ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

THE GENERALIZED FRACTIONAL BENJAMIN BONA MAHONY EQUATION: ANALYTICAL AND NUMERICAL RESULTS

Ph.D. THESIS

Göksu ORUÇ

Department of Mathematical Engineering

Mathematical Engineering Programme

JULY 2021

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GENELLEŞTİRİLMİŞ KESİRLİ BENJAMİN BONA MAHONY DENKLEMİ: ANALİTİK VE SAYISAL SONUÇLAR

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Göksu ORUÇ, a Ph.D. student of ITU Graduate School 509142205 successfully defended the thesis entitled “THE GENERALIZED FRACTIONAL BENJAMIN BONA MAHONY EQUATION: ANALYTICAL AND NUMERICAL RESULTS”, which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To the memory of my grandmother,

FOREWORD

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ABBREVIATIONS

FFT **:** Fast Fourier Transform
IFFT **:** Inverse Fast Fourier Transform

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**THE GENERALIZED FRACTIONAL
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ANALYTICAL AND NUMERICAL RESULTS**

SUMMARY

In this thesis study we consider the generalized fractional Benjamin-Bona-Mahony (gfBBM) equation

$$u_t + u_x + \frac{1}{2}(u^{p+1})_x + \frac{3}{4}D^\alpha u_x + \frac{5}{4}D^\alpha u_t = 0,$$

where x and t represents spatial coordinate and time, respectively. This equation is derived to model the propagation of small amplitude long unidirectional waves in a nonlocally and nonlinearly elastic medium. The gfBBM equation has a general power-type nonlinearity and two fractional-type terms. Thanks to these properties, the gfBBM equation is noticed as a satisfactory and interesting model in the literature.

The aim of this thesis study is to perform various mathematical and numerical analyses for the gfBBM equation and to understand the influence of nonlinearity and fractional dispersion on the dynamics of solutions.

The thesis study is organized in the following way:

In the first chapter, we briefly introduce the general background on the fractional type nonlinear partial differential equations with lower dispersion such as fractional Korteweg de Vries (fKdV) and fractional Benjamin-Bona-Mahony (fBBM) and gfBBM equations. Then, we propose derivation and some properties of the gfBBM equation. We also state the analytical and numerical methods used to solve this equation. Furthermore, the literature overview on gfBBM and related equations is given in this chapter.

The second chapter is devoted to the analytical results for the gfBBM equation. In the first section of this chapter we recall the preliminaries. This section contains useful definitions related to functional analysis, lemmas and theorems used in the thesis. In the second section, we derive conserved quantities of the gfBBM equation. We also find constraints on the order α of the fractional term. The aim of the third section is to prove the local well-posedness of the Cauchy problem for the gfBBM equation together with the initial condition

$$u(x,0) = u_0(x).$$

For the case $1 \le \alpha \le 2$, we prove the local well-posedness of the solutions by using contraction mapping principle. On the other hand, for the case $0 < \alpha < 1$, we use the approaches given for the fBBM equation by He and Mammeri (2018). Therefore, we consider the regularization of the Cauchy problem for the gfBBM equation and then use the convergence of regularized solutions to the solutions of main problem. The section 4 presents the conditions for the non-existence of solitary wave solutions to the

gfBBM equation. Existence and uniqueness of solitary wave solutions are obtained by using the result of Frank and Lenzmann (2013). We also consider the restrictions on the α and speed of wave c so that the gfBBM equation admits positive or negative solitary waves. Finally, we derive exact solitary wave solutions to the gfBBM equation for the special cases $\alpha = 1$ and $\alpha = 2$ when $p = 1$. In the last section of this chapter we discuss the stability properties of solitary wave solutions associated to the gfBBM equation. We first give the Hamiltonian formulation of the equation. Then, we prove the orbital stability of solitary wave solutions by using approach given by Grillakis Shatah Strauss (GSS) (1987) and for the stability we obtain following conditions when $1 \leq p \leq 4$:

1. $\frac{p}{p+2} < \alpha < \frac{p}{2}$ and $c > c_{1,p} > 1$,

2. $\frac{p}{2} < \alpha < 2$ and $c > 1$ or $\frac{3}{5} > c > c_{2,p}$,

with $c_{1,p} = \frac{6\alpha+2p+3\alpha p+\sqrt{2}p\sqrt{2\alpha-p+\alpha p}}{5(2\alpha+\alpha p)}$ and $c_{2,p} = \frac{6\alpha+2p+3\alpha p-\sqrt{2}p\sqrt{2\alpha-p+\alpha p}}{5(2\alpha+\alpha p)}$.

In the last chapter, we present the numerical results for the gfBBM equation. We first state efficient numerical algorithms for gfBBM equation and then carry out various numerical experiments. The Petviashvili method is proposed for the generation of the solitary wave solutions that cannot be obtained analytically. We numerically investigate the effects of the relation between the nonlinearity and the dispersion on the solutions. The evolution of generated wave profiles in time is investigated numerically by Fourier pseudo-spectral method. The efficiency of the methods will be demonstrated by various numerical simulations.

GENELLEŞTİRİLMİŞ KESİRLİ BENJAMİN BONA MAHONY DENKLEMİ: ANALİTİK VE SAYISAL SONUÇLAR

ÖZET

Genelleştirilmiş kesirli Benjamin-Bona-Mahony (gkBBM) denklemi

$$u_t + u_x + \frac{1}{2}(u^{p+1})_x + \frac{3}{4}D^{\alpha}u_x + \frac{5}{4}D^{\alpha}u_t = 0$$

ile verilmektedir. Burada x ve t sırası ile uzay ve zaman koordinatlarını temsil etmektedir. Denklemde kuvvet tipinde doğrusal olmayan terimin mertebesi $p > 1$ olup, kesirli mertebeden türev operatörü $D^{\alpha} = (-\Delta)^{\frac{\alpha}{2}}, \alpha \in \mathbb{R}$ ise Riesz potansiyeli olarak adlandırılmakta ve keyfi bir $q(x)$ fonksiyonu için Fourier dönüşümü yardımı ile

$$\widehat{D^{\alpha}q}(k) = |k|^{\alpha}\hat{q}(k)$$

şeklinde tanımlanmaktadır. Bu denklem ilk defa Erbay ve diğerleri (2016) tarafından türetilmiş olup literatürde iyi bilinen kesirli Korteweg deVries (kKdV) ve kesirli Benjamin-Bona-Mahony (kBBM) gibi benzer denklemlerin aksine fiziksel bir modelden doğmuştur. Genelleştirilmiş kesirli Benjamin-Bona-Mahony denklemi yerel ve doğrusal olmayan elastik bir ortamda yayılan küçük genlikli uzun dalgaları modellemektedir. Ayrıca iki farklı kesirli türev içeren terim ve kuvvet tipinde doğrusal olmayan terim içermektedir. Söz konusu denklem bu özelliklerinden dolayı literatürdeki benzer denklemler ile karşılaştırıldığında daha güçlü bir model olarak dikkat çekmektedir.

Bu tez çalışmasının amacı gkBBM denklemi için çeşitli matematiksel ve sayısal analizler gerçekleştirmek, ayrıca zayıf dispersiyon ve kuvvet tipinde doğrusal olmayan terim etkisi altında çözümlerin davranışının nasıl değiştiğini incelemektir.

Bu tez çalışması aşağıda sunulan şekilde düzenlenmiştir.

Tez çalışmasının ilk bölümü olan giriş bölümünde kesirli dispersif terime sahip doğrusal olmayan kısmi türevli diferansiyel denklemler kısaca tanıtılmıştır. Bu tipteki denklemlere örnek olarak kesirli kKdV, kBBM ve gkBBM denklemleri verilmiştir. Ayrıca bu tezde gkBBM denklemin çözümlerini incelemek için ele alınan analitik ve sayısal yöntemlerden bahsedilmiştir. Son olarak, gkBBM denklemi ve benzer denklemler hakkında literatürde var olan çalışmalar tanıtılmıştır.

Çalışmanın ikinci bölümünde gkBBM denklemi için elde edilen analitik sonuçlara yer verilmiştir. Analitik sonuçlar bölümünün içeriği aşağıda sunulan şekildedir.

İkinci bölümün birinci kısmı bu tez çalışmasında kullanılacak olan ön bilgilere ayrılmıştır. Fonksiyonel analiz ile ilgili temel tanımlar, özel fonksiyon uzayları, Fourier dönüşümleri ve tezde kullanılan tanımlar hatırlatılmış; gelecek kısımlarda yararlanılacak olan lemma ve teoremler sunulmuştur.

İkinci kısımda, ilk olarak gfBBM denkleminin korunan büyüklükleri türetilmiştir. Bu aşamada Hamiltonyen'in iyi tanımlı olabilmesi için kesirli türevin mertebesi olan α üzerinde bazı kısıtlar elde edilmiştir. Çalışmada ele alınan gkBBM denkleminin gerçek çözümlerinin bilinmediği durumlarda denklem için önerilen sayısal yöntemin doğruluğunu gösteren hata hesabı yapılamamaktadır. Bu sebep ile korunan büyüklüklerin zaman ile değişimi sayısal yöntemin doğruluğunu test etmek için önemli olmaktadır.

Üçüncü kısımda, gkBBM denklemi için

$$u(x,0) = u_0(x)$$

başlangıç koşulu seçimi altında karşı gelen Cauchy problemi tanımlanmış ve bu problemin çözümlerinin yerel iyi tanımlılığı incelenmiştir. İlk olarak $1 \leq \alpha \leq 2$ bölgesi ele alınmıştır ve standart büzülme dönüşümü prensibi ile Cauchy probleminin çözümlerinin iyi tanımlılığını gösterilmiştir. Daha sonra $0 < \alpha < 1$ seçimi altında çözümlerin iyi tanımlılığını göstermek üzere He ve Mammeri (2018) tarafından kBBM denklemi için önerilen yaklaşım kullanılmıştır. Bu yaklaşımdaki fikir, gkBBM denklemine düzgünleştirici bir terim eklenerek çözümlerin yerel iyi tanımlılığının incelenmesi ve düzgünleştirilmiş problemin çözümlerinin, gkBBM için tanımlanan Cauchy probleminin çözümlerine yakınsadığının gösterilmesi şeklinde özetlenebilir.

Dördüncü kısımda, gkBBM denkleminin yalnız dalgalarının hangi şartlar altında var olmadığı ispatlanmıştır. Söz konusu dalgaların varlığı ve tekliği ise Frank ve Lenzmann (2013) tarafından elde edilen sonuçlar kullanılarak gösterilmiştir. Bu kısımda ayrıca α, p ve dalga hızı c parametrelerinin seçimine bağlı olarak gkBBM denkleminin pozitif veya negatif yalnız dalgalar ürettiği gösterilmiştir. Son olarak $p = 1$ iken $\alpha = 1$ ve $\alpha = 2$ özel durumları için gkBBM denkleminin gerçek yalnız dalga çözümleri türetilmiştir.

Beşinci kısımda, gkBBM denklemi tarafından üretilen yalnız dalga çözümlerinin yörüngesel kararlılık özellikleri tartışılmıştır. Bunun için önce gkBBM denkleminin Hamilton formu verilmiştir. Denklemin bu formda yazılması ile Grillakis, Shatah ve Strauss (1987) tarafından yalnız dalgaların yörüngesel kararlılığını göstermek için önerilen yaklaşım uygulanabilir hale gelmiştir. Bu tez çalışmasında gkBBM denkleminin yalnız dalga çözümlerinin aşağıda verilen şartlar altında yörüngesel kararlı olduğu gösterilmiştir:

1. $\frac{p}{p+2} < \alpha < \frac{p}{2}$ and $c > c_{1,p} > 1$,
2. $\frac{p}{2} < \alpha < 2$ and $c > 1$ or $\frac{3}{5} > c > c_{2,p}$.

Burada $1 \leq p \leq 4$ için $c_{1,p} = \frac{6\alpha+2p+3\alpha p+\sqrt{2}p\sqrt{2\alpha-p+\alpha p}}{5(2\alpha+\alpha p)}$ ve $c_{2,p} = \frac{6\alpha+2p+3\alpha p-\sqrt{2}p\sqrt{2\alpha-p+\alpha p}}{5(2\alpha+\alpha p)}$ şeklinde hesaplanmıştır.

