + \varepsilon\left(\frac{\partial X}{\partial \varepsilon}(x;\varepsilon)|_{\varepsilon=0}\right) + O(\varepsilon^2). & \text{(2.2.2b)} \end{aligned}$$

Let

$$\xi(x) = \frac{\partial X}{\partial \varepsilon}(x;\varepsilon)|_{\varepsilon=0}. \tag{2.2.3}$$

The transformation $x + \varepsilon\xi(x)$ is called the infinitesimal transformation of the Lie group of transformations (2.2.1)*; the components of $\xi(x)$ are called the infinitesimals of* (2.2.1).

Theorem 2.2.1 (First Fundamental Theorem of Lie) *There exists a parametrization $\tau(\varepsilon)$ such that the Lie group of transformations* (2.2.1) *is equivalent to the solution of the initial value problem for the system of first order differential equations*

$$\frac{d\tilde{x}}{d\tau} = \xi(\tilde{x}), \tag{2.2.4}$$

with

$$\tilde{x} = x \quad when \quad \tau = 0. \tag{2.2.5}$$

In particular

$$\begin{aligned} \tau(\varepsilon) &= \int_0^{\varepsilon} \Gamma(s)ds, && (2.2.6a)\\ where \quad \Gamma(\varepsilon) &= \frac{\partial\varphi(a,b)}{\partial b}|_{(a,b)=(\varepsilon^{-1},\varepsilon)} && (2.2.6b)\\ and \quad \Gamma(0) &= 1. && (2.2.6c) \end{aligned}$$

(ε^{-1} denotes the inverse element to ε).

Definition 2.2.2 (Infinitesimal Generator) *The infinitesimal generator of the one-parameter Lie group of transformations* (2.2.1) *is the operator*

$$X = X(x) = \xi(x)\cdot\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i} \tag{2.2.7}$$

where ∇ is the gradient operator,

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}); \tag{2.2.8}$$

for any differentiable function $F(x) = F(x_1, x_2, \dots, x_n)$,

$$XF(x) = \xi(x)\cdot\nabla F(x) = \sum_{i=1}^{n} \xi_i(x)\frac{\partial F(x)}{\partial x_i}.$$

Theorem 2.2.2 *The one-parameter Lie group of transformations* (2.2.1) *is equivalent to*

$$\tilde{x} = e^{\varepsilon X}x = x + \varepsilon Xx + \frac{\varepsilon^2}{2}X^2x + \cdots = [1 + \varepsilon X + \frac{\varepsilon^2}{2}X^2 + \dots]x = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!}X^k x \tag{2.2.9}$$

where the operator $X = X(x)$ is defined by (2.2.7) *and the operator $X^k = XX^{k-1}$, $k = 1, 2, \dots$; in particular $X^kF(x)$ is the function obtained by applying the operator X to the function $X^{k-1}F(x)$, $X^k = XX^{k-1}$, $k = 1, 2, \dots$ with $X^0F(x) \equiv F(x)$.*

Corollary 2.2.1 *If $F(x)$ is infinitely differentiable, then for a Lie group of transformations* (2.2.1) *with infinitesimal generator $X = X(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}$,*

$$F(\tilde{x}) = F(e^{\varepsilon X} x) = e^{\varepsilon X} F(x). \tag{2.2.10}$$

Definition 2.2.3 (Invariant Function) *An infinitely differentiable function $F(x)$ is an invariant function of the Lie group of transformation* (2.2.1) *if*

$$F(\tilde{x}) \equiv F(x). \tag{2.2.11}$$

If $F(x)$ is an invariant function of (2.2.1), *then $F(x)$ is called an invariant of* (2.2.1) *and $F(x)$ is said to be invariant under* (2.2.1) .

Theorem 2.2.3 *$F(x)$ is invariant under* (2.2.1) *if and only if*

$$XF(x) \equiv 0. \tag{2.2.12}$$

Theorem 2.2.4 *For a Lie group of transformations* (2.2.1), *the identity*

$$F(\tilde{x}) \equiv F(x) + \varepsilon \tag{2.2.13}$$

holds if and only if $F(x)$ is such that

$$XF(x) \equiv 1 \tag{2.2.14}$$

(Notation for Change of Coordinates) Suppose we make a change of coordinates (one to one and continuously differentiable in a some appropriate domain)

$$y = Y(x) = (y_1(x), y_2(x), \ldots, y_n(x)). \tag{2.2.15}$$

The infinitesimal generator with respect to the new coordinates (2.2.15) is

$$Y = \sum_{i=1}^{n} \eta_i(y) \frac{\partial}{\partial y_i}. \tag{2.2.16}$$

The infinitesimal with respect to coordinate y is

$$\eta(y) = (\eta_1(y), \eta_2(y), \ldots, \eta_n(y)) = Yy. \tag{2.2.17}$$

Theorem 2.2.5 *With respect to new coordinates y given by* (2.2.15), *the Lie group transformations* (2.2.1) *is*

$$\tilde{y} = e^{\varepsilon Y} y \tag{2.2.18}$$

Definition 2.2.4 *(Canonical Coordinates) A change of coordinates* (2.2.15) *defines a set of canonical coordinates for the one-parameter Lie group transformations* (2.2.1) *if in terms of such coordinates the group* (2.2.1) *becomes*

$$\tilde{y}_i = y_i, \qquad i = 1,2,\dots,n-1, \tag{2.2.19a}$$

$$\tilde{y}_n = y_n + \varepsilon. \tag{2.2.19b}$$

Theorem 2.2.6 *(Existence of Canonical Coordinates) For any Lie group of transformations* (2.2.1) *there exists a set of canonical coordinates* $y = (y_1, y_2, \dots, y_n)$ *such that* (2.2.1) *is equivalent to* (2.2.19).

Theorem 2.2.7 (Infinitesimal Generator for Canonical Coordinates) *In terms of any set of canonical coordinates* $y = (y_1, y_2, \dots, y_n)$*, the infinitesimal generator of the one-parameter Lie group of transformations* (2.2.1) *is*

$$Y = \frac{\partial}{\partial y_n}. \tag{2.2.20}$$

Definition 2.2.5 (Invariant Surface) *A surface* $F(x) = 0$ *is an invariant surface for a one parameter Lie group of transformations* (2.2.1) *if and only if* $F(\tilde{x}) = 0$ *when* $F(x) = 0$.

Definition 2.2.6 (Invariant Curve) *A curve* $F(x,y) = 0$ *is an invariant curve for a one-parameter Lie group of transformations*

$$\tilde{x} = X(x,y;\varepsilon) = x + \varepsilon\xi(x,y) + O(\varepsilon^2), \tag{2.2.21a}$$

$$\tilde{y} = Y(x,y;\varepsilon) = y + \varepsilon\eta(x,y) + O(\varepsilon^2), \tag{2.2.21b}$$

with the infinitesimal generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{2.2.22}$$

if and only if $F(\tilde{x},\tilde{y}) = 0$ *when* $F(x,y) = 0$.