Tezin son bölümünde, gkBBM denklemi için önerilen sayısal yöntemlere ve çeşitli sayısal deneylere yer verilmiştir. Bu bölümde ilk olarak gkBBM denkleminin açık formda elde edilemeyen yalnız dalga çözümleri üretilmiştir. Daha sonra tezin ikinci bölümünde yalnız dalga çözümleri için elde edilen analitik sonuçların doğruluğu test edilmiştir. Ayrıca doğrusal olmayan terimin mertebesi p ile kesirli dispersif terimin mertebesi α arasındaki ilişkinin, üretilen çözümler üzerindeki etkisi ve yalnız dalgaların hızı ile genliği arasındaki ilişkiler sayısal olarak incelenmiştir. Deneylerde denklemin yalnız dalga profilleri Petviashvili yöntemi ile türetilmiş; bir Fourier sözde spektral yöntemi yardımı ile türetilen yalnız dalgaların zamanda ilerlemesi incelenmiştir. Petviashvili şemasının doğruluğu çeşitli sayısal kontrol parametreleri yardımı ile test edilmiştir. Fourier sözde spektral şemanın doğruluğunu test etmek için ise önceki bölümde türetilen korunan büyüklüklerden faydalanılmıştır. Bu çalışmada gkBBM denkleminin sayısal çözümlerini elde etmek için önerilen her iki şemanın da oldukça etkin yöntemler olduğu gösterilmiştir.

1. INTRODUCTION

Many years before the invention of the word “fractal" by Mandelbrot [1], the Japanese artist Hokusai published some artworks which are very close to fractal images. The artwork entitled The Great Wave off Kanagawa is illustrated in Figure 1.1. The

Figure 1.1 : Hokusai’s "The Great Wave off Kanagawa".

fractals are known as geometric figures with non-integer dimension, whereas the term fractional derivative defines the differential operator with non-integer order. The fractality concept has been increasingly appreciated in the literature because of the fractal nature of the world. Especially, there has been great interest in fractional differential equations, since they are used to model wide variety of phenomena in engineering and science such as fluid mechanics, finance, viscoelasticity, biology, sociology and image processing.

Many considerable researches have been carried out on fractional type nonlinear partial differential equations with lower dispersion, which is described by the help of fractional Laplacian operator. Now, we introduce a precise definition for the fractional Laplacian operator $D^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ in terms of Fourier transform as

$$\widehat{D^\alpha q}(k) = |k|^\alpha \hat{q}(k).$$

Here $\hat{q}$ is the Fourier transform of a suitable function $q(x)$. Throughout this study, we consider functions of single variable in space. In this case, the fractional Laplacian operator turns into the ordinary second order derivative when $\alpha = 2$. An example for the function $q(x) = \operatorname{sech}^2(x)$ is illustrated in Figure 1.2. Physical motivation and

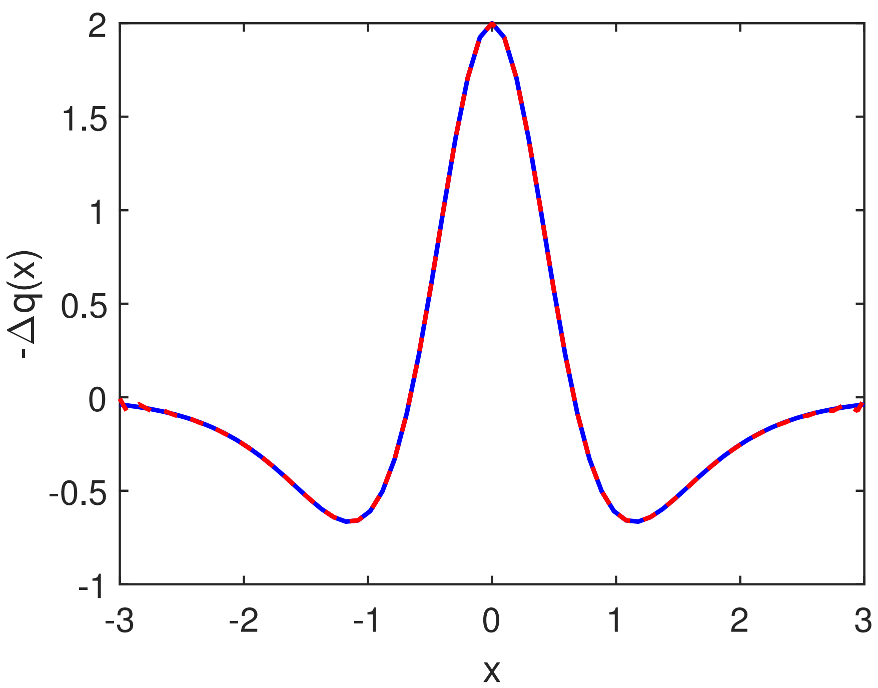

Figure 1.2 : Second order derivative of $q(x)$ using the fractional Laplacian operator (dashed line) and the exact derivative (solid line).

additional information for the fractional Laplacian operator can be found in [2]. Here, we also note that there are other fractional derivative definitions derived in fractional calculus such as Riemann-Luoville, Caputo and Grünwald-Letnikov. To prevent the confusion, the fractional Laplacian is also known as the Riesz potential derivative. The fractional time derivative will not be considered in this thesis study.

The fractional Camassa Holm (fCH) equation

$$u_t + u_x + uu_x + \frac{3}{4}D^\alpha u_x + \frac{5}{4}D^\alpha u_t = -\frac{1}{4}[2D^\alpha(uu_x) + uD^\alpha u_x]$$

was first presented in the study [3]. Here, the operator $D^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ denotes the Riesz potential for any $\alpha \in \mathbb{R}$. Then, same authors introduced the generalized fractional CH equation

$$u_t + u_x + \frac{1}{2}(u^{p+1})_x + \frac{3}{4}D^\alpha u_x + \frac{5}{4}D^\alpha u_t = -\frac{p+1}{8}[2D^\alpha(u^p u_x) + u^p D^\alpha u_x]$$

in [4]. Here, p is an integer. This equation derived from the unidirectional, small-but-finite amplitude, long wave solutions of the fractional improved Boussinesq equation

$$u_{tt} - u_{xx} + D^{\alpha} u_{tt} = (u^{p+1})_{xx}$$

by using double asymptotic expansion technique. Neglecting higher order terms in asymptotic expansion, the generalized fractional Benjamin-Bona-Mahony (gfBBM) equation is obtained in the same study as

$$u_t + u_x + \frac{1}{2}(u^{p+1})_x + \frac{3}{4}D^{\alpha} u_x + \frac{5}{4}D^{\alpha} u_t = 0. \tag{1.1}$$

This equation is proposed as main concern of the current study for $0 < \alpha \leq 2$.

The gfBBM equation is associated with the fractional Korteweg-de Vries (fKdV) and fractional Benjamin-Bona-Mahony (fBBM) equation in the literature. If the nonlinear term of the standard KdV equation was fixed as quadratic uu_x, and the dispersive term u_{xxx} was replaced by $D^{\alpha} u_x$ in the KdV equation, then we get fKdV equation as follows

$$u_t + u_x + uu_x + D^{\alpha} u_x = 0. \tag{1.2}$$

If the nonlinear term of the standard BBM equation was also fixed as quadratic uu_x, and the term u_{txx} was replaced by $D^{\alpha} u_t$ in the BBM equation, then we obtain fBBM equation as follows

$$u_t + u_x + uu_x + D^{\alpha} u_t = 0. \tag{1.3}$$

The gfBBM equation includes both, the term $D^{\alpha} u_x$ of the fKdV equation and the term $D^{\alpha} u_t$ of the fBBM equation. Although low dispersion fKdV and fBBM equations appear to be more relevant physically than KdV and BBM equations, they have not been derived from a physical model but are toy models to understand the interesting mathematical behaviour [5]. On the other hand, the equation with both u_{xxx} of KdV and u_{xxt} of the BBM equations appear to be a better approximation for the full Euler equation than both KdV and BBM equations [6]. Since this equation is a special form of the gfBBM equation, it is a better approach to use both terms as fractional. Furthermore, as the presence of the term $D^{\alpha} u_t$ of BBM equation weakens the nonlinearity, the nonlinearity-dispersion interaction will be more interesting in the presence of these two fractional terms.

A problem for a partial differential equation is well-posed if the following assumptions are satisfied: *i*) The problem has a solution for each choice of data in an appropriate space, *ii*) This solution is unique for each choice of data in an appropriate space, *iii*)

The solution depends continuously on the data given in the problem. If we have the existence and uniqueness of the solution in a small time interval near the origin $t = 0$, then we say the solution is locally well-posed. For the case $1 \leq \alpha \leq 2$, one can prove the local well-posedness of solutions by using a standard tool of the nonlinear analysis: the contraction mapping principle. However, it is not possible for the more complicated case $0 < \alpha < 1$. Therefore, the vanishing viscosity method is used to obtain the local well-posedness of solutions when $0 < \alpha < 1$.

Travelling wave solutions appear due to the balance between nonlinearity and dispersion and they have the form

$$u(x,t) = Q_c(\xi) \qquad \xi = x - ct,$$

where $c \in \mathbb{R}$ is speed of the wave. Solitary waves are particular class of the travelling waves with

$$\lim_{|\xi|\to\infty} Q_c(\xi) = 0.$$

Furthermore, solitary waves have some special properties: They are localized solutions and have single hump.

In historical perspective, John Scott-Russel was the first person to observe empirically the solitary waves in water, in 1834, on the Edinburgh-Glasgow canal. Then, Joseph Boussinesq proposed well-known Boussinesq (bidirectional) equation which admits the solitary wave solutions. He showed that the solitary waves remain stable, namely, under a slight perturbation solitary waves propagate without evolving into any other wave form. Indeed, this discovery was the birth of stability theory of solitary wave solutions. Here we can state the standard definition of stability which is also known as Lyapunov stability.

Definition 1. *(Lyapunov Stability) A solution $u^*(t)$ with a given initial value $u^*(t_0)$ is stable if, given any $\varepsilon > 0$, a $\delta > 0$ can be found such that for any solution $u(t)$ satisfying $||u^*(t_0) - u(t_0)|| < \delta$, $||u^*(t) - u(t)|| < \varepsilon$ is true for all $t \geq t_0$.*

Stability notion for the solitary waves was first proposed by Benjamin in [7]: Orbital stability of solitary waves which is also known as stability in shape. Afterwards Grillakis-Shatah-Strauss (GSS) developed a systematic approach to prove the stability of solitary waves in [8]. Later this method is called as GSS's approach and made a significant impact on the researchers. In brief, the orbital stability of the corresponding solitary wave solutions can be determined by the convexity of a suitable scalar function if the following assumptions are satisfied: *i*) local well-posedness of corresponding initial value problem, *ii*) existence of solitary wave solutions, *iii*) existence of a Hamiltonian operator that is defined by the help of conserved quantities which satisfies certain spectral properties.

Solitary waves play an important role in simulation of wave phenomena. This special solutions can be easily constructed and so they are widely used in the experiments. Since the analytical solitary wave solutions of fractional differential equations are still unknown in general, the numerical wave generation has become very significant for the researchers. In this sense, Petviashvili iteration method is noticed as a quite efficient technique to construct the solitary wave solutions of the nonlinear dispersive equations, numerically. This method was first used by V. I. Petviashvili to generate solitary wave solution of the Kadomtsev-Petviashvili (KP) equation in [9]. Although it seems as a basic fixed point iteration technique, it has been improved by introducing a stabilizing factor. In this way, Petviashvili method is easy for implementation and converges rapidly. Moreover, it converges only to the ground state solution of nonlinear wave equations. For further information we refer to the studies [10, 11].

The other numerical method discussed in this thesis study is the Fourier pseudo-spectral method. It is proposed to analyze gfBBM equation, numerically. The main idea behind the spectral method depends on approximation to the solution of the problem with finite sum of basis functions. Spectral methods have great advantages such as exponential convergence and high order accuracy. Moreover, the fractional derivative in the gfBBM equation is defined by a Fourier multiplier. Thus, the Fourier spectral method is noticed as the most convenient method for investigating the evolution of the solution in time.

1.1 Literature Overview

The motivation for contributing originally to the literature leads us to use results for the fKdV and fBBM equations. Therefore, we first give literature overview about fKdV and fBBM equations in this subsection. These equations have been intensively studied in the past years for $\alpha \geq 1$ in terms of global well-posedness, stability and blow-up etc. We refer to [12–14] for a more detailed discussion and review. The research on the more delicate case $\alpha \in (0,1)$ has been increased in the last few years. For the local well-posedness of Cauchy problem to both equations, we refer the reader to [5] and [15]. The missing uniqueness result in [5] is recovered by using the regularization of fBBM equation in [16]. Additionally, the extension of life span for these equations are obtained in [17] and [18]. The global well-posedness and regularity of solutions to the fKdV equation are proved in [19] and [20], respectively. In [21], the solitary wave solutions in terms of existence and stability are investigated. The stability and linear instability results for a general nonlinearity are obtained in [14]. Blow-up issues for both equations are observed in [22], numerically. A Fourier spectral method is studied for fKdV equation and its solitary wave solutions are constructed numerically in [23] by using Petviashvili method. For the convergence analysis of the Petviashvili method we refer to [10, 24]. Recently, the existence, uniqueness and spectral stability of periodic waves for the fKdV and fBBM equations have been proved in [25] and [26].

On the other hand, the literature about gfBBM equation is still very limited. This equation is derived from a physical model in [3, 4] as explained above. In [27], the gfBBM equation is studied in both mathematical and numerical aspects. Stability properties of the solitary wave solution to the same equation is proposed in [28].

Literature overview of the fKdV, fBBM and gfBBM equations has been alluded. In literature, the authors studied only quadratic nonlinearity for the corresponding equations. However, we consider power type nonlinearity in this study. At that point the question arises what are the effects of both power type nonlinear terms and the fractional differentiation operator on dispersive terms.

To the best of our knowledge, the problem proposed in this thesis will be the first in literature for the gfBBM equation. In addition, since special cases of the gfBBM equation contain a large class of equations we expect that the results will attract interest and give rise to new research problems.

2. ANALYTICAL RESULTS FOR THE GENERALIZED FRACTIONAL BENJAMIN-BONA-MAHONY EQUATION

2.1 Preliminaries and Notations

In this section, the notations and basic definitions will be presented. Then, useful inequalities, lemmas and theorems that we utilized in this chapter will be recalled.

Definition 2. ***($L^p(\mathbb{R})$ Space)*** *Let f be an arbitrary function defined on the domain $\mathbb{R}$. $L^p(\mathbb{R})$ is the normed vector space endowed with the norm*

$$\|u\|_{L^p(\mathbb{R})} = \begin{cases} (\int_{\mathbb{R}} |f|^p dx)^{\frac{1}{p}} < \infty & \text{if } 1 < p < \infty \\ \\ ess\ sup\ \mid f \mid & \text{if } p = \infty \end{cases}$$

The inner product of functions $f,\, g \in L^2(\mathbb{R})$ is defined by

$$(f,g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)dx.$$

Due to simplicity, we keep to use notation $(.,.)$ for the inner product on $L^2(\mathbb{R})$.

Definition 3. ***(Test Function)*** *[29] Let $C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions $\varphi : \Omega \subset (R^n) \to \mathbb{C}$, with compact support in Ω, an open subset of R^n. A function f belonging to $C_c^\infty(\Omega)$ is called a test function.*

Definition 4. ***(Compactly Contained Set)*** *[29] Let Ω and V denote open subsets of R^n. We write*

$$V \subset\subset \Omega$$

and say V is compactly contained in Ω if the closure of V belongs to Ω.

Definition 5. ***(Locally Integrable Functions)*** *[29] Let $1 \le p \le \infty$. $L^p_{loc}(\Omega)$ is the set of locally integrable functions, $L^p_{loc}(\Omega) = \{u : \Omega \to C | u \in L^p(V) \text{ for each } V \subset\subset \Omega\}$, i.e $u \in L^p_{loc}(\Omega)$ if $u : \Omega \to C$ satisfies $u \in L^p(V)$ for all $V \subset\subset \Omega$.*

Definition 6. ***(Weak Derivative)*** *[29] Suppose $u, v \in L^1_{loc}(\Omega)$ and $\bar{\alpha} = (\bar{\alpha}_1, ..., \bar{\alpha}_n)$ is multiindex. If the equality*

$$\int_\Omega u \bar{D}^{\bar{\alpha}} \phi dx = (-1)^{|\bar{\alpha}|} \int_\Omega v \phi dx,$$

with

$$\bar{D}^{\bar{\alpha}} \phi = \frac{\partial^{|\bar{\alpha}|} \phi}{\partial x_1^{\bar{\alpha}_1} ... \partial x_n^{\bar{\alpha}_n}}$$

is satisfied for all test functions $\phi \in C_c^\infty$, then v is called the weak derivative of order $| \bar{\alpha} |$ of the function u in the domain Ω and is denoted by $\bar{D}^{\bar{\alpha}} u$ i.e. $v = \bar{D}^{\bar{\alpha}} u$.

Definition 7. ***(Sobolev Space)*** *[29] The Sobolev space $W^{s,p}(\Omega)$ consists of all integrable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex $\bar{\alpha}$ with $| \bar{\alpha} | \leq s$, $\bar{D}^{\bar{\alpha}} u$ exists in the weak sense and belongs to $L^p(\Omega)$. Similarly we define the space $W^{s,p}_{loc}(\Omega)$ using locally integrable functions instead of integrable ones. We introduce a natural norm on the Sobolev space:*

$$\| u \|_{W^{s,p}} = \sum_{|\bar{\alpha}| \leq s} \| \bar{D}^{\bar{\alpha}} u \|_{L^p} .$$

Throughout this study, we consider $p = 2$, space dimension $n = 1$.

Remark: Let $p = 2$ and $n = 1$, then we write usual notation

$$H^s(\mathbb{R}) = W^{s,2}(\mathbb{R})$$

with the representation of norm in Fourier space

$$\|u\|^2_{H^s} = \int_{\mathbb{R}} (1 + |k|^2)^s |\hat{u}(k)|^2 dk$$

for $s \in \mathbb{R}$. Here, it is useful to give the Fourier transform and its inverse for a function $u \in L^2(\mathbb{R})$. They are defined by

$$\hat{u}(k) = \int_{\mathbb{R}} u(x) e^{-ikx} dx$$

and

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(k) e^{ikx} dk,$$

respectively. Here, i is the imaginary unit and k denotes wave number in Fourier space.

Theorem 1. ***(Sobolev Imbedding Theorem)*** *Let $r > 0$ and $1 \leq p \leq \infty$. We have*

- *If* $2r < 1$, *then* $H^r(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ *for every* $p \leq \frac{2}{1-2r}$.

- *If* $2r = 1$, *then* $H^r(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ *for every* $p < \infty$.

- *If* $2r > 1$, *then* $H^r(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Definition 8. ***(Hölder's Inequality)*** *Assume that* $p \in [1,\infty]$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $u \in L^p(\mathbb{R})$ *and* $v \in L^q(\mathbb{R})$, *then* $uv \in L^1(\mathbb{R})$ *and*

$$\|uv\|_{L^1(\mathbb{R})} \leq \|u\|_{L^p(\mathbb{R})} \|v\|_{L^q(\mathbb{R})}.$$

Definition 9. ***(Minkowski's Inequality)*** *Assume that* $p \in [1,\infty]$. *If* $u,\ v \in L^p(\mathbb{R})$, *then* $u + v \in L^p(\mathbb{R})$ *with*

$$\|u+v\|_{L^p(\mathbb{R})} \leq \|u\|_{L^p(\mathbb{R})} + \|v\|_{L^p(\mathbb{R})}.$$

Definition 10. ***(Gronwall's Inequality - Differential Form)*** *Suppose that* $b(t)$ *and* $f(t)$ *are continuous functions for* $t > a$ *and* $u(t)$ *is a differentiable function for* $t \geq a$. *If the following inequalities are satisfied*

$$\begin{aligned} u'(t) &= b(t)u(t) + f(t), \quad t \geq a, \\ u(a) &\leq u_0, \end{aligned}$$

then

$$u(t) \leq u_0 \exp\left(\int_a^t b(\tau)d\tau\right) + \int_a^t f(\tau)\exp\left(\int_\tau^t b(s)ds\right)d\tau, \qquad \forall t \geq a.$$

Definition 11. ***(Gronwall's Inequality - Integral Form)*** *Suppose that* $u(t)$ *and* $f(t)$ *are nonnegative continuous functions for* $t > 0$ *and satisfy the following inequality*

$$u(t) \leq C + \int_0^t u(\tau)f(\tau), \quad t \geq 0,$$

with $C \geq 0$. *Then*

$$u(t) \leq C\exp\left(\int_0^t f(\tau)d\tau\right), \qquad \forall t \geq 0.$$

Definition 12. ***(Fractional Leibniz Rule)*** *[30] Let* $\alpha = \alpha_1 + \alpha_2 \in (0,1)$ *with* $\alpha_i \geq 0$ *and* $p,q,r \in (1,\infty)$. *Then,*

$$\| D^\alpha(fg) - fD^\alpha g - gD^\alpha f \|_{L^p} \lesssim \| D^{\alpha_1} f \|_{L^q} \| D^{\alpha_2} g \|_{L^r}$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$.

Here, the notation $x \lesssim y$ is used to denote $x \leq Cy$ for any quantities x, y and constant $C > 0$.

Lemma 1. *[31] For $r > 1/2$, $H^r(\mathbb{R})$ is an algebra with respect to the product of functions. That is, if $u, v \in H^r(\mathbb{R})$ then $uv \in H^r(\mathbb{R})$ and*

$$\|uv\|_{H^r(\mathbb{R})} \leq C\|u\|_{H^r(\mathbb{R})}\|v\|_{H^r(\mathbb{R})}. \tag{2.1}$$

Lemma 2. *[32] Assume that $f \in C^k(\mathbb{R})$, $u, v \in H^r(\Omega) \cap L^\infty(\Omega)$ and $k = [r]+1$, where $r \geq 0$. Then we have*

$$\|f(u) - f(v)\|_r \leq K(M)\|u - v\|_r \tag{2.2}$$

if $\|u\|_\infty \leq M$, $\|v\|_\infty \leq M$, $\|u\|_r \leq M$ and $\|v\|_r \leq M$, where $K(M)$ is a constant depending on M and s.

Definition 13. ***(Skew Symmetric Operator)*** *An operator A is said to be skew-symmetric with respect to the inner product on $L^2(\mathbb{R})$, if it satisfies*

$$(Au, v) = -(u, Av).$$

Definition 14. ***(Variational Derivative)*** *The variational derivative of sufficiently smooth functional G of u is defined by*

$$(G'(u), v) = \frac{d}{d\varepsilon}G(u + \varepsilon v)|_{\varepsilon=0}.$$

Throughout this study, $C = C_i (i \in \mathbb{N})$ denotes a generic constant.

2.2 Conserved Quantities

In this section, we derive the conserved quantities of the gfBBM equation (1.1) for the smooth enough solutions with

$$u(x,t) \to 0, \qquad x \to \mp\infty.$$

First, we rewrite the equation (1.1) in the conservative like form

$$(I + \frac{5}{4}D^\alpha)u_t + (u + \frac{1}{2}u^{p+1} + \frac{3}{4}D^\alpha u)_x = 0.$$

By using the fact that the operator $(I + \frac{5}{4}D^\alpha)$ is invertible, we are allowed to write

$$u_t + \partial_x(I + \frac{5}{4}D^\alpha)^{-1}(u + \frac{1}{2}u^{p+1} + \frac{3}{4}D^\alpha u) = 0. \tag{2.3}$$

Therefore, we have obviously first conserved quantity as

$$I = \int_{\mathbb{R}} u(x,t)dx. \tag{2.4}$$

Multiplying the equation (1.1) by u

$$uu_t + uu_x + \frac{1}{2}u(u^{p+1})_x + \frac{3}{4}uD^\alpha u_x + \frac{5}{4}uD^\alpha u_t = 0,$$

and integrating result on the whole line, we have

$$\int_{\mathbb{R}} uu_t dx + \int_{\mathbb{R}} uu_x dx + \frac{1}{2}\int_{\mathbb{R}} u(u^{p+1})_x dx + \frac{3}{4}\int_{\mathbb{R}} uD^\alpha u_x dx + \frac{5}{4}\int_{\mathbb{R}} uD^\alpha u_t dx = 0.$$

By the help of integration by parts technique, we rewrite the above equation as

$$\begin{aligned}\int_{\mathbb{R}} (\frac{u^2}{2})_t dx + \int_{\mathbb{R}} (\frac{u^2}{2})_x dx &+ \frac{1}{2}\frac{p+1}{p+2}\int_{\mathbb{R}} (u^{p+2})_x dx \\ &+ \frac{3}{4}\int_{\mathbb{R}} uD^\alpha u_x dx + \frac{5}{4}\int_{\mathbb{R}} uD^\alpha u_t dx = 0.\end{aligned} \tag{2.5}$$

To deal with the last two terms of the equation (2.5), we use Plancherel's formula which yields that

$$\begin{aligned}\int_{\mathbb{R}} (D^\alpha u_t)u dx &= \int_{\mathbb{R}} |\xi|^\alpha \hat{u}_t(\xi,t)\overline{\hat{u}(\xi,t)}d\xi, \\ &= \int_{\mathbb{R}} |\xi|^{\frac{\alpha}{2}}\hat{u}_t|\xi|^{\frac{\alpha}{2}}\bar{\hat{u}}d\xi, \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |D^{\frac{\alpha}{2}}u|^2 dx\end{aligned} \tag{2.6}$$

and

$$\frac{3}{4}\int_{\mathbb{R}} (D^\alpha u_x)u dx = \frac{3}{8}\int_{\mathbb{R}} |D^{\frac{\alpha}{2}}u|_x^2 dx. \tag{2.7}$$