Theorem 2.2.8 (Invariance Conditions for Solved Forms) *(i) A surface written in a solved form* $F(x) = x_n - f(x_1, x_2, \dots, x_{n-1}) = 0$*, is an invariant surface for* (2.2.1) *if and only if*

$$XF(x) = 0 \quad when \quad F(x) = 0. \tag{2.2.23}$$

(ii) A curve written in a solved form $F(x,y) = y - f(x) = 0$*, is an invariant curve for* (2.2.21) *if and only if*

$$XF(x,y) = \eta(x,y) - \xi(x,y)f'(x) = 0 \quad when \quad F(x,y) = y - f(x) = 0. \tag{2.2.24}$$

Definition 2.2.7 (Invariant Family of Surfaces) *The family of surfaces*

$$\omega(x) = const = c$$

is an invariant family of surfaces for (2.2.1) *if and only if*

$$\omega(\tilde{x}) = const = \tilde{c} \quad when \quad \omega(x) = c$$

2.3 Extended Transformations (Prolongations)

(Notation for Prolongations) To study the invariance properties of kth order ordinary differential equation with independent variable x and dependent variable y one needs to find admitted one-parameter Lie groups of transformations of the form

$$\tilde{x} = X(x,y;\varepsilon), \tag{2.3.1a}$$

$$\tilde{y} = Y(x,y;\varepsilon) \tag{2.3.1b}$$

where $y = y(x)$. The equations (2.3.1) are naturally extended to $(x,y,y_1,\ldots,y_k)$-space, $k = 1,2,\ldots$, by the demanding that (2.3.1) preserve the contact conditions relating differentials $dx, dy, dy_1, dy_2, \ldots$:

$$dy = y_1 dx, \tag{2.3.2a}$$

$$\text{and} \quad dy_k = y_{k+1} dx, \quad k = 1,2,\ldots \tag{2.3.2b}$$

In particular under the group action of the group transformations (2.3.1) the transformed derivatives $\tilde{y}_k$, $k = 1,2,\ldots$ are defined successively by

$$d\tilde{y} = \tilde{y}_1 d\tilde{x}, \tag{2.3.3a}$$

$$d\tilde{y}_k = \tilde{y}_{k+1} d\tilde{x} \tag{2.3.3b}$$

where $\tilde{x}$ and $\tilde{y}$ defined by (2.3.1). After simplifying above equations one can obtain following equation:

$$\tilde{y}_1 = Y_1(x,y,y_1;\varepsilon) = \frac{\frac{\partial Y}{\partial x}(x;\varepsilon) + y_1 \frac{\partial Y}{\partial y}(x;\varepsilon)}{\frac{\partial X}{\partial x}(x;\varepsilon) + y_1 \frac{\partial X}{\partial y}(x;\varepsilon)} \tag{2.3.4}$$

.

Theorem 2.3.1 *The Lie group of transformations* (2.3.1) *acting on (x,y)-space (naturally) extends to the following one-parameter Lie group of transformations acting on* (x,y,y_1)*-space:*

$$\tilde{x} = X(x,y;\varepsilon), \tag{2.3.5a}$$
$$\tilde{y} = Y(x,y;\varepsilon). \tag{2.3.5b}$$
$$\tilde{y}_1 = Y_1(x,y,y_1;\varepsilon), \tag{2.3.5c}$$

where $Y_1(x,y,y_1;\varepsilon)$ *is given by* (2.3.4).

Theorem 2.3.2 *The Lie group of transformations* (2.3.1) *extends to its kth extension,* $k \geq 2$*, which is the following one-parameter Lie group of transformations acting on* $(x,y,y_1,\ldots,y_k)$*-space:*

$$\tilde{x} = X(x,y;\varepsilon), \tag{2.3.6a}$$
$$\tilde{y} = Y(x,y;\varepsilon), \tag{2.3.6b}$$
$$\tilde{y}_1 = Y_1(x,y,y_1;\varepsilon), \tag{2.3.6c}$$
$$\vdots \tag{2.3.6d}$$
$$\tilde{y}_k = Y_k(x,y,y_1,\ldots,y_k;\varepsilon) = \frac{\frac{\partial Y_{k-1}}{\partial x} + y_1\frac{\partial Y_{k-1}}{\partial y} + \cdots + y_k\frac{\partial Y_{k-1}}{\partial y_{k-1}}}{\frac{\partial X}{\partial x}(x;\varepsilon) + y_1\frac{\partial X}{\partial y}(x;\varepsilon)} \tag{2.3.6e}$$

where $Y_1 = Y_1(x,y,y_1;\varepsilon)$ *is defined by* (2.3.4)*, and* $Y_{k-1} = Y_{k-1}(x,y,y_1,\ldots,y_{k-1};\varepsilon)$.

Theorem 2.3.3 *(i)* $\eta^{(k)}$ *is linear in* y_k*,* $k = 2,3,\ldots$.
(ii) $\eta^{(k)}$ *is a polynomial in* $y_1,y_2,\ldots,y_k$ *whose coefficients are linear homogeneous in* $(\xi(x,y),\eta(x,y))$ *up to their kth order partial derivatives.*

Theorem 2.3.4

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \quad i = 1,2,\ldots,n; \tag{2.3.7a}$$
$$\eta^{(k)}_{i_1 i_2 \ldots i_k} = D_{i_k}\eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k}\xi_j)u_{i_1 i_2 \ldots i_{(k-1)}}, \quad i_l = 1,2,\ldots,n \tag{2.3.7b}$$

for $l = 1,2,\ldots,k$ *with* $k = 2,3,\ldots$.

2.4 Lie Algebras

For an r-parameter Lie group of transformations,

$$\tilde{x} = X(x;\varepsilon), \tag{2.4.1}$$

let $x = (x_1, x_2, \cdots, x_n)$ and let the parameters be denoted by $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r)$. Let the law of composition of parameters be denoted by

$$\varphi(\varepsilon,\delta) = (\varphi_1(\varepsilon,\delta), \varphi_2(\varepsilon,\delta), \cdots, \varphi_r(\varepsilon,\delta))$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_r)$; $\varphi(\varepsilon,\delta)$ satisfies group axioms with $\varepsilon = 0$ corresponding to the identity $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_r = 0$; $\varphi(\varepsilon,\delta)$ is assumed to be analytic in its domain of definition.

Definition 2.4.1 *The infinitesimal generator X_α, corresponding to the parameter ε_α of r-parameter Lie group of transformations* (2.4.1) *is*

$$X_\alpha = \sum_{j=1}^{n} \xi_{\alpha j}(x)\frac{\partial}{\partial x_j}, \quad \alpha = 1,2,\ldots,r. \tag{2.4.2}$$

The exponentiation of any infinitesimal generator is a one-parameter Lie group of transformations which is a subgroup of the r-parameter Lie group of transformations.

Definition 2.4.2 *Consider an r-parameter Lie group of transformation* (2.4.1) *with infinitesimal generators X_α, $\alpha = 1,2,\ldots,r$, defined by* (2.4.2)*. The commutator of X_α and X_β is another first order operator*

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha = \sum_{j=1}^{n} \eta_j(x)\frac{\partial}{\partial x_j} \tag{2.4.3}$$

where

$$\eta_j(x) = \sum_{i=1}^{n} [\xi_{\alpha i}(x)\frac{\partial \xi_{\beta j}(x)}{\partial x_i} - \xi_{\beta i}(x)\frac{\partial \xi_{\alpha j}(x)}{\partial x_i}].$$

It follows that

$$[X_\alpha, X_\beta] = -[X_\beta, X_\alpha] \tag{2.4.4}$$

Any three infinitesimal generator $X_\alpha, X_\beta, X_\gamma$, satisfy Jacobi's identity:

$$[X_\alpha, [X_\beta, X_\gamma]] + [X_\beta, [X_\gamma, X_\alpha]] + [X_\gamma, [X_\alpha, X_\beta]] = 0. \tag{2.4.5}$$

Theorem 2.4.1 (Second Fundamental Theorem of Lie) *The commutator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator, in particular*

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta} X_\gamma, \tag{2.4.6}$$

where the coefficients $C^\gamma_{\alpha\beta}$ are called structure constants.

Definition 2.4.3 *A Lie algebra $\mathscr{L}$ is a vector space over some field $\mathscr{F}$ with an additional law of combination of elements in $\mathscr{L}$ (the commutator) satisfying the properties* (2.4.4), (2.4.5) *and closedness with respect to commutation.*

In particular the infinitesimal generators $X_\alpha, \quad \alpha = 1, 2, \cdots, r$, of an r-parameter Lie group of transformations (2.4.1) form an r-dimensional Lie algebra $\mathscr{L}^r$ over the field $\mathbb{R}$.