Using (2.6) and (2.7) in (2.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (u^2 + \frac{5}{4}|D^{\frac{\alpha}{2}}u|^2)dx + \frac{1}{2}\int_{\mathbb{R}} (u^2 + \frac{p+1}{p+2}u^{p+2} + \frac{3}{4}|D^{\frac{\alpha}{2}}u|^2)_x dx = 0.$$

Finally, the last equation allows us to write second conserved quantity in the following form

$$F = \int_{\mathbb{R}} (u^2 + \frac{5}{4}|D^{\frac{\alpha}{2}}u|^2)dx. \tag{2.8}$$

On the other hand, the equivalent form (2.3) of the equation (1.1) gives us the Hamiltonian formulation

$$u_t + JH'(u) = 0. \tag{2.9}$$

Here, $'$ denotes variational derivative. J is skew-adjoint operator with

$$J = \partial_x (I + \frac{5}{4} D^{\alpha})^{-1}, \tag{2.10}$$

and H is the Hamiltonian with

$$H(u) = \frac{1}{2} \int_{\mathbb{R}} \left(u^2 + \frac{u^{p+2}}{p+2} + \frac{3}{4} |D^{\frac{\alpha}{2}} u|^2\right) dx. \tag{2.11}$$

We note that some restriction on α occurs so that $H(u)$ is well-defined. According to Sobolev imbedding theorem we have

$$H^{\frac{\alpha}{2}} \hookrightarrow L^{p+2}, \quad \text{for} \quad p+2 \leq \frac{2}{1 - \frac{2\alpha}{2}}.$$

In particular, the Hamiltonian together with the L^{p+2} norm is able to control the $H^{\alpha/2}$ norm when $\alpha \geq p/(p+2)$. This implies that the Hamiltonian does not make sense when $0 < \alpha < p/(p+2)$.

2.3 Local Well-posedness

In this chapter, we address the question on the local existence and uniqueness of solutions to the Cauchy problem for the gfBBM equation

$$u_t + u_x + \frac{1}{2}(u^{p+1})_x + \frac{3}{4} D^{\alpha} u_x + \frac{5}{4} D^{\alpha} u_t = 0, \tag{2.12}$$

$$u(x,0) = u_0(x). \tag{2.13}$$

To prove the local existence and uniqueness of the Cauchy problem we separate it into the following cases: $1 \leq \alpha \leq 2$ and $0 < \alpha < 1$.

2.3.1 The case $1 \leq \alpha \leq 2$

The Cauchy problem (2.12)-(2.13) is rewritten as

$$u_t + (I + \frac{3}{4} D^{\alpha})(I + \frac{5}{4} D^{\alpha})^{-1} u_x = -\frac{1}{2}(I + \frac{5}{4} D^{\alpha})^{-1} (u^{p+1})_x \tag{2.14}$$

$$u(x,0) = u_0(x). \tag{2.15}$$

Here, we note that the operator $I + \frac{5}{4} D^{\alpha}$ is invertible, as its Fourier transform is never zero. Due to the Duhamel formula we can extend the solution of homogenous problem to the nonhomogeneous one. Thus, this formula implies that u is the solution of the

Cauchy problem (2.14)-(2.15) if, and only if, u satisfies the integral equation $u = \Phi u$ where

$$(\Phi u)(x,t) = S(t)u_0(x) - \frac{1}{2}\int_0^t S(t-\tau)\big[\partial_x(I+\frac{5}{4}D^\alpha)^{-1}u^{p+1}\big](x,\tau)d\tau, \tag{2.16}$$

with

$$S(t)u = \mathscr{F}^{-1}\left(e^{-\frac{1+\frac{3}{4}|\xi|^\alpha}{1+\frac{5}{4}|\xi|^\alpha}i\xi t}\right) * u(x).$$

The symbol $*$ denotes the convolution operation.

Now, we prove the existence and uniqueness of the local solution for the problem (2.12)-(2.13) by using the contraction mapping principle for $1 \leq \alpha \leq 2$.

Theorem 2. *Let $1 \leq \alpha \leq 2$ be fixed and $u_0 \in H^r(\mathbb{R})$ with $r \geq \alpha/2$. Then there exists a unique local solution $u \in C([0,T'), H^r(\mathbb{R}))$ of the Cauchy problem* (2.12)-(2.13)*, where $T' = T'(||u_0||_{H^r}) > 0$.*

Proof. For $T > 0$, define $\overline{B}_T$ as the closed ball

$$\overline{B}_T = \{u \in C([0,T], H^r(\mathbb{R})), \|u\|_{L^\infty([0,T],H^r(\mathbb{R}))} \leq 2\|u_0\|_{H^r}\}.$$

We first prove Φ maps $\overline{B}_T$ into $\overline{B}_T$ for $T > 0$ small enough and a convenient choice of $a > 0$. From (2.16), one gets

$$\begin{aligned}\|\Phi u(t)\|_{H^r} &= \|S(t)u_0 - \frac{1}{2}\int_0^t S(t-\tau)\big[\partial_x(I+\frac{5}{4}D^\alpha)^{-1}(u)^{p+1}\big](\tau)d\tau\|_{H^r} \\ &\leq \|S(t)u_0\|_{H^r} + \frac{1}{2}\int_0^t \|S(t-\tau)\big[\partial_x(I+\frac{5}{4}D^\alpha)^{-1}u^{p+1}(\tau)\big]\|_{H^r}d\tau.\end{aligned} \tag{2.17}$$

The definition of Sobolev norm allows us to estimate the following term

$$\begin{aligned}\|S(t)u_0\|_{H^r}^2 &= \int_{\mathbb{R}}(1+|k|^2)^r|e^{-\frac{1+\frac{3}{4}|k|^\alpha}{1+\frac{5}{4}|k|^\alpha}ikt}\hat{u}_0(k)|^2dk, \\ &\leq \int_{\mathbb{R}}(1+|k|^2)^r|\hat{u}_0(k)|^2dk, \\ &= \|u_0\|_{H^r}^2.\end{aligned} \tag{2.18}$$

Using Lemma 1 and $1+\frac{5}{4}|k|^{\alpha} \geq |k|$ for $\alpha \geq 1$, one gets similarly to (2.18) that

$$\begin{aligned} \|S(t-\tau)\left[\partial_x(I+\tfrac{5}{4}D^{\alpha})^{-1}u^{p+1}(t)\right]\|_{H^r}^2 &\leq \int_{\mathbb{R}}(1+|k|^2)^r\left|\frac{ik}{1+\frac{5}{4}|k|^{\alpha}}\widehat{u^{p+1}}(k,t)\right|^2 dk \\ &\leq \int_{\mathbb{R}}(1+|k|^2)^r|\widehat{u^{p+1}}(k,t)|^2 dk \\ &= \|u^{p+1}(t)\|_{H^r}^2 \\ &\leq C\|u(t)\|_{H^r}^{2(p+1)}. \end{aligned} \tag{2.19}$$

Using (2.18) and (2.19) in (2.17), we obtain

$$\|\Phi u(t)\|_{H^r} \leq \|u_0\|_{H^r} + \frac{CT}{2}\sup_{t\in[0,T]}\|u(t)\|_{H^r}^{p+1} \leq 2\|u_0\|_{H^r}$$

By choosing $T=T'>0$ small enough and satisfying $T \leq \dfrac{2}{C\|u_0\|_{H^r}^{p}}$, we have that Φ maps $\overline{B}_T$ into $\overline{B}_T$.

In the second step of proof we show that Φ is a strict contraction. Let u_1, $u_2 \in \overline{B}_T$. The Duhamel formula (2.16) gives

$$\|\Phi u_1(t)-\Phi u_2(t)\|_{H^r} \leq \frac{1}{2}\int_0^t \|S(t-\tau)\partial_x(I+\frac{5}{4}D^{\alpha})^{-1}\left[u_1^{p+1}-u_2^{p+1}\right](\tau)\|_{H^r}d\tau \tag{2.20}$$

and Lemma 2 imply

$$\begin{aligned} \|\Phi u_1(t)-\Phi u_2(t)\|_{H^r} &\leq C\int_0^t K(M)\|u_1(\tau)-u_2(\tau)\|_{H^r}d\tau, \\ &\leq CTK(M)\sup_{t\in[0,T]}\|u_1(t)-u_2(t)\|_{H^r}. \end{aligned}$$

where $\|u_1\|_{H^r} \leq M$ and $\|u_2\|_{H^r} \leq M$. If we choose $T < \frac{1}{CK(M)}$, then Φ is strictly contractive.

To obtain the continuity with respect to the initial data, we consider the solutions u and v in $H^r(\mathbb{R})$ corresponding to initial conditions u_0 and v_0, respectively, with $\|u_0\|_{H^r} \leq M$ and $\|v_0\|_{H^r} \leq M$. Computations similar as above show that

$$\begin{aligned} &\|u(t)-v(t)\|_{H^r} \\ &\quad \leq \|S(t)\big(u_0-v_0\big)\|_{H^r} + \int_0^t \|S(t-\tau)\partial_x(I+\frac{5}{4}D^{\alpha})^{-1}\big(u^{p+1}-v^{p+1}\big)(\tau)\|_{H^r}d\tau \\ &\quad \leq \|u_0-v_0\|_{H^r} + CTK(M)\sup_{t\in[0,T]}\|u(t)-v(t)\|_{H^r}. \end{aligned}$$

The inequality

$$\sup_{t\in[0,T]} \|u(t)-v(t)\|_{H^r} \leq \frac{1}{1-CTK(M)}\|u_0-v_0\|_{H^r}$$

with $1-CTK(M)>0$ yields that the solution depends continuously on the given initial data as it is bounded by a continuous function related to the difference of the initial data. □

2.3.2 The case $0<\alpha<1$

The local well-posedness of solutions in Sobolev spaces for the Cauchy problem to the fBBM equation with the quadratic nonlinearity

$$\begin{aligned} &u_t+u_x+uu_x+D^\alpha u_t=0,\\ &u(x,0)=u_0(x), \end{aligned}$$

is studied by using energy estimation technique given in [5] in which the authors consider $0<\alpha<1$. However, this method does not provide the uniqueness since one of the terms can not be controlled by the appropriate Sobolev norm. The same problem is also observed for the equation (1.1). Therefore, we use the vanishing viscosity method given in [33] and consider the following regularization of the equation (1.1)

$$u_t^\varepsilon+u_x^\varepsilon+\frac{1}{2}[(u^\varepsilon)^{p+1}]_x+\frac{3}{4}D^\alpha u_x^\varepsilon+\frac{5}{4}D^\alpha u_t^\varepsilon-\varepsilon u_{xx}^\varepsilon=0, \tag{2.21}$$

$$u^\varepsilon(x,0)=u_0^\varepsilon(x). \tag{2.22}$$

The equation (2.21) is rewritten as

$$(I+\frac{5}{4}D^\alpha)u_t^\varepsilon+(I+\frac{3}{4}D^\alpha)u_x^\varepsilon-\varepsilon u_{xx}^\varepsilon=-\frac{1}{2}[(u^\varepsilon)^{p+1}]_x.$$

Since the operator $(I+\frac{5}{4}D^\alpha)$ is invertible, we are allowed to write above equation in the following form

$$u_t^\varepsilon+(I+\frac{3}{4}D^\alpha)(I+\frac{5}{4}D^\alpha)^{-1}u_x^\varepsilon-(I+\frac{5}{4}D^\alpha)^{-1}\varepsilon u_{xx}^\varepsilon=-\frac{1}{2}(I+\frac{5}{4}D^\alpha)^{-1}[(u^\varepsilon)^{p+1}]_x. \tag{2.23}$$

According to Duhamel formula u^ε is the solution of the Cauchy problem (2.21)-(2.22) if and only if u^ε is the solution of the integral equation $u^\varepsilon=\Phi^\varepsilon u^\varepsilon$ where

$$(\Phi^\varepsilon u^\varepsilon)(x,t)=S(t)u_0^\varepsilon(x)-\frac{1}{2}\int_0^t S(t-\tau)\Big[\frac{\partial_x}{(I+\frac{5}{4}D^\alpha)}(u^\varepsilon)^{p+1}\Big](x,\tau)d\tau, \tag{2.24}$$

where

$$S(t)u = \mathscr{F}^{-1}\left(e^{-\left[\frac{1+\frac{3}{4}|k|^{\alpha}}{1+\frac{5}{4}|k|^{\alpha}}ik+\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^{\alpha}}\right]t}\right) * u(x).$$

We need the following lemmas in order to proceed with the fixed point theorem:

Lemma 3. *Let* $0 < \alpha < 1$ *and* $r \geq 0$. *We have*

$$\|\frac{\partial_x}{(I+\frac{5}{4}D^{\alpha})}S(t)(uv)\|_{H^r} \leq C(\varepsilon,t)\|u\|_{H^r}\|v\|_{H^r}, \tag{2.25}$$

where

$$S(t)u = \mathscr{F}^{-1}\left(e^{-\left[\frac{1+\frac{3}{4}|k|^{\alpha}}{1+\frac{5}{4}|k|^{\alpha}}ik+\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^{\alpha}}\right]t}\right) * u(x), \qquad C(\varepsilon,t) = C(\varepsilon t)^{-\frac{\frac{3}{2}-\alpha}{2-\alpha}}. \tag{2.26}$$

Proof. By duality, proving the lemma is equivalent to proving

$$\int_{\mathbb{R}} \frac{\partial_x}{(I+\frac{5}{4}D^{\alpha})}S(t)(u(x)v(x))\overline{w}(x)dx \leq C(\varepsilon,t)\|u\|_{H^r}\|v\|_{H^r}\|w\|_{H^{-r}}$$

for all $w \in \mathscr{S}(\mathbb{R})$. Thanks to the Plancherel's identity and the nice property of convolution opeator, one gets

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\partial_x}{(I+\frac{5}{4}D^{\alpha})}S(t)(u(x)v(x))\overline{w}(x)dx \\
&\qquad = \int_{\mathbb{R}} \frac{ik}{1+\frac{5}{4}|k|^{\alpha}}\left[e^{-\left(\frac{1+\frac{3}{4}|k|^{\alpha}}{1+\frac{5}{4}|k|^{\alpha}}ik+\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^{\alpha}}\right)t}\right]\widehat{u(k)v(k)}\widehat{\overline{w}}(k)dk \\
&\qquad = \int_{\mathbb{R}} \frac{ik}{1+\frac{5}{4}|k|^{\alpha}}\left[e^{-\left(\frac{1+\frac{3}{4}|k|^{\alpha}}{1+\frac{5}{4}|k|^{\alpha}}ik+\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^{\alpha}}\right)t}\right](\widehat{u}*\widehat{v})(k)\widehat{\overline{w}}(k)dk \\
&\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{ik}{1+\frac{5}{4}|k|^{\alpha}}\left[e^{-\left(\frac{1+\frac{3}{4}|k|^{\alpha}}{1+\frac{5}{4}|k|^{\alpha}}ik+\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^{\alpha}}\right)t}\right]\widehat{u}(k-\eta)\widehat{v}(\eta)\widehat{\overline{w}}(k)d\eta dk.
\end{aligned}$$

Let us define

$$\widehat{U} = \langle k\rangle^r\widehat{u}, \qquad \widehat{V} = \langle k\rangle^r\widehat{v}, \qquad \widehat{W} = \langle k\rangle^{-r}\widehat{w}, \tag{2.27}$$

where $\langle k\rangle = (1+k^2)^{1/2}$. By using the triangle inequality $\langle k\rangle^r \leq C\langle k-\eta\rangle^r\langle\eta\rangle^r$, we have

$$
\begin{aligned}
&| \int_{\mathbb{R}} \frac{\partial_x}{(I+\frac{5}{4}D^\alpha)} S(t)(u(x)v(x))\overline{w}(x)dx \,| \\
&= \left| \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{ik}{1+\frac{5}{4}|k|^\alpha} e^{-\left[\frac{1+\frac{3}{4}|k|^\alpha}{1+\frac{5}{4}|k|^\alpha} ik + \frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}\right]t} \frac{\langle k\rangle^r}{\langle k-\eta\rangle^r \langle \eta\rangle^r} \widehat{U}(k-\eta)\widehat{V}(\eta)\widehat{\overline{W}}(k)d\eta dk \right| \\
&\le C\|W\|_{L^2} \;\; \| \frac{ik}{1+\frac{5}{4}|k|^\alpha} e^{-\left[\frac{1+\frac{3}{4}|k|^\alpha}{1+\frac{5}{4}|k|^\alpha} ik + \frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}\right]t} \widehat{U}*\widehat{V}\|_{L^2} \\
&\le C\|W\|_{L^2}\|\widehat{U}*\widehat{V}\|_{L^\infty} \| \frac{|k|}{1+\frac{5}{4}|k|^\alpha} e^{-\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}t} \|_{L^2} \\
&\le C\|W\|_{L^2}\|U\|_{L^2}\|V\|_{L^2} \| \frac{|k|}{1+\frac{5}{4}|k|^\alpha} e^{-\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}t} \|_{L^2}
\end{aligned}
$$

The definition of L^2-norm provides

$$
\begin{aligned}
\| \frac{|k|}{1+\frac{5}{4}|k|^\alpha} e^{-\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}t} \|_{L^2}^2 &= \int_{\mathbb{R}} \left(\frac{|k|}{1+\frac{5}{4}|k|^\alpha} e^{-\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}t} \right)^2 dk \\
&= 2\int_0^1 \left(\frac{|k|}{1+\frac{5}{4}|k|^\alpha} e^{-\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}t} \right)^2 dk + 2\int_1^\infty \left(\frac{|k|}{1+\frac{5}{4}|k|^\alpha} e^{-\frac{\varepsilon k^2}{1+\frac{5}{4}|k|^\alpha}t} \right)^2 dk \\
&= I + II.
\end{aligned}
$$

We have

$$
\begin{aligned}
I &= 2\int_0^1 \left(\frac{|k|}{1+\frac{5}{4}|k|^\alpha} \right)^2 \frac{1}{\left(e^{\frac{k^2}{1+\frac{5}{4}|k|^\alpha}}\right)^{2\varepsilon t}} dk \\
&\le C\int_0^1 \left(\frac{|k|}{1+\frac{5}{4}|k|^\alpha} \right)^2 dk \\
&\le C\int_0^1 \frac{1}{|k|^{2(\alpha-1)}} dk.
\end{aligned}
$$

Above integral is convergent for $\alpha < \frac{3}{2}$. The inequality $e^{-x} \le \dfrac{1}{x^\theta}$ for $x > 0$ and $\theta \in \mathbb{R}$ yields that

$$
\begin{aligned}
II &= 2\int_1^\infty \left(\frac{|k|}{1+\frac{5}{4}|k|^\alpha} \right)^2 \frac{1}{\left(\frac{\varepsilon k^2}{(1+\frac{5}{4}|k|^\alpha)} t \right)^{2\theta}} dk \\
&\le C\frac{1}{(\varepsilon t)^{2\theta}} \int_1^\infty \frac{1}{|k|^{2(\alpha-1)}} |k|^{2\theta(\alpha-2)} dk \\
&= C\frac{1}{(\varepsilon t)^{2\theta}} \int_1^\infty \frac{1}{|k|^{2(\alpha-1)+2\theta(2-\alpha)}} dk
\end{aligned}
$$

Here we note that the last integral is convergent for $\theta > \frac{\frac{3}{2}-\alpha}{2-\alpha}$. Finally we obtain

$$| \int_{\mathbb{R}} \frac{\partial_x}{(I+\frac{5}{4}D^\alpha)} S(t)(u(x)v(x))\overline{w}(x)dx | \leq C(\varepsilon,t)\|W\|_{L^2}\|U\|_{L^2}\|V\|_{L^2}$$

for $C(\varepsilon,t) = C(\varepsilon t)^{-\theta}$ with $\theta > \frac{\frac{3}{2}-\alpha}{2-\alpha}$. By using the definitions in (2.27), we obtain

$$| \int_{\mathbb{R}} \frac{\partial_x}{(I+\frac{5}{4}D^\alpha)} S(t)(u(x)v(x))\overline{w}(x)dx | \leq C(\varepsilon,t)\|w\|_{H^{-r}}\|u\|_{H^r}\|v\|_{H^r}.$$

□

Lemma 1 leads to the following corollary.

Corollary 1. *Let* $0 < \alpha < 1$ *and* $r > 1/2$. *We have*

$$\|\frac{\partial_x}{(I+\frac{5}{4}D^\alpha)} S(t)(u^p)\|_{H^r} \leq C(\varepsilon,t)\|u\|^p_{H^r}, \tag{2.28}$$

where the operator $S(t)$ *and* $C(\varepsilon,t)$ *are given in the equation* (2.26).

Now, we prove the existence and uniqueness of the local solution for the reqularized problem (2.21)-(2.22) by using the contraction mapping principle.

Theorem 3. *Assume that* $u_0^\varepsilon \in H^r(\mathbb{R})$ *with* $r > 1/2$ *and* $0 < \alpha < 1$. *Then, for any* $\varepsilon > 0$ *there exists a unique solution* $u^\varepsilon \in C([0,T_\varepsilon], H^r(\mathbb{R}))$ *to the Cauchy problem* (2.21)-(2.22)*, where* $T_\varepsilon = T(\|u_0^\varepsilon\|, \varepsilon)$.

Proof. Let $\overline{B}_T$ with $T > 0$ be the closed ball

$$\overline{B}_T = \{u \in C([0,T], H^r(\mathbb{R})), \|u\|_{L^\infty([0,T],H^r(\mathbb{R}))} \leq 2\|u_0\|_{H^r(\mathbb{R})}\}.$$

We first prove Φ^ε maps $\overline{B}_T$ into $\overline{B}_T$ for T small enough. From (2.24), one gets

$$\begin{aligned} \|\Phi^\varepsilon u^\varepsilon(t)\|_{H^r} &= \|S(t)u_0^\varepsilon - \frac{1}{2}\int_0^t S(t-\tau)\big[\frac{\partial_x}{(I+\frac{5}{4}D^\alpha)}(u^\varepsilon)^{p+1}\big](\tau)d\tau\|_{H^r} \\ &\leq \|u_0^\varepsilon\|_{H^r} + \frac{1}{2}\int_0^t \|S(t-\tau)\big[\frac{\partial_x}{(I+\frac{5}{4}D^\alpha)}(u^\varepsilon)^{p+1}(\tau)\big]\|_{H^r}d\tau. \end{aligned}$$

The Corollary 1 implies that

$$\|S(t-\tau)\big[\frac{\partial_x}{(I+\frac{5}{4}D^\alpha)}(u^\varepsilon)^{p+1}(\tau)\big]\|_{H^r} \leq C(\varepsilon(t-\tau))^{-\frac{\frac{3}{2}-\alpha}{2-\alpha}}\|u^\varepsilon(t)\|^{p+1}_{H^r}. \tag{2.29}$$

Using (2.29), we have

$$\|\Phi^\varepsilon u^\varepsilon(t)\|_{H^r} \leq \|u_0^\varepsilon\|_{H^r} + C_{\varepsilon,r} T \sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{H^r}^{p+1}.$$

By choosing T small enough to satisfy $T \leq \dfrac{1}{C_{\varepsilon,r} 2^{p+1} \|u_0^\varepsilon\|_{H^r}^p}$ gives that Φ^ε maps $\overline{B}_T$ into $\overline{B}_T$.