Definition 2.4.4 *A subalgebra $\mathscr{I} \subset \mathscr{L}$ is called an ideal or normal subalgebra of $\mathscr{L}$ if for any $X \in \mathscr{I}, Y \in \mathscr{L}$, we have $[X,Y] \in \mathscr{L}$.*

Definition 2.4.5 *$\mathscr{L}^q$ is a q-dimensional solvable Lie algebra if there exists a chain of subalgebras*

$$\mathscr{L}^{(1)} \subset \mathscr{L}^{(2)} \subset \cdots \subset \mathscr{L}^{(q-1)} \subset \mathscr{L}^{(q)} = \mathscr{L}^q \tag{2.4.7}$$

such that $\mathscr{L}^{(k)}$ is a k-dimensional Lie algebra and $\mathscr{L}^{(k-1)}$ is an ideal of $\mathscr{L}^{(k)}, \quad k = 1, 2, \ldots, q$.

Definition 2.4.6 *$\mathscr{L}$ is called an Abelian Lie algebra if for any $X_\alpha, X_\beta \in \mathscr{L}$, we have $[X_\alpha, X_\beta] = 0$.*

Theorem 2.4.2 *Every Abelian Lie algebra is a solvable Lie algebra.*

Theorem 2.4.3 *Every two dimensional Lie algebra is solvable.*

2.5 Partial Differential Equations

Definition 2.5.1 *The one-parameter Lie group of transformations*

$$\tilde{x} = X(x,u;\varepsilon), \quad (2.5.1a)$$

$$\tilde{u} = U(x,u;\varepsilon), \quad (2.5.1b)$$

leaves a PDE invariant if and only if its kth extension leaves the surface equation of the PDE invariant.

Theorem 2.5.1 *(Infinitesimal Criterion for Invariance of a PDE). Let*

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} \quad (2.5.2)$$

be the infinitesimal generator of (2.5.1). *Let*

$$X^{(k)} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x,u,u_1)\frac{\partial}{\partial u_i} + \cdots + \eta_{i_1 i_2 \ldots i_k}^{(k)}(x,u,u_1,u_2,\ldots,u_k)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}} \quad (2.5.3)$$

be the kth extended infinitesimal generator of (2.5.2) *where* $\eta_i^{(1)}$ *is given by Theorem 2.3.5,* $i_j = 1,2,\ldots,n$ *for* $j = 1,2,\ldots,k$, *in terms of* $(\xi(x,u),\eta(x,u))$.
Then (2.5.1) *is admitted by a PDE if and only if*

$$X^{(k)}F(x,u,u_1,\ldots,u_k) = 0$$

when

$$F(x,u,u_1,\ldots,u_k) = 0.$$

Remark 2.5.1 *The content of this Chapter 2 serves as a preliminary and brief discussion of the theory of Lie group analysis of differential equations related to the analysis we are going to perform in Chapter 3. Each and every information available in this Chapter 2 was collected directly or by paraphrasing from the source [16] therefore is not to be understood as an original content submitted by the author of this thesis in fulfillment of this degree.*

3. THE LIE ALGEBRA AND REDUCTIONS OF THE HIGHER ORDER BOUSSINESQ EQUATION

3.1 The Symmetry Algebra of the Equation

In what follows it is assumed that $\eta_1 \neq 0$, $\eta_2 \neq 0$, $f_{uu} \neq 0$. The infinitesimal generator is of the form

$$V = \phi_1(t,x,u)\partial_t + \phi_2(t,x,u)\partial_x + \phi_3(t,x,u)\partial_u. \tag{3.1.1}$$

We find

$$V = \tau(t)\partial_t + \xi(x)\partial_x + \phi(t,x,u)\partial_u, \tag{3.1.2}$$

where

$$\phi = Q(x,t) + \Big(\frac{1}{2}\tau_t + \frac{3}{2}\xi_x + \phi_0\Big)u, \tag{3.1.3a}$$

$$\tau_{ttt} = 0, \tag{3.1.3b}$$

$$2\eta_1\xi_x - 5\eta_2\xi_{xxx} = 0, \tag{3.1.3c}$$

$$8\xi_x - 3\eta_1\xi_{xxx} + 3\eta_2\xi_{xxxxx} = 0, \tag{3.1.3d}$$

$$2(\xi_x + \tau_t)f_u + \phi f_{uu} = 0, \tag{3.1.3e}$$

$$\xi_{xx}f_u + \phi_x f_{uu} = 0, \tag{3.1.3f}$$

$$Q_{tt} - \eta_1 Q_{xxtt} + \eta_2 Q_{xxxxtt} - \Big(Q_{xx} + \frac{3}{2}u\xi_{xxx}\Big)f_u = 0, \tag{3.1.3g}$$

where ϕ_0 is an arbitrary constant. If we differentiate (3.1.3e) with respect to x and subtract it from (3.1.3f), we get $\xi_{xx} = 0$ and $\phi_x = 0$, and hence $Q_x = 0$. Eq. (3.1.3c) gives $\xi_x = 0$, so $\xi(x) = \xi_0$, a constant. After these, the infinitesimal generator is of the form

$$V = \tau(t)\partial_t + \xi_0\partial_x + \phi(t,u)\partial_u \tag{3.1.4}$$

with

$$\phi = Q(t) + \Big(\frac{1}{2}\tau_t + \phi_0\Big)u, \tag{3.1.5a}$$

$$\tau_{ttt} = 0, \quad Q_{tt} = 0, \tag{3.1.5b}$$

$$2\tau_t f_u + \phi f_{uu} = 0. \tag{3.1.5c}$$

It is seen that when f is arbitrary, we have the two symmetries

$$X_1 = \partial_t, \quad X_2 = \partial_x \tag{3.1.6}$$

and the algebra is the Abelian two-dimensional Lie algebra. One can proceed and solve the system above for different cases. It has been taken another approach and played a little bit on (3.1.5c) to get an equation involving only f.Differentiating (3.1.5c) with respect to u,we obtain

$$2\tau_t f_{uu} + \phi_u f_{uu} + \phi f_{uuu} = 0. \tag{3.1.7}$$

Using (3.1.5c) and (3.1.7) we can eliminate the term with τ_t and get

$$\phi f_u f_{uuu} + \phi_u f_u f_{uu} - \phi f_{uu}^2 = 0. \tag{3.1.8}$$

Again if we differentiate (3.1.8) with respect to u and we get

$$2\phi_u f_u f_{uuu} + \phi f_u f_{uuuu} - \phi f_{uu} f_{uuu} = 0. \tag{3.1.9}$$

If we eliminate ϕ between (3.1.8) and (3.1.9), we obtain

$$f_u f_{uu} f_{uuuu} + f_{uu}^2 f_{uuu} - 2 f_u f_{uuu}^2 = 0, \tag{3.1.10}$$

which is exactly the same equation for $f(u)$ that was obtained in [15]. Compatible with their findings, Eq. (3.1.10) is solved by the following different forms of f:

$$\text{(a)} \quad f(u) = \alpha e^{\beta u} + \gamma, \tag{3.1.11a}$$

$$\text{(b)} \quad f(u) = \alpha \ln(\beta u + \delta) + \gamma, \tag{3.1.11b}$$

$$\text{(c)} \quad f(u) = \alpha (\beta u + \delta)^n + \gamma, \quad n \neq 0, 1. \tag{3.1.11c}$$

Here $\alpha, \beta, \gamma, \delta, n$ are arbitrary constants where $\alpha\beta \neq 0$. Actually, the constant γ has no significance when we consider the HBq equation (1.1.1). By a transformation $u = \delta_1 \bar{u} + \delta_2$, $\bar{x} = \mu x$, $\bar{t} = \lambda t$ and relabeling the constants, Eq. (1.1.1) with the above forms of $f(u)$ can be converted to an equation with

$$\text{(A)} \quad f(u) = \alpha e^u, \qquad \alpha = \mp 1, \tag{3.1.12a}$$

$$\text{(B)} \quad f(u) = \alpha \ln(u), \qquad \alpha = \mp 1, \tag{3.1.12b}$$

$$\text{(C)} \quad f(u) = \alpha u^n, \qquad \alpha = \mp 1, \quad \mathbb{R} \ni n \neq 0, 1. \tag{3.1.12c}$$

3.2 Canonical Classes, the Optimal System and the Reduced Equations

Now we concentrate on these simplified forms of the nonlinearity $f(u)$. Note that, in the remaining part of the thesis, for all of the cases (A), (B) and (C), we did not restrict the constant α to ∓ 1 in our calculations, therefore one can use the following results for any nonzero constant α.

Case A: $\mathbf{f(u) = \alpha e^{u}}, \quad \alpha = \mp 1$.

Equation (1.1.1) is of the form

$$u_{tt} = \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + \alpha (e^u)_{xx}. \tag{3.2.1}$$

The Lie algebra L_A of this equation is three dimensional, $L_A = \{X_1, X_2, X_3\}$, generated by the vector fields

$$X_1 = \partial_t, \quad X_2 = t\partial_t - 2\partial_u, \quad X_3 = \partial_x. \tag{3.2.2}$$

The nonzero commutation relation is

$$[X_1, X_2] = X_1, \tag{3.2.3}$$

therefore the Lie algebra has the structure $L_A = A_2 \oplus A_1 = \{X_1, X_2\} \oplus X_3$. The optimal system of one-dimensional subalgebras of $A_2 \oplus A_1$ is given in [18]. Therefore, the optimal system of one-dimensional subalgebras of L_A is

$$\{X_1\}, \quad \{X_1 + \varepsilon X_3\}, \quad \{-X_2 \cos\theta + X_3 \sin\theta\} \tag{3.2.4}$$

with $\varepsilon = \mp 1$ and $0 \leq \theta < \pi$. The reductions through the last subalgebra should be analyzed carefully. *(i)* When $\theta = \frac{\pi}{2}$, we have the generator $X_3 = \partial_x$. Solutions invariant under the group of transformations generated by this subalgebra are time-dependent ones, $u = u(t)$. Not only for (3.2.1), but for any form of f in (1.1.1), these solutions are found from $u_{tt} = 0$ hence $u = at + b$. This subalgebra will not be considered in any of the subcases. *(ii)* When $\theta \in [0, \pi) - \{\frac{\pi}{2}\}$, we have $\{-X_2 \cos\theta + X_3 \sin\theta\} \simeq \{X_2 - (\tan\theta) X_3\}$, for which it is simply written $\{X_2 + cX_3\}$, $c \in \mathbb{R}$. It is observed that when $c = 0$, the reduction obtained is 2 less in the order than the order of the reduced equation that is obtained through $X_2 + cX_3$; therefore, for this subalgebra, the cases $c = 0$ and $c \neq 0$ are considered separately.

(i) The Subalgebra $X_1 = \partial_t$. The solutions will have the form $u = u(x)$ and from (3.2.1) we get $u = \ln(ax + b)$, where a, b are arbitrary constants.

(ii) The Subalgebra $X_1 + \varepsilon X_3 = \partial_t + \varepsilon \partial_x$, $\varepsilon = \mp 1$. The invariant solution will have the form $u(x,t) = F(\xi) = F(x - \varepsilon t)$. This generator produces traveling wave solutions, and will appear in other forms of the nonlinearity $f(u)$. Instead of working on (3.2.1), let us do the reduction for (1.1.1), which will be useful for other cases of $f(u)$. (Furthermore, see that since ∂_t and ∂_x are symmetries of (1.1.1) for any form of $f(u)$, so is the generator $\partial_t + \varepsilon \partial_x$.) Substituting $u = F(\xi)$, $\xi = x - \varepsilon t$ in (1.1.1), it reduces to

$$F'' = \eta_1 F^{(4)} - \eta_2 F^{(6)} + [f(F)]'' \tag{3.2.5}$$

which is integrated to

$$\eta_2 F^{(4)} - \eta_1 F'' + F - f(F) = K_1 \xi + K_0. \tag{3.2.6}$$

Here K_0, K_1 are arbitrary constants and the derivatives are with respect to the variable ξ. Therefore, for $f(u) = \alpha e^u$, the reduced equation is

$$\eta_2 F^{(4)} - \eta_1 F'' + F - \alpha e^F = K_1 \xi + K_0. \tag{3.2.7}$$

(iii) The Subalgebra $X_2 = t\partial_t - 2\partial_u$. The invariant solution is of the form $u = -2\ln t + F(x)$, of which substitution into (3.2.1) gives $F(x) = \ln(\frac{1}{\alpha}x^2 + ax + b)$ and hence

$$u(x,t) = \ln\Big[\frac{1}{t^2}\big(\frac{1}{\alpha}x^2 + ax + b\big)\Big]. \tag{3.2.8}$$

(iv) The Subalgebra $X_2 + cX_3 = t\partial_t + c\partial_x - 2\partial_u$. The group-invariant solution will have the form $u = -2\ln t + F(\xi)$, $\xi = x - c\ln t$. From (3.2.1) we get, after a further integration,

$$\eta_2 c^2 F^{(5)} + c\eta_2 F^{(4)} - \eta_1 c^2 F^{(3)} - \eta_1 c F'' + c^2 F' - \alpha (e^F)' + cF + 2\xi = K \tag{3.2.9}$$

with K being the integration constant.

Case B: $\mathbf{f(u)} = \alpha \ln \mathbf{u}$, $\alpha = \mp 1$.

It is summarized the results in Table (3.1). For this case of $f(u)$, the Lie algebra L_B of the equation is again three-dimensional, and the basis of the algebra is presented in Table 1. Let us note that the nonzero commutation relation for this algebra is exactly the same as (3.2.3); therefore, the same Lie algebra is realized as the Case A by the vector fields that generate the group of transformations of the related equation.

Case C: $\mathbf{f(u)} = \alpha \mathbf{u^n}$, $\alpha = \mp 1$, $\mathbb{R} \ni n \neq 0, 1$. This case has two different branches.

C.1: $n \neq -3$. In that case, the Lie symmetry algebra $L_{C.1}$ is 3-dimensional, with the generators given in Table 1. The canonical form of $f(u)$ is with $\alpha = \mp 1$, but for an arbitrary α we find the symmetries and the reduced equation. The structure of the Lie algebra is the same with L_A and L_B. The nonzero commutation relation is as in (3.2.3).

C.2: $n = -3$, $f(u) = \alpha u^{-3}$. The symmetry generators of $L_{C.1}$ are also admitted in this case. Besides, there arises a new symmetry generator and the equation admits a 4-dimensional Lie algebra $L_{C.2} = \{X_1, X_2, X_3, X_4\}$, and the basis of the Lie algebra is presented in Table 1. The nonzero commutation relations are

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \tag{3.2.10}$$

In [18] it is seen this algebra as $A_{3.8} \oplus A_1$ and the optimal system of one-dimensional subalgebras is

$$\{X_1\}, \quad \{X_4\}, \quad \{X_2 + cX_4\}, \quad \{-X_1 + X_3 + dX_4\}, \quad \{X_1 + \varepsilon X_4\}, \tag{3.2.11}$$

where $c \geq 0$ and $d \in \mathbb{R}$. The ODEs that are satisfied by the group-invariant solutions of (1.1.1) under the transformations generated by these one-dimensional subalgebras are presented in Table (3.1).