The next step is to prove that Φ^ε is contractive. Let u_1^ε, $u_2^\varepsilon \in \overline{B}_T$. The Duhamel formula (2.24) gives

$$\|\Phi^\varepsilon u_1^\varepsilon(t) - \Phi^\varepsilon u_2^\varepsilon(t)\|_{H^r} \leq \frac{1}{2}\int_0^t \|S(t-\tau)\frac{\partial_x}{(I+\frac{5}{4}D^\alpha)}\left[(u_1^\varepsilon)^{p+1} - (u_2^\varepsilon)^{p+1}\right](\tau)\|_{H^r} d\tau \tag{2.30}$$

and Corollary 1 and Lemma 2 imply

$$\begin{aligned} \|\Phi^\varepsilon u_1^\varepsilon(t) - \Phi^\varepsilon u_2^\varepsilon(t)\|_{H^r} &\leq C\int_0^t (\varepsilon(t-\tau))^{-\frac{\frac{3}{2}-\alpha}{2-\alpha}} \|(u_1^\varepsilon)^{p+1}(\tau) - (u_2^\varepsilon)^{p+1}(\tau)\|_{H^r} d\tau \\ &\leq C_{\varepsilon,r} T K(M) \sup_{t\in[0,T]} \|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{H^r}. \end{aligned}$$

where $\|u_1^\varepsilon\|_{H^r} \leq M$ and $\|u_2^\varepsilon\|_{H^r} \leq M$. If we choose $T < \frac{1}{C_{\varepsilon,r}K(M)}$, then Φ^ε is strictly contractive such that

$$\|\Phi^\varepsilon u_1^\varepsilon(t) - \Phi^\varepsilon u_2^\varepsilon(t)\|_{H^r} \leq \|u_1^\varepsilon(t) - u_2^\varepsilon(t)\|_{H^r}.$$

To obtain the continuity with respect to the initial data, we consider the solutions u^ε and v^ε in $H^r(\mathbb{R})$ corresponding to initial conditions u_0^ε and v_0^ε, respectively, with $\|u_0^\varepsilon\|_{H^r} \leq M$ and $\|v_0^\varepsilon\|_{H^r} \leq M$. Similar computations show that

$$\begin{aligned} &\|u^\varepsilon(t) - v^\varepsilon(t)\|_{H^r} \\ &\quad \leq \|S(t)\big(u_0^\varepsilon - v_0^\varepsilon\big)\|_{H^r} + \int_0^t \|S(t-\tau)\frac{\partial_x}{(I+\frac{5}{4}D^\alpha)}\big((u^\varepsilon)^{p+1} - (v^\varepsilon)^{p+1}\big)(\tau)\|_{H^r} d\tau \\ &\quad \leq \|u_0^\varepsilon - v_0^\varepsilon\|_{H^r} + C_{\varepsilon,r} T K(M) \sup_{t\in[0,T]} \|u^\varepsilon(t) - v^\varepsilon(t)\|_{H^r}. \end{aligned}$$

The inequality

$$\sup_{t\in[0,T]} \|u^\varepsilon(t) - v^\varepsilon(t)\|_{H^r} \leq \frac{1}{1 - C_{\varepsilon,r} T K(M)} \|u_0^\varepsilon - v_0^\varepsilon\|_{H^r}$$

with $1 - C_{\varepsilon,r} T K(M) > 0$ yields that the solution depends continuously on the given initial data since it is bounded by a continuous function related to the difference of the initial data. □

Theorem 4. *Let* $0<\alpha<1$ *and the initial data in* $u_0^\varepsilon \in H^r(\mathbb{R})$ *with* $r>\frac{3}{2}-\alpha$. *Then the unique regularized solution* $C([0,T],H^r(\mathbb{R}))$ *to the equations* (2.21)-(2.22) *satisfies the energy inequality*

$$\frac{d}{dt}\|u^\varepsilon(t)\|^2_{H^r(\mathbb{R})} \le C\|u^\varepsilon(t)\|^{p+2}_{H^r(\mathbb{R})}. \tag{2.31}$$

Proof. Let us set $r=s+\frac{\alpha}{2}$. Applying the Bessel potential J^s to the equation (2.21), multiplying both sides of the equation by $J^s u^\varepsilon$ and then integrating on the whole line, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(|J^s u^\varepsilon|^2+\frac{5}{4}|J^{s+\frac{\alpha}{2}}u^\varepsilon|^2)dx+\varepsilon\frac{d}{dt}\int_{\mathbb{R}}(J^s u_x^\varepsilon)^2dx=-\frac{p+1}{2}\int_{\mathbb{R}}J^s u^\varepsilon J^s\big((u^\varepsilon)^p u_x^\varepsilon\big)dx. \tag{2.32}$$

Using the fractional Leibniz rule, the term $J^s\big((u^\varepsilon)^p u_x^\varepsilon\big)$ is written as

$$J^s\big((u^\varepsilon)^p u_x^\varepsilon\big)=(u^\varepsilon)^p J^s u_x^\varepsilon+u_x^\varepsilon J^s(u^\varepsilon)^p+R.$$

Here, R denotes remainder terms. Then, the equation (2.32) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(|J^s u^\varepsilon|^2 \quad &+ \quad \frac{5}{4}|J^{s+\frac{\alpha}{2}}u^\varepsilon|^2)dx+\varepsilon\frac{d}{dt}\int_{\mathbb{R}}(J^s u_x^\varepsilon)^2dx\\ &= \quad -\frac{p+1}{2}\left(\int_{\mathbb{R}}(u^\varepsilon)^p J^s u_x^\varepsilon J^s u^\varepsilon dx+\int_{\mathbb{R}}u_x^\varepsilon J^s(u^\varepsilon)^p J^s u^\varepsilon dx+\int_{\mathbb{R}}RJ^s u^\varepsilon dx\right).\end{aligned} \tag{2.33}$$

Using the integration by parts first term of RHS can be written as

$$\int_{\mathbb{R}}(u^\varepsilon)^p J^s u_x^\varepsilon J^s u^\varepsilon dx=-\frac{1}{2}\int_{\mathbb{R}}[(u^\varepsilon)^p]_x(J^s u^\varepsilon)^2dx. \tag{2.34}$$

By using the Hölder's inequality, (2.34) is estimated as

$$\begin{aligned}\int_{\mathbb{R}}|[(u^\varepsilon)^p]_x(J^s u^\varepsilon)^2|dx &\le C\|[(u^\varepsilon)^p]_x\|_{L^{q_1}}\|J^s u^\varepsilon\|^2_{L^{q_2}}\\ &\le C\|[(u^\varepsilon)^p]_x\|_{H^{s+\frac{\alpha}{2}-1}}\|J^s u^\varepsilon\|^2_{H^{\frac{\alpha}{2}}}\\ &\le C\|(u^\varepsilon)^p\|_{H^{s+\frac{\alpha}{2}}}\|J^s u^\varepsilon\|^2_{H^{\frac{\alpha}{2}}}\\ &\le C\|u^\varepsilon\|^p_{H^{s+\frac{\alpha}{2}}}\|J^s u^\varepsilon\|^2_{H^{\frac{\alpha}{2}}}\\ &\le C\|u^\varepsilon\|^{p+2}_{H^{s+\frac{\alpha}{2}}}.\end{aligned} \tag{2.35}$$

Here, we have used the following Sobolev imbeddings

$$\begin{aligned}H^{s+\frac{\alpha}{2}-1} &\hookrightarrow L^{q_1}, \text{ for } q_1\le\frac{2}{3-2s-\alpha},\\ H^{\frac{\alpha}{2}} &\hookrightarrow L^{q_2}, \text{ for } q_2\le\frac{2}{1-\alpha},\end{aligned} \tag{2.36}$$

where $\frac{1}{q_1}+\frac{2}{q_2}\leq 1$ implies $s\geq\frac{3}{2}(1-\alpha)$. By Lemma 1, $u^\varepsilon\in H^{s+\frac{\alpha}{2}}(\mathbb{R})$ ensures $(u^\varepsilon)^p\in H^{s+\frac{\alpha}{2}}(\mathbb{R})$, as the condition $s\geq\frac{3}{2}(1-\alpha)$ guarantees that $H^{s+\frac{\alpha}{2}}(\mathbb{R})$ is an algebra. The estimation of the second term in RHS of the equation (2.33) is given as

$$\begin{aligned}\int_{\mathbb{R}}|u_x^\varepsilon J^s u^\varepsilon J^s(u^\varepsilon)^p|dx &\leq \|u_x^\varepsilon\|_{L^{q_3}}\|J^s u^\varepsilon\|_{L^{q_4}}\|J^s(u^\varepsilon)^p\|_{L^{q_5}}\\ &\leq C\|u_x^\varepsilon\|_{H^{s+\frac{\alpha}{2}-1}}\|J^s u^\varepsilon\|_{H^{\frac{\alpha}{2}}}\|J^s(u^\varepsilon)^p\|_{H^{\frac{\alpha}{2}}}\\ &\leq C\|u^\varepsilon\|^{p+2}_{H^{s+\frac{\alpha}{2}}},\end{aligned}\tag{2.37}$$

where $\frac{1}{q_3}+\frac{1}{q_4}+\frac{1}{q_5}\leq 1$. Here, the equations

$$\begin{aligned}\|J^s u\|^2_{H^{\frac{\alpha}{2}}} &= \int_{\mathbb{R}}(1+k^2)^{\frac{\alpha}{2}}|\widehat{J^s u}|^2dk\\ &= \int_{\mathbb{R}}(1+k^2)^{\frac{\alpha}{2}+s}|\widehat{u}|^2dk=\|u\|^2_{H^{s+\frac{\alpha}{2}}},\end{aligned}$$

and

$$\|J^s u^p\|^2_{H^{\frac{\alpha}{2}}} = \|u^p\|^2_{H^{s+\frac{\alpha}{2}}}$$

help us for the estimation (2.37). Additionally, the Sobolev imbeddings used in (2.37)

$$\begin{aligned}H^{s+\frac{\alpha}{2}-1} &\hookrightarrow L^{q_3},\ \text{for}\ q_3\leq\frac{2}{3-2s-\alpha},\\ H^{\frac{\alpha}{2}} &\hookrightarrow L^{q_4},\ \text{for}\ q_4\leq\frac{2}{1-\alpha},\\ H^{\frac{\alpha}{2}} &\hookrightarrow L^{q_5},\ \text{for}\ q_5\leq\frac{2}{1-\alpha},\end{aligned}$$

provide $s\geq\frac{3}{2}(1-\alpha)$. The estimation of last term of the equation (2.33),

$$\int_{\mathbb{R}}|RJ^s u^\varepsilon|dx\leq\|R\|_{L^{2/(1+\alpha)}}\|J^s u^\varepsilon\|_{L^{2/(1-\alpha)}}\leq C\|R\|_{L^{2/(1+\alpha)}}\|u^\varepsilon\|_{H^{s+\frac{\alpha}{2}}},$$

follows directly from Hölder's inequality and the Sobolev imbedding $H^{\frac{\alpha}{2}}\hookrightarrow L^{2/(1-\alpha)}$. Following [5, 30] and using

$$\begin{aligned}H^{\frac{\alpha}{2}+\mu}(\mathbb{R}) &\hookrightarrow L^{4/(1+\alpha)}, & \frac{4}{1+\alpha}&\leq\frac{2}{1-2(\frac{\alpha}{2}+\mu)},\\ H^{s+\frac{\alpha}{2}-1-\mu}(\mathbb{R}) &\hookrightarrow L^{4/(1+\alpha)}, & \frac{4}{1+\alpha}&\leq\frac{2}{1-2(s+\frac{\alpha}{2}-1-\mu)}\end{aligned}$$

for any $0<\mu<s$, one gets

$$\begin{aligned}\|R\|_{L^{2/(1+\alpha)}} &\leq C\|J^{s-\mu}(u^\varepsilon)^p\|_{L^{4/(1+\alpha)}}\|J^\mu(u^\varepsilon)_x\|_{L^{4/(1+\alpha)}}\\ &\leq C\|u^\varepsilon\|^p_{H^{s+\frac{\alpha}{2}}}\|u^\varepsilon\|_{H^{s+\frac{\alpha}{2}}},\end{aligned}$$

and finally

$$\int_{\mathbb{R}} |RJ^s u^\varepsilon| dx \leq C\|u^\varepsilon\|^{p+2}_{H^{s+\frac{\alpha}{2}}}. \tag{2.38}$$

Choosing a suitable μ, last restriction provides $s \geq \frac{3}{2}(1-\alpha)$ as above. Combining the equations (2.35), (2.37) and (2.38), we have

$$\frac{d}{dt}\|J^{s+\frac{\alpha}{2}}u^\varepsilon(t)\|^2_{L^2} \leq C\|J^{s+\frac{\alpha}{2}}u^\varepsilon(t)\|^{p+2}_{L^2}.$$

From $\|J^s u\|_{L^2} = \|u\|_{H^s}$, the energy estimate is given by

$$\frac{d}{dt}\|u^\varepsilon(t)\|^2_{H^r} \leq C\|u^\varepsilon(t)\|^{p+2}_{H^r}. \tag{2.39}$$

According to (2.39), one can write $\|u^\varepsilon(t)\|^2_{H^r} \leq y(t)$ where $y(t)$ is the solution of the following initial value problem

$$\begin{aligned} y'(t) &= C[y(t)]^{\frac{p+2}{2}} \\ y(0) &= \|u_0^\varepsilon\|^2_{H^r}. \end{aligned} \tag{2.40}$$

The energy bound is given by

$$y(t) = \frac{y(0)}{\{2 - Cp[y(0)]^{p/2}\, t\}^{2/p}} = \frac{\|u_0^\varepsilon\|^2_{H^r}}{\{2 - Cp\|u_0^\varepsilon\|^p_{H^r}\, t\}^{2/p}}. \tag{2.41}$$

□

Theorem 5. *Let* $0 < \alpha < 1,\ r \geq 2 - \frac{\alpha}{2}$ *and* $u_0 \in H^r(\mathbb{R})$. *Then there exists a time* $T > 0$, *the solution* u^ε *of the Cauchy problem for the equations* (2.21)-(2.22) *converges uniformly to* u *of the Cauchy problem for the equations* (2.12)-(2.13) *in* $C([0,T], H^r(\mathbb{R}))$ *as* $\varepsilon \to 0$.

Proof. We show that $(u^\varepsilon(t))_{\varepsilon\geq 0}$ is a Cauchy sequence in $[0,T]$. Let $\varepsilon, \delta \geq 0$ and u^ε, v^δ be the respective solutions of the equations (2.21)-(2.22). The difference $w = u^\varepsilon - v^\delta$ satisfies

$$w_t + w_x + \frac{3}{4}D^\alpha w_x + \frac{3}{4}D^\alpha w_t + \frac{1}{2}\{[(u^\varepsilon)^{p+1}]_x - [(v^\delta)^{p+1}]_x\} = (\varepsilon - \delta)u^\varepsilon_{xx} + \delta w_{xx}. \tag{2.42}$$

Multiplying both sides of the equation (2.42) by w and then integrating on the whole line, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}[w^2+(D^{\alpha/2}w)^2]dx \\
&= -\int_{\mathbb{R}}\left\{w\left[(u^{\varepsilon})^p+(u^{\varepsilon})^{p-1}v^{\delta}+....+(v^{\delta})^p\right]\right\}_x wdx+\int_{\mathbb{R}}\left[2(\varepsilon-\delta)u^{\varepsilon}_{xx}w-2\delta(w_x)^2\right]dx \\
&\leq \int_{\mathbb{R}}\left[(u^{\varepsilon})^p+(u^{\varepsilon})^{p-1}v^{\delta}+....+(v^{\delta})^p\right]ww_xdx+\int_{\mathbb{R}}\left[2(\varepsilon-\delta)u^{\varepsilon}_{xx}w\right]dx \\
&\leq \frac{1}{2}\int_{\mathbb{R}}\mid\left[(u^{\varepsilon})^p+(u^{\varepsilon})^{p-1}v^{\delta}+....+(v^{\delta})^p\right]_x w^2\mid dx+2(\varepsilon-\delta)\int_{\mathbb{R}}\mid u^{\varepsilon}_{xx}w\mid dx.
\end{aligned}
\tag{2.43}
$$

By using the Hölder's inequality, the first term of RHS of the equation (2.43) is estimated as

$$
\begin{aligned}
&\int_{\mathbb{R}}\mid\left[(u^{\varepsilon})^p+(u^{\varepsilon})^{p-1}v^{\delta}+....+(v^{\delta})^p\right]_x w^2\mid dx \\
&\qquad \leq C\|\,[(u^{\varepsilon})^p+(u^{\varepsilon})^{p-1}v^{\delta}+....+(v^{\delta})^p]_x\,\|_{L^{q_1}}\|w\|^2_{L^{q_2}} \\
&\qquad \leq C\|(u^{\varepsilon})^p+(u^{\varepsilon})^{p-1}v^{\delta}+....+(v^{\delta})^p\|_{H^{s+\frac{\alpha}{2}}}\|w\|^2_{H^{\frac{\alpha}{2}}}.
\end{aligned}
\tag{2.44}
$$

Here, we have used the Sobolev imbeddings in (2.36). By Theorem 4, u^{ε} and v^{δ} are bounded. Then, we deduce that

$$
\int_{\mathbb{R}}\mid\left[(u^{\varepsilon})^p+(u^{\varepsilon})^{p-1}v^{\delta}+....+(v^{\delta})^p\right]_x w^2\mid dx\leq C\|w\|^2_{H^{\frac{\alpha}{2}}}.
\tag{2.45}
$$

On the other hand, the second term in RHS of (2.43) is estimated as

$$
\begin{aligned}
\int_{\mathbb{R}}\mid u^{\varepsilon}_{xx}w\mid dx &\leq \|u^{\varepsilon}_{xx}\|_{L^{q_6}}\|w\|_{L^{q_7}} \\
&\leq \|u^{\varepsilon}_{xx}\|_{H^{s+\frac{\alpha}{2}-2}}\|w\|_{H^{\frac{\alpha}{2}}} \\
&\leq \|u^{\varepsilon}\|_{H^{s+\frac{\alpha}{2}}}\|w\|_{H^{\frac{\alpha}{2}}}.
\end{aligned}
$$

Here, we have used the following Sobolev imbeddings

$$
\begin{aligned}
H^{s+\frac{\alpha}{2}-2} &\hookrightarrow L^{q_6},\ \text{ for }\ q_6\leq\frac{2}{5-2s-\alpha}, \\
H^{\frac{\alpha}{2}} &\hookrightarrow L^{q_7},\ \text{ for }\ q_7\leq\frac{2}{1-\alpha},
\end{aligned}
$$

where $\frac{1}{q_6}+\frac{1}{q_7}\leq 1$ implies $s\geq 2-\alpha$. Since $r=s+\frac{\alpha}{2}$, we obtain $r\geq 2-\frac{\alpha}{2}$. By Theorem 4, it follows that

$$
\int_{\mathbb{R}}\mid u^{\varepsilon}_{xx}w\mid dx\leq C\|w\|_{H^{\frac{\alpha}{2}}}.
\tag{2.46}
$$

Combining the estimates (2.45) and (2.46), we get

$$\frac{d}{dt}\|w\|^2_{H^{\frac{\alpha}{2}}} \leq C\|w\|^2_{H^{\frac{\alpha}{2}}} + C(\varepsilon - \delta)\|w\|_{H^{\frac{\alpha}{2}}}.$$

The Gronwall's inequality implies that $(u^{\varepsilon}(t))_{\varepsilon \geq 0}$ is a Cauchy sequence in the complete space $H^{\frac{\alpha}{2}}(\mathbb{R})$ and it converges to a limit $u(t)$. Moreover, $u^{\varepsilon}(t)$ is continuous with respect to time and uniformly bounded by Theorem 4, the sequence $u^{\varepsilon}(t)$ is also weakly convergent in $H^{s+\frac{\alpha}{2}}(\mathbb{R})$ to the limit $u(t)$. □

2.4 Solitary Wave Solutions

Solitary waves play an important role in the context of wave phenomena. In this chapter, the existence and non-existence of the solitary wave solutions of the equation (1.1) are studied and some properties of the corresponding solutions are investigated.

A localized solitary wave solution of the equation (1.1) is a solution of the form

$$u(x,t) = Q_c(x-ct), \qquad \xi = x - ct,$$

with

$$\lim_{|\xi|\to\infty} Q_c(\xi) = 0.$$

Substituting this into (1.1) one gets ordinary differential equation

$$-cQ_c' + Q_c' + \frac{1}{2}(Q_c^{p+1})' + \frac{3}{4}D^\alpha Q_c' - c\frac{5}{4}D^\alpha Q_c' = 0.$$

Here $'$ denotes the derivative with respect to ξ. Integrating the above equation, we have

$$(\frac{5}{4}c - \frac{3}{4})D^\alpha Q_c + (c-1)Q_c - \frac{1}{2}(Q_c)^{p+1} = 0. \qquad (2.47)$$

2.4.1 Non-Existence of Solitary Waves

In this section, we derive the sufficient conditions for the non-existence of the nontrivial solutions which satisfy the equation (2.47). The following theorem states our results.

Theorem 6. *Assume that one of the following cases*

i. $c \in (\frac{3}{5}, 1)$ *and* $\alpha \geq \frac{p}{p+2}$,

ii. $c \notin (\frac{3}{5}, 1)$ *and* $\alpha \leq \frac{p}{p+2}$,

iii. $c = \frac{3}{5}$ *or* $c = 1$,

is satisfied. Then, (2.47) *does not posses any nontrivial solution* Q_c *in the class* $H^{\frac{\alpha}{2}}(\mathbb{R}) \bigcap L^{p+2}(\mathbb{R})$.

Proof: Let Q_c be any nontrivial solution of (2.47) in the class $H^{\frac{\alpha}{2}}(\mathbb{R})\bigcap L^{p+2}(\mathbb{R})$. If we multiply the equation (2.47) by Q_c and integrate on $\mathbb{R}$, we get

$$(\frac{3}{4}-\frac{5}{4}c)\int_{\mathbb{R}} Q_c D^{\alpha} Q_c d\xi + (1-c)\int_{\mathbb{R}} Q_c^2 d\xi + \frac{1}{2}\int_{\mathbb{R}} Q_c^{p+2} d\xi = 0. \tag{2.48}$$

From Plancherel's formula we have

$$\begin{aligned}\int_{\mathbb{R}} Q_c D^{\alpha} Q_c d\xi &= \int_{\mathbb{R}} |k|^{\alpha} \hat{Q}_c(k)\overline{\hat{Q}_c(k)}dk,\\ &= \int_{\mathbb{R}} |k|^{\frac{\alpha}{2}} \hat{Q}_c(k)|k|^{\frac{\alpha}{2}}\overline{\hat{Q}_c(k)}dk,\\ &= \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi.\end{aligned}$$

Now, the equation (2.48) becomes

$$(\frac{3}{4}-\frac{5}{4}c)\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi + (1-c)\int_{\mathbb{R}} Q_c^2 d\xi = -\frac{1}{2}\int_{\mathbb{R}} Q_c^{p+2} d\xi. \tag{2.49}$$

On the other hand, multiplying the equation (2.47) by $\xi Q_c'$ and integrating on $\mathbb{R}$ we have

$$(\frac{3}{4}-\frac{5}{4}c)\int_{\mathbb{R}} \xi Q_c' D^{\alpha} Q_c d\xi + (1-c)\int_{\mathbb{R}} \xi Q_c' Q_c d\xi + \frac{1}{2}\int_{\mathbb{R}} \xi Q_c' Q_c^{p+1} d\xi = 0. \tag{2.50}$$

To write the above equation in a simple form, we now consider the first term. By the help of integration by parts, Plancherel's formula and some properties of Fourier transform we gather

$$\begin{aligned}\int_{\mathbb{R}} \xi Q_c' D^{\alpha} Q_c d\xi &= -\int_{\mathbb{R}} \widehat{D^{\alpha} Q_c}(k)\frac{d}{dk}(k\overline{\hat{Q}_c(k)})dk,\\ &= -\int_{\mathbb{R}} |k|^{\alpha}|\hat{Q}_c(k)|^2 dk - \int_{\mathbb{R}} |k|^{\alpha} k\hat{Q}_c \frac{d}{dk}\overline{\hat{Q}_c(k)}dk,\\ &= -\int_{\mathbb{R}} |k|^{\alpha}|\hat{Q}_c(k)|^2 dk + (\alpha+1)\int_{\mathbb{R}} |k|^{\alpha}|\hat{Q}_c(k)|^2 dk + \int_{\mathbb{R}} |k|^{\alpha} k\frac{d}{dk}\hat{Q}_c\overline{\hat{Q}_c(k)}dk,\\ &= \alpha\int_{\mathbb{R}} |k|^{\alpha}|\hat{Q}_c(k)|^2 dk + \int_{\mathbb{R}} (k\frac{d}{dk}\hat{Q}_c)(|k|^{\alpha}\overline{\hat{Q}_c(k)})dk,\\ &= \alpha\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi - \int_{\mathbb{R}} \frac{d}{d\xi}(\xi Q_c)(D^{\alpha} Q_c)d\xi,\\ &= (\alpha-1)\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi - \int_{\mathbb{R}} \xi Q_c' D^{\alpha} Q_c d\xi,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}} \xi Q_c' D^{\alpha} Q_c d\xi = \frac{\alpha-1}{2}\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi. \tag{2.51}$$

For the second and third terms in the equation (2.50) we use integration by parts technique. Then we obtain

$$\int_{\mathbb{R}} \xi Q_c' Q_c d\xi = -\int_{\mathbb{R}} \frac{Q_c^2}{2} d\xi, \tag{2.52}$$

and

$$\int_{\mathbb{R}} \xi Q_c' Q_c^{p+1} d\xi = -\int_{\mathbb{R}} \frac{Q_c^{p+2}}{p+2} d\xi. \tag{2.53}$$

The equalities (2.52), (2.53) and (2.51) turn the equation (2.50) into the Pohozaev type identity

$$(\frac{3}{4}-\frac{5}{4}c)\frac{\alpha-1}{2}\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi - \frac{1-c}{2}\int_{\mathbb{R}} Q_c^2 d\xi - \frac{1}{2(p+2)}\int_{\mathbb{R}} Q_c^{p+2} d\xi = 0. \tag{2.54}$$

Multiplying (2.49) by $\frac{1}{p+2}$ and then adding to (2.54), we obtain

$$\left(\frac{3}{4}-\frac{5}{4}c\right)\left(\frac{1}{p+2}+\frac{\alpha-1}{2}\right)\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi - \left(\frac{1-c}{2}-\frac{1-c}{p+2}\right)\int_{\mathbb{R}} Q_c^2 d\xi = 0.$$

Rewriting above equation

$$\left(\frac{3}{4}-\frac{5}{4}c\right)(\alpha(p+2)-p)\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi = p(1-c)\int_{\mathbb{R}} Q_c^2 d\xi,$$

we finally obtain

$$\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi = \frac{p(1-c)}{(\frac{3}{4}-\frac{5}{4}c)(\alpha(p+2)-p)}\int_{\mathbb{R}} Q_c^2 d\xi. \tag{2.55}$$

The proof of items (**i**) and (**ii**) of the theorem follows directly from the positivity of the left hand integral. It is possible when $c > 1$ or $c = 3/5$ for the case $\alpha > \dfrac{p}{p+2}$. To prove item (**iii**) we first set $c = 1$ in (2.55) which gives $D^{\frac{\alpha}{2}} Q_c = 0$ and then the assumption $\lim_{|\xi|\to\infty} Q_c(\xi) = 0$ shows that the solution is trivial. Lastly, if we rewrite (2.55) as

$$\frac{(5c-3)[\alpha(p+2)-p]}{4p(c-1)}\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 d\xi = \int_{\mathbb{R}} Q_c^2 d\xi$$

and set $c = 3/5$ we directly have that $Q_c = 0$. Finally, we complete the proof of Theorem (6).