Table 3.1 : Table of Case B and Case C

Subalgebra	$u_{tt}=\eta_1 u_{xxtt}-\eta_2 u_{xxxxtt}+(f(u))_{xx}$	Similarity variable
Case B The equation $L_B=\{X_1,X_2,X_3\}$ Reduction by X_1	$f(u)=\alpha\ln u,\quad \alpha=\mp 1$ $u_{tt}=\eta_1 u_{xxtt}-\eta_2 u_{xxxxtt}+\alpha(\ln u)_{xx}$ $X_1=\partial_t,\quad X_2=t\partial_t+2u\partial_u,\quad X_3=\partial_x$ $u(x)=ae^{bx}$	
$X_1+\varepsilon X_3$, $\varepsilon=\mp 1$	$\eta_2 F^{(4)}-\eta_1 F''+F-\alpha\ln F=K_1\xi+K_0$	$u=F(\xi)=F(x-\varepsilon t)$
X_2+cX_3, $c\in\mathbb{R}$	$\eta_2 c^2 F^{(6)}-3\eta_2 cF^{(5)}+(2\eta_2-\eta_1 c^2)F^{(4)}+3\eta_1 cF^{(3)}$ $+(c^2-2\eta_1)F''-\alpha(\ln F)''-3cF'+2F=0$	$u=t^2F(\xi)$ $\xi=x-c\ln t$
X_2	$\eta_2 F^{(4)}-\eta_1 F''-\frac{\alpha}{2}(\ln F)''+F=0$	$u=t^2F(x)$
Case C.1 The equation $L_{C.1}=\{X_1,X_2,X_3\}$ Reduction by X_1	$f(u)=\alpha u^n,\quad \alpha=\mp 1,\quad n\neq 0,1,\quad n\in\mathbb{R}$ $u_{tt}=\eta_1 u_{xxtt}-\eta_2 u_{xxxxtt}+\alpha(u^n)_{xx}$ $X_1=\partial_t,\quad X_2=t\partial_t+\dfrac{2}{1-n}u\partial_u,\quad X_3=\partial_x$ $u(x)=(ax+b)^{1/n}$	
$X_1+\varepsilon X_3$, $\varepsilon=\mp 1$	$\eta_2 F^{(4)}-\eta_1 F''+F-\alpha F^n=K_1\xi+K_0$	$u=F(\xi)=F(x-\varepsilon t)$
X_2+cX_3, $c\in\mathbb{R}$	$\eta_2 c^2(n-1)^2F^{(6)}+\eta_2 c(n-1)(n+3)F^{(5)}$ $+\left[2\eta_2(n+1)-\eta_1 c^2(n-1)^2\right]F^{(4)}$ $-\eta_1 c(n-1)(n+3)F^{(3)}+2(n+1)F$ $+\left[c^2(n-1)^2-2\eta_1(n+1)\right]F''$ $-\alpha(n-1)^2(F^n)''+c(n-1)(n+3)F'=0$	$u=t^{2/(1-n)}F(\xi)$ $\xi=x-c\ln t$
X_2	$2(n+1)\left(\eta_2 F^{(4)}-\eta_1 F''+F\right)-\alpha(n-1)^2(F^n)''=0$	$u=t^{2/(1-n)}F(x)$
Case C.2 The equation $L_{C.2}$ $=\{X_1,X_2,X_3,X_4\}$ Reduction by X_1	$f(u)=\alpha u^{-3},\quad \alpha=\mp 1$ $u_{tt}=\eta_1 u_{xxtt}-\eta_2 u_{xxxxtt}+\alpha(u^{-3})_{xx}$ $X_1=\partial_t,\quad X_2=t\partial_t+\frac{1}{2}u\partial_u,$ $X_3=t^2\partial_t+tu\partial_u,\quad X_4=\partial_x$ $u(x)=(ax+b)^{-1/3}$	
X_4	$u(t)=at+b$	
X_2+cX_4 $(c\geq 0)$	$4\eta_2 c^2F^{(6)}-(\eta_2+4\eta_1 c^2)F^{(4)}$ $+(\eta_1+4c^2)F''-4\alpha(F^{-3})''-F=0$	$u=t^{1/2}F(\xi)$ $\xi=x-c\ln t$
X_2	$\eta_2 F^{(4)}-\eta_1 F''+4\alpha(F^{-3})''+F=0$	$u=t^{1/2}F(x)$
$-X_1+X_3+dX_4$ $(d\in\mathbb{R})$	$\eta_2 d^2F^{(6)}-(\eta_2+\eta_1 d^2)F^{(4)}+(\eta_1+d^2)F''$ $-\alpha(F^{-3})''-F=0$	$u=\sqrt{\lvert t^2-1\rvert}F(\xi)$ $\xi=x+d\tanh^{-1}t$
$X_1+\varepsilon X_4$, $\varepsilon=\mp 1$	$\eta_2 F^{(4)}-\eta_1 F''+F-\alpha F^{-3}=K_1\xi+K_0$	$u=F(\xi)=F(x-\varepsilon t)$

The main result of this thesis is presented in the following Theorem.

Theorem 3.2.1 *The Lie symmetry algebra L of the higher order Boussinesq equation* (1.1.1) *can be 2-dimensional, 3-dimensional, or 4-dimensional.*

(i) The Abelian two-dimensional Lie algebra $2A_1$ *is admitted as the invariance the algebra of Eq.* (1.1.1) *for any* $f(u)$*, and is realized by the Lie algebra with basis* $\{X_1, X_2\} = \{\partial_t, \partial_x\}$.

(ii) The three-dimensional Lie algebra $A_2 \oplus A_1$ *(where* A_1 *is the one-dimensional Lie algebra and* A_2 *is the two-dimensional non-Abelian algebra) is admitted as the symmetry algebra of Eq.* (1.1.1) *if* $f(u)$ *respects one of the forms given in* (3.1.11), *or, equivalently,* (3.1.12). *The related generators of the Lie algebras for these cases are given in, respectively, for case A in* (3.2.2)*, and in case B and case C.1 of Table 1 for the latter two. In all of these cases, the Lie algebra has the decomposition* $\{X_1, X_2\} \oplus X_3$.

(iii) If $f(u) = \alpha(\beta u + \delta)^{-3} + \gamma$*, or,equivalently, if* $f(u) = \alpha u^{-3}$*,* $\alpha = \mp 1$*, then L is 4-dimensional, which is denoted as case C.2 in Table 1. The symmetry algebra has the structure*

$$L_{C.2} = \{X_1, X_2, X_3\} \oplus X_4 \simeq \mathrm{sl}(2, \mathbb{R}) \oplus \mathbb{R} \tag{3.2.12}$$

which contains the simple algebra $\mathrm{sl}(2, \mathbb{R})$ *as a subalgebra.*

(iv) According to these results, maximal dimension of the Lie algebra of a higher-order Boussinesq equation belonging to the class (1.1.1) *can be 4.*

Remark 3.1 Let us have a more close look to the case C.2, i.e., when $f(u) = \alpha u^{-3}$. The Lie algebra with the basis

$$L_{C.2} = \{X_1 = \partial_t,\, X_2 = t\partial_t + \frac{1}{2}u\partial_u,\, X_3 = t^2\partial_t + tu\partial_u,\, X_4 = \partial_x\} \tag{3.2.13}$$

is the symmetry algebra of the equation

$$u_{tt} = \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + \alpha(u^{-3})_{xx} \tag{3.2.14}$$

regardless of the values of η_1, η_2 and α. Therefore, when $\eta_2 = 0$, the symmetry algebra of the equation

$$u_{tt} = \eta_1 u_{xxtt} + \alpha(u^{-3})_{xx} \tag{3.2.15}$$

is also 4-dimensional. Eq. (3.2.15) falls into the class (1.2.11), the generalized modified Boussinesq equation, analyzed in [15]. In their classification of the symmetry algebras of Eq. (1.2.11), they arrive at the same forms of $f(u)$ given in (3.1.11), and, according to their results, for these forms of $f(u)$ the symmetry algebras are at most three-dimensional. As far as we can see, this work does not consider the case $n = -3$, $f(u) = \alpha u^{-3}$ separately and seems to miss the fourth symmetry generator $X_3 = t^2\partial_t + tu\partial_u$ appearing.

Therefore, it should be stated that, the Theorem 3.2.1 is also valid when $\eta_2 = 0$, hence the maximal dimension of the symmetry algebra of the generalized modified Boussinesq equation (1.2.11), studied in [15], is equal to 4. The simple algebra $L_{C.2}$ with the decomposition (3.2.12) is also admitted as an invariance algebra in the case $f(u) = \alpha u^{-3}$.

4. SOME EXACT SOLUTIONS

In this Chapter, some exact solutions to the Equation (1.1.1) is presented.

4.1 Employing the Traveling Wave Ansatz

We consider the nonlinearity $f(u) = \alpha u^2 + u$, which gives rise to the equation

$$u_{tt} = u_{xx} + \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + \alpha(u^2)_{xx} \tag{4.1.1}$$

where $\alpha \neq 0$ is any constant. The reason for the inclusion of the term u_{xx} is obvious, as one can see from the literature review. For our analysis above, we had considered the u_{xx} term to covered by the nonlinearity $f(u)$, just on a purpose of bookkeeping. The quadratic nonlinearity u^2 can be interpreted like that one considers the stress-strain function of the physical model to be having a quadratic nonlinearity; see [2].

We aim at finding traveling wave solutions to (4.1.1), therefore we assume $u = F(\xi)$ with $\xi = kx - ct$ (which amounts to finding the group-invariant solutions under the action of the transformation produced by the generator $c\partial_x + k$dt). Putting this ansatz in (4.1.1) and integrating thrice, we obtain

$$\eta_2 k^4 c^2 \left[F'''F' - \frac{1}{2}(F'')^2 \right] - \frac{\eta_1 k^2 c^2}{2}(F')^2 - \frac{\alpha k^2}{3} F^3 + \frac{c^2 - k^2}{2} F^2 = K_0. \tag{4.1.2}$$

We chose the coefficients of the first two integrations as zero and kept only the last one, K_0. Since this equation does not contain the independent variable ξ, it can be integrated once by setting $F' = W(F)$ and treating W as the dependent variable and F as independent. However, the resulting equation is so complicated that we could not proceed with it further.

At this point, let us briefly outline the results of [19] in their Section 4, in which they consider a $2+1$-dimensional Boussinesq type equation

$$U_{tt} - U_{xx} - U_{yy} - \alpha(U^2)_{xx} - \alpha U_{xxxx} - \alpha \varepsilon^2 U_{xxxxxx} = 0. \tag{4.1.3}$$

In order to find traveling wave solutions of this equation, they propose the following ansatz:

$$U = a_0 + a_1\varphi(\zeta) + a_2\varphi^2(\zeta) + a_3\varphi^3(\zeta) + a_4\varphi^4(\zeta), \quad (4.1.4a)$$

$$\zeta(x,y,t) = x + y - \kappa t, \quad (4.1.4b)$$

$$\left(\frac{d\varphi}{d\zeta}\right)^2 = c_0 + c_1\varphi(\zeta) + c_2\varphi^2(\zeta) + c_3\varphi^3(\zeta) + c_4\varphi^4(\zeta). \quad (4.1.4c)$$

Notice that, if successful, this ansatz will produce trigonometric, hyperbolic or elliptic type solutions due to the Eq. (4.1.4c) that $\varphi(\zeta)$ satisfies. It is easy to see that, under the traveling wave ansatz, Eq. (4.1.1) with $u = F(kx - ct)$ and Eq. (4.1.3) with $U = U(x + y - \kappa t)$ reduce to ordinary differential equations which are the same up to coefficients. Therefore we adapt the methodology in [19] find the exact solutions to (4.1.1). Let us stress that we obtained some more solutions which were not mentioned there. Therefore, for (4.1.2) we propose

$$F(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi) + a_3\varphi^3(\xi) + a_4\varphi^4(\xi), \quad (4.1.5a)$$

$$\left(\frac{d\varphi}{d\xi}\right)^2 = c_0 + c_1\varphi(\xi) + c_2\varphi^2(\xi) + c_3\varphi^3(\xi) + c_4\varphi^4(\xi) = P(\varphi(\xi)) \quad (4.1.5b)$$

where $\xi(x,t) = kx - ct$. Upon this substitution, in the resulting expression we express all derivatives of $\varphi(\xi)$ in terms of φ using (4.1.5b). Afterwards, we look for the possibility that coefficients of φ^j, $j = 0, 1, 2, ...$ vanish. Below are the several cases we examined.

4.2 Hyperbolic and Trigonometric Solutions

We assume $c_0 = c_1 = c_3 = 0$ and $a_1 = a_2 = a_3 = 0$. We find two main branches for the remaining constants a_0, a_4, c_2, c_4, k and c.

The first set of parameters is

$$a_0 = 0, \quad (4.2.1a)$$

$$a_4 = \frac{840c^4c_4^2(169\eta_2 - 36\eta_1^2)}{169\alpha}, \quad (4.2.1b)$$

$$c_2 = \frac{13\eta_1}{4c^2(169\eta_2 - 36\eta_1^2)}, \quad (4.2.1c)$$

$$k^2 = c^2\left(1 - \frac{36\eta_1^2}{169\eta_2}\right) \quad (4.2.1d)$$

and the second set of possible parameters is

$$\begin{aligned} a_0 &= -\frac{36\eta_1^2}{\alpha(169\eta_2+36\eta_1^2)}, && (4.2.2a)\\ a_4 &= \frac{840c^4c_4^2(169\eta_2+36\eta_1^2)}{169\alpha}, && (4.2.2b)\\ c_2 &= \frac{13\eta_1}{4c^2(169\eta_2+36\eta_1^2)}, && (4.2.2c)\\ k^2 &= c^2\Big(1+\frac{36\eta_1^2}{169\eta_2}\Big). && (4.2.2d) \end{aligned}$$

In both cases, c and c_4 are arbitrary. Equation (4.1.5b) reduces to $\frac{d\varphi}{|\varphi|\sqrt{c_2+c_4\varphi^2}} = d\xi$, and it is integrated in three different ways depending on the signs of c_2 and c_4. Observe that $c_2 = \eta_1/(52k^2\eta_2)$. Although the physical derivation of (1.1.1) gives $\eta_1, \eta_2 > 0$ and the case $c_2 < 0$ seems irrelevant, we include this case also, for completeness.

Case I.a In case $c_2 > 0$, $c_4 > 0$, we obtain

$$\varphi(\xi) = \left(\frac{c_2}{c_4}\right)^{1/2} \operatorname{cosech}\Big(\varepsilon\sqrt{c_2}(\xi-\xi_0)\Big) \tag{4.2.3}$$

and the solution to (4.1.1) is

$$u(x,t) = a_0 + \frac{a_4c_2^2}{c_4^2}\operatorname{cosech}^4\Big(\varepsilon\sqrt{c_2}(kx-ct-\xi_0)\Big) \tag{4.2.4}$$

with $\varepsilon = \mp 1$. The set of four constants a_0, a_4, c_2, c_4 can be chosen as in (4.2.1) or (4.2.2).

Case I.b If $c_2 > 0$, $c_4 < 0$, we obtain

$$\varphi(\xi) = \left(\frac{c_2}{-c_4}\right)^{1/2} \operatorname{sech}\Big(\varepsilon\sqrt{c_2}(\xi-\xi_0)\Big) \tag{4.2.5}$$

and hence

$$u(x,t) = a_0 + \frac{a_4c_2^2}{c_4^2}\operatorname{sech}^4\Big(\varepsilon\sqrt{c_2}(kx-ct-\xi_0)\Big). \tag{4.2.6}$$

with $\varepsilon = \mp 1$. When $a_0 = 0$, this result is in the same form with the exact solution presented in [3].

Case I.c Finally, when $c_2 < 0$, $c_4 > 0$ we find

$$\varphi(\xi) = \left(\frac{-c_2}{c_4}\right)^{1/2} \sec\Big(\varepsilon\sqrt{-c_2}(\xi-\xi_0)\Big). \tag{4.2.7}$$

(4.2.7) also appears in [19]. The solution becomes

$$u(x,t) = a_0 + \frac{a_4 c_2^2}{c_4^2} \sec^4\left(\varepsilon\sqrt{-c_2}(kx - ct - \xi_0)\right). \tag{4.2.8}$$

In the solutions (4.2.4), (4.2.6) and (4.2.8) the valid sets of parameters are those given in (4.2.1) and (4.2.2).

4.3 Elliptic Type Solutions

We assume $c_4 = 0$ and $a_1 = a_3 = a_4 = 0$. We find the following values for the remaining constants a_0, a_2, c_0, c_1 and c_2;

$$a_0 = \frac{169\eta_2(c^2 - k^2 + 42c^2k^4c_1c_3\eta_2) - 36c^2\eta_1^2}{338k^2\alpha\eta_2}, \tag{4.3.1a}$$

$$a_2 = \frac{105c^2c_3^2k^2\eta_2}{2\alpha}, \tag{4.3.1b}$$

$$c_0 = \frac{4c_1\eta_1}{65k^2c_3\eta_2}, \tag{4.3.1c}$$

$$c_1 = \varepsilon_0\frac{R}{c_3}, \tag{4.3.1d}$$

$$c_2 = \frac{\eta_1}{13k^2\eta_2}, \tag{4.3.1e}$$

$$R = \frac{\sqrt{28561(c^2-k^2)^2\eta_2^2 - 1296c^4\eta_1^4}}{507\sqrt{161}c^2k^4\eta_2^2}, \tag{4.3.1f}$$

where $\varepsilon_0 = \pm 1$ and c_3 is arbitrary.

Now that we have determined the constants appearing in (4.1.5) successfully, we need to integrate (4.1.5b), which takes the form

$$\dot{\varphi}^2 = c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 = P(\varphi), \tag{4.3.2}$$

and find $\varphi(\xi)$ and hence $u(x,t)$. Evaluation of the integral of (4.3.2) depends on the factorization of the polynomial $P(\varphi)$. Assume that φ_1, φ_2 and φ_3 are zeros of the equation $P(\varphi) = 0$, for which the discriminant is

$$\Delta = 18c_0c_1c_2c_3 + c_1^2c_2^2 - 27c_0^2c_3^2 - 4c_3c_1^3 - 4c_0c_2^3. \tag{4.3.3}$$

Making use of (4.3.1) we obtain

$$\Delta = -\frac{\varepsilon_0 R(80\eta_1^4 + 7943\varepsilon_0k^4R\eta_1^2\eta_2^2 + 2856100k^8R^2\eta_2^4)}{714025c_3^2k^8\eta_2^4}. \tag{4.3.4}$$

$\Delta = 0$ if $R = 0$. When we analyze this branch, the coefficients in (4.3.1) give results the same as in Case I.

Let $\varepsilon_0 = -1$. The sign of Δ is determined by the sign of the term inside the paranthesis in (4.3.4). When we consider this term as a second-degree polynomial in R and calculate its discriminant, we see it is negative, therefore the polynomial is always positive. Hence $\Delta > 0$. Therefore the polynomial (4.3.2) has three distinct real zeros $\varphi_1, \varphi_2, \varphi_3$. Then we can factorize (4.3.2) as

$$\dot{\varphi}^2 = P(\varphi) = c_3(\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3). \tag{4.3.5}$$

<u>Case II.a</u> Let $c_3 > 0$. In order that (4.3.5) makes sense, the right hand side must be nonnegative. Therefore we should consider the intervals $\varphi > \varphi_1 > \varphi_2 > \varphi_3$ and $\varphi_1 > \varphi_2 > \varphi > \varphi_3$ when integrating (4.3.5). Let us first write

$$\frac{d\varphi}{\sqrt{c_3(\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)}} = \varepsilon d\xi \tag{4.3.6}$$

where $\varepsilon = \mp 1$. In the first hand, when $\varphi > \varphi_1 > \varphi_2 > \varphi_3$, using the results available in the handbook [20], we obtain

$$\int_{\varphi_1}^{\varphi} \frac{d\tau}{\sqrt{c_3(\tau - \varphi_1)(\tau - \varphi_2)(\tau - \varphi_3)}} = \frac{1}{\sqrt{c_3}} g\,\mathrm{sn}^{-1}\Big(\sqrt{\frac{\varphi - \varphi_1}{\varphi - \varphi_2}}, m\Big) \tag{4.3.7}$$

for the integration of the left hand side of (4.3.6), where $g = \dfrac{2}{\sqrt{\varphi_1 - \varphi_3}}$, $m^2 = \dfrac{\varphi_2 - \varphi_3}{\varphi_1 - \varphi_3}$. This gives rise to the elliptic function solution φ to (4.3.2),

$$\varphi(\xi) = \varphi_1 \mathrm{nc}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(\xi - \xi_0), m\Big) - \varphi_2 \mathrm{tn}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(\xi - \xi_0), m\Big), \tag{4.3.8}$$

and hence the solution to (4.1.1) can be written as follows

$$u(x,t) = a_0 + a_2\left[\varphi_1 \mathrm{nc}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(kx - ct - \xi_0), m\Big) - \varphi_2 \mathrm{tn}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(kx - ct - \xi_0), m\Big)\right]^2. \tag{4.3.9}$$

<u>Case II.b</u> When the coefficient $c_3 > 0$, for $\varphi_1 > \varphi_2 > \varphi > \varphi_3$ we obtain

$$\int_{\varphi_3}^{\varphi} \frac{d\tau}{\sqrt{c_3(\varphi_1 - \tau)(\varphi_2 - \tau)(\tau - \varphi_3)}} = \frac{1}{\sqrt{c_3}} g\,\mathrm{sn}^{-1}\Big(\sqrt{\frac{\varphi - \varphi_3}{\varphi_2 - \varphi_3}}, m\Big) \tag{4.3.10}$$

where $g = \dfrac{2}{\sqrt{\varphi_1 - \varphi_3}}$, $m^2 = \dfrac{\varphi_2 - \varphi_3}{\varphi_1 - \varphi_3}$. After we find

$$\varphi(\xi) = \varphi_2 \mathrm{sn}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(\xi - \xi_0), m\Big) + \varphi_3 \mathrm{cn}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(\xi - \xi_0), m\Big) \tag{4.3.11}$$

and hence the solution to (4.1.1) can be written as follows:

$$u(x,t) = a_0 + a_2\left[\varphi_2 \mathrm{sn}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(kx - ct - \xi_0), m\Big) + \varphi_3 \mathrm{cn}^2\Big(\varepsilon\frac{\sqrt{c_3}}{g}(kx - ct - \xi_0), m\Big)\right]^2. \tag{4.3.12}$$

Case II.c If the coefficient $c_3 < 0$, working on the interval $\varphi_1 > \varphi_2 > \varphi_3 > \varphi$ we find the following:

$$\int_{\varphi}^{\varphi_3} \frac{d\tau}{\sqrt{-c_3(\varphi_1-\tau)(\varphi_2-\tau)(\varphi_3-\tau)}} = \frac{1}{\sqrt{-c_3}} g\,\mathrm{sn}^{-1}\left(\sqrt{\frac{\varphi_3-\varphi}{\varphi_2-\varphi}}, m\right) \tag{4.3.13}$$

where $g = \frac{2}{\sqrt{\varphi_1-\varphi_3}}$, $m^2 = \frac{\varphi_1-\varphi_2}{\varphi_1-\varphi_3}$. This gives us

$$\varphi(\xi) = \varphi_3 \mathrm{nc}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(\xi-\xi_0), m\right) - \varphi_2 \mathrm{tn}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(\xi-\xi_0), m\right) \tag{4.3.14}$$

therefore the solution to (4.1.1) can be written as follows:

$$u(x,t) = a_0 + a_2\left[\varphi_3 \mathrm{nc}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(kx-ct-\xi_0), m\right) - \varphi_2 \mathrm{tn}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(kx-ct-\xi_0), m\right)\right]^2. \tag{4.3.15}$$

Case II.d For $c_3 < 0$, on the interval $\varphi_1 > \varphi > \varphi_2 > \varphi_3$ we see that we can proceed to obtain

$$\int_{\varphi_2}^{\varphi} \frac{d\tau}{\sqrt{-c_3(\varphi_1-\tau)(\tau-\varphi_2)(\tau-\varphi_3)}} = \frac{1}{\sqrt{-c_3}} g\,\mathrm{sn}^{-1}\left(\sqrt{\frac{(\varphi_1-\varphi_3)(\varphi-\varphi_2)}{(\varphi_1-\varphi_2)(\varphi-\varphi_3)}}, m\right) \tag{4.3.16}$$

where $g = \frac{2}{\sqrt{\varphi_1-\varphi_3}}$, $m^2 = \frac{\varphi_1-\varphi_2}{\varphi_1-\varphi_3}$. This immediately results in

$$\varphi(\xi) = \varphi_2 \mathrm{nd}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(\xi-\xi_0), m\right) - \varphi_3 m^2 \mathrm{sd}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(\xi-\xi_0), m\right) \tag{4.3.17}$$

producing the solution to (4.1.1) as

$$u(x,t) = a_0 + a_2\left[\varphi_2 \mathrm{nd}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(kx-ct-\xi_0), m\right) - \varphi_3 m^2 \mathrm{sd}^2\left(\varepsilon\frac{\sqrt{-c_3}}{g}(kx-ct-\xi_0), m\right)\right]^2. \tag{4.3.18}$$

In case $\varepsilon_0 = 1$ we have $\Delta < 0$. Therefore the polynomial (4.3.2) has one real zero φ_1 and two complex conjugate zeros φ_2, φ_3.

Case II.e If the coefficient $c_3 > 0$ we can obtain

$$\int_{\varphi_1}^{\varphi} \frac{d\tau}{\sqrt{c_3(\tau-\varphi_1)[(\tau-b_1)^2+a_1^2]}} = \frac{1}{\sqrt{c_3}} g\,\mathrm{cn}^{-1}\left(\frac{A+\varphi_1-\varphi}{A-\varphi_1+\varphi}, m\right) \tag{4.3.19}$$

where $b_1 = \frac{\varphi_2+\varphi_3}{2}$, $a_1^2 = -\frac{(\varphi_2-\varphi_3)^2}{4}$, $A^2 = (b_1-\varphi_1)^2 + a_1^2$, $g = \frac{1}{\sqrt{A}}$, $m^2 = \frac{A+b_1-\varphi_1}{2A}$. After that we find

$$\varphi(\xi) = \varphi_1 + A\frac{1-\mathrm{cn}\left(\varepsilon\frac{\sqrt{c_3}}{g}(\xi-\xi_0), m\right)}{1+\mathrm{cn}\left(\varepsilon\frac{\sqrt{c_3}}{g}(\xi-\xi_0), m\right)} \tag{4.3.20}$$

and hence the solution to (4.1.1) can be written as

$$u(x,t) = a_0 + a_2 \left[\varphi_1 + A \frac{1 - \operatorname{cn}\left(\varepsilon \frac{\sqrt{c_3}}{g}(kx - ct - \xi_0), m\right)}{1 + \operatorname{cn}\left(\varepsilon \frac{\sqrt{c_3}}{g}(kx - ct - \xi_0), m\right)} \right]^2. \tag{4.3.21}$$

Case II.f If the coefficient $c_3 < 0$ one can proceed to get

$$\int_{\varphi}^{\varphi_1} \frac{d\tau}{\sqrt{-c_3(\varphi_1 - \tau)[(\tau - b_1)^2 + a_1^2]}} = \frac{1}{\sqrt{-c_3}} g \operatorname{cn}^{-1}\left(\frac{A - \varphi_1 + \varphi}{A + \varphi_1 - \varphi}, m\right) \tag{4.3.22}$$

where $b_1 = \dfrac{\varphi_2 + \varphi_3}{2}$, $a_1^2 = -\dfrac{(\varphi_2 - \varphi_3)^2}{4}$, $A^2 = (b_1 - \varphi_1)^2 + a_1^2$, $g = \dfrac{1}{\sqrt{A}}$, $m^2 = \dfrac{A - b_1 + \varphi_1}{2A}$.

After this we get

$$\varphi(\xi) = \varphi_1 - A \frac{1 - \operatorname{cn}\left(\varepsilon \frac{\sqrt{-c_3}}{g}(\xi - \xi_0), m\right)}{1 + \operatorname{cn}\left(\varepsilon \frac{\sqrt{-c_3}}{g}(\xi - \xi_0), m\right)} \tag{4.3.23}$$

and hence the solution to (4.1.1) turns out to be

$$u(x,t) = a_0 + a_2 \left[\varphi_1 - A \frac{1 - \operatorname{cn}\left(\varepsilon \frac{\sqrt{-c_3}}{g}(kx - ct - \xi_0), m\right)}{1 + \operatorname{cn}\left(\varepsilon \frac{\sqrt{-c_3}}{g}(kx - ct - \xi_0), m\right)} \right]^2. \tag{4.3.24}$$

5. CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In this thesis, higher-order Boussinesq (HBq) equations of the form

$$u_{tt} = \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + (f(u))_{xx} \tag{5.1.1}$$

are classified according to the Lie symmetry algebras the equation admits depending on the formulation of the nonlinearity $f(u)$. It is shown that, for an arbitrary $f(u)$, (5.1.1) admits the two-dimensional abelian algebra as the invariance algebra. For the following three canonical possible forms of $f(u)$,

$$\text{(A)} \quad f(u) = \alpha e^u, \qquad \alpha = \mp 1, \tag{5.1.2a}$$

$$\text{(B)} \quad f(u) = \alpha \ln(u), \qquad \alpha = \mp 1, \tag{5.1.2b}$$

$$\text{(C)} \quad f(u) = \alpha u^n, \qquad \alpha = \mp 1, \quad \mathbb{R} \ni n \neq 0, 1. \tag{5.1.2c}$$

(or, equivalently, for the forms of $f(u)$ given in (3.1.11)), the symmetry algebra is three-dimensional. It is also shown that for a specific form of $f(u)$, the HBq has a four-dimensional symmetry algebra and this result is also valid when $\eta_2 = 0$. The results for the cases obtained by the algorithm due to Lie are shown in a table. As indicated in the table, the sixth-order PDE (5.1.1) is reduced to ODEs of fourth and sixth-order by similarity reductions.

Moreover, some exact solutions to the HBq equation are obtained by traveling wave ansatz method. When $f(u) = u + \alpha u^2$, (5.1.1) is reduced to

$$\eta_2 k^4 c^2 \left[F'''F' - \frac{1}{2}(F'')^2\right] - \frac{\eta_1 k^2 c^2}{2}(F')^2 - \frac{\alpha k^2}{3}F^3 + \frac{c^2 - k^2}{2}F^2 = K_0 \tag{5.1.3}$$

by this method. Some trigonometric, hyperbolic and elliptic type solutions to the HBq equation which do not exist in the literature are obtained.

5.2 Future Discussions

In this thesis, we restricted ourselves to a subclass of (3.1.11c), by searching for the exact solution in the case $f(u) = u + \alpha u^2$. Actually, the analysis of the reduced equation for $n = 2$ for Case C.1 in Table 1 which were obtained by the infinitesimal generator $X_1 + \varepsilon X_3$ would follow similar lines to the analysis in Section 3. Mainly due to the complicated nature of the reduced equations, we do not perform a further analysis for the reduced equations in this work. Regarding the families (3.1.11) or equivalently (3.1.12), which means some HBq equations with certain symmetries, one can ask another question: Do these canonical forms of nonlinearities $f(u)$ have any physical meaning? As far as we know, the answer is affirmative when f has power-type nonlinearities like $f(u) = u + \alpha u^2$, etc., and that has been the main reason for writing Chapter 4 of this thesis. The analysis of the other reduced equations remains still open.

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