Combining the results of Theorem (6) and the condition $\alpha \geq \dfrac{p}{p+2}$ which ensures that the Hamiltonian (2.11) is well-posed we conclude that in order to have a non-trivial solution we must have $\alpha > \dfrac{p}{p+2}$ and $c < 3/5$ or $c > 1$.

On the other hand, we now investigate the cases $c < 3/5$ and $c > 1$, in detail. First, we focus on the case $c < 3/5$ and assume that the solution $Q_c(x-ct)$ is positive, then the equation (2.49) gives a contradiction while the RHS of the equation is positive and the LHS is negative. Therefore, the gfBBM equation has no positive solitary wave solutions for $c < 3/5$. More preciously, we see that the equation (2.49) is satisfied if the solution $Q_c(x-ct)$ is the negative and p is an odd number. Second, we focus on the case $c>1$. Then, the equation (2.49) is satisfied if the solution $Q_c(x-ct)$ is positive for any p. Otherwise, it gives contradiction.

Next, we note that if $Q_c(x-ct)>0$ is a solitary wave solution for the gfBBM equation, then $-Q_c(x-ct)$ is also a solution when $c>1$ and p is even. As it is obvious that the dynamics for the solutions Q_c and $-Q_c$ are the same, we do not consider these type of negative solutions in the current study.

2.4.2 Existence of Solitary Waves

In this section we show the existence and uniqueness of the both positive and negative solitary wave solutions to the gfBBM equation. To this end, we first recall the results of Frank and Lenzmann [34].

Definition 15. *(Definition 2.1 of [34], Definition 1.1 of [14]) Let $Q \in H^{\alpha/2}(\mathbb{R})$ be an even and positive solution of the equation*

$$D^{\alpha}Q+Q-Q^{p+1}=0. \tag{2.56}$$

If

$$J^{\alpha,p}(Q)=\inf\{J^{\alpha,p}(u)|u\in H^{\alpha/2}(\mathbb{R})\backslash\{0\}\} \tag{2.57}$$

then $Q \in H^{\alpha/2}(\mathbb{R})$ is a positive ground state solution of the equation (2.56) *where $J^{\alpha,p}$ is the Weinstein functional defined by*

$$J^{\alpha,p}(u)=\left(\int_{\mathbb{R}}|u|^{p+2}dx\right)^{-1}\left(\int_{\mathbb{R}}|D^{\alpha/2}u|^2dx\right)^{p/2\alpha}\left(\int_{\mathbb{R}}|u|^2dx\right)^{p(\alpha-1)/2\alpha+1}.$$

First, we prove the existence of positive solitary wave solutions that the gfBBM equation admits for $c>1$ and $\alpha>\dfrac{p}{p+2}$. The scaling

$$Q_c(\xi)=(2(c-1))^{1/p}Q\left(\left(\frac{4(c-1)}{5c-3}\right)^{1/\alpha}\xi\right) \tag{2.58}$$

converts the equation (2.47) into the equation (2.56). The condition $c>1$ guarantees the positivity of the coefficients in this scaling. The definition (15) gives the existence and uniqueness of the positive ground state solutions of the equation (2.56) when $0<\alpha<2$ and $0<p<p_{max}$ holds, where the critical exponent is defined as

$$p_{\max}(\alpha)=\begin{cases}\frac{2\alpha}{1-\alpha}, & \text{for } 0<\alpha<1\\ \infty, & \text{for } 1\leq\alpha<2.\end{cases} \tag{2.59}$$

Here the condition (2.59) coincides with the above condition $\alpha \geq \frac{p}{p+2}$ for the Hamiltonian to be well-defined. Therefore, the equation (2.47) has a unique positive ground state solution $Q_c \in H^{\alpha/2}(\mathbb{R})$ for $c > 1$ and $\alpha > \frac{p}{p+2}$. In the next chapters, we will choose the parameters c, α and p satisfying these conditions so that we obtain positive solitary wave solutions.

Now, we prove the existence and uniqueness of negative solitary wave solutions by following the similar idea. For this aim, we consider $c < 3/5$, $\alpha > \frac{p}{p+2}$ and p is odd. To employ the result of existence and uniqueness to the solitary waves in [34], the equation (2.47) is written in terms of positive solitary wave solutions. Setting $Q_c = -R_c$ where $R_c > 0$, equation (2.47) becomes

$$(\frac{3}{4} - \frac{5}{4}c)D^\alpha R_c + (1-c)R_c - \frac{1}{2}R_c^{p+1} = 0. \tag{2.60}$$

A scaling argument as

$$R_c(\xi) = (2(1-c))^{1/p} Q\left(\left(\frac{4(c-1)}{5c-3}\right)^{1/\alpha} \xi\right) \tag{2.61}$$

converts (2.60) into the equation (2.56). Here the condition $c < 3/5$ guarantees the positivity of the coefficients in this scaling. Therefore, we obtain the existence and uniqueness of a positive even solution R_c of the equation (2.60) and consequently a negative solution Q_c of the equation (2.47) when $c < 3/5$ and p is odd for $0 < \alpha \leq 2$. In the next chapters, we will choose the parameters c, α and p satisfying these conditions to obtain negative solitary wave solutions.

Finally, the all results are summarized in the Figure 2.5.

2.4.3 Exact Solitary Wave Solution

This thesis study concerns numerical behaviours to the solutions besides the qualitative properties. Therefore we need exact solitary wave solutions to compare with numerical ones. The exact solitary wave solutions to the equation (1.1) is still unknown for $\alpha \in (0,1)$. For a special case $\alpha = 1$, however, it is possible to construct by using solitary wave solutions of

$$u_t + uu_x + D^\alpha u_x = 0. \tag{2.62}$$

A solitary wave solution $Q(\xi)$ of (2.62) satisfies

$$D^\alpha Q + cQ - \frac{1}{2}Q^2 = 0, \quad c > 0. \tag{2.63}$$

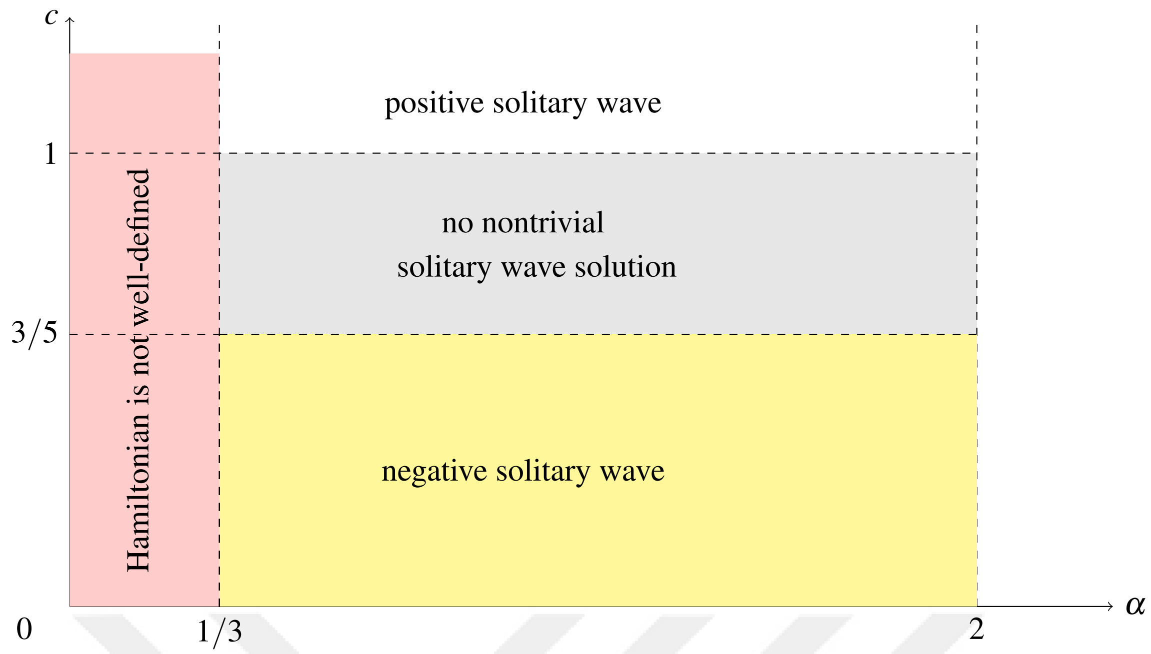

Figure 2.1 : Existence of solitary wave solutions and their dependence on α and c when $p = 1$.

In the literature, it is shown that (2.63) has exact solution in the form

$$Q(\xi) = \frac{4c}{1 + c^2\xi^2},$$

for $\alpha = 1$. Thus, solitary wave solution Q_c of the equation (1.1) for $\alpha = 1$ and $p = 1$ must also satisfy (2.63) after a simple change of variables:

$$c \to \frac{c-1}{\frac{5c}{4} - \frac{3}{4}} \qquad u \to \frac{u}{\frac{5c}{4} - \frac{3}{4}}.$$

Finally, exact solitary wave solution of (1.1) can be obtained as

$$u(x,t) = \frac{4(c-1)}{1 + (\frac{c-1}{\tilde{c}})^2(x-ct)^2}, \qquad \tilde{c} = \frac{5}{4}c - \frac{3}{4}. \tag{2.64}$$

Additionally, it is also possible to find exact solitary wave solution of gfBBM equation by using similar idea for the case of $\alpha = 2$ and $p = 1$. For this case, the equation (1.1) has the solitary wave solution

$$u(x,t) = 3(c-1)\text{sech}^2\left(\frac{1}{2}\sqrt{\frac{4(c-1)}{5c-3}}(x-ct)\right). \tag{2.65}$$

Here we also note that both of the exact solutions defined in (2.64) and (2.65) agree with the above result: The solution is positive when $c > 1$, it is negative when $c < 3/5$.

2.5 Orbital Stability

In this part, we will investigate the orbital stability properties of both positive and negative solitary wave solutions to the equation (1.1).Depending on the parameters c, α and p included in the equation, stability properties change. We first state the method in an abstract setting. Later we give the proof of stability.

2.5.1 Grillakis Shatah Strauss Approach

For the orbital stability analysis we follow the approach given by Grillakis, Shatah and Strauss in [8]. The stability theory in their study applies to the evolutionary partial differential equation which is of the form (2.9). To use this method, following assumptions are required:

(A1) For every $u_0 \in H^r(\mathbb{R})$, $r > \frac{1}{2}$ and $0 < \alpha < 2$, there exists a solution u of (1.1) in $[0,T)$ such that $u(0) = u_0$ where $u \in C([0,T], H^r(\mathbb{R}))$. Moreover, there exist conserved functionals $F(u)$ and $H(u)$ for solutions of (1.1).

(A2) For all $c > 1$ when $\frac{p}{p+2} < \alpha < 2$, there exists a positive travelling wave solution $Q_c \in H^{\frac{\alpha}{2}}(\mathbb{R})$ of (1.1), where $Q_c > 0$ and $Q_c' \neq 0$. Furthermore $cF'(Q_c) + H'(Q_c) = 0$, where F' and H' denote the variational derivatives of F and H, respectively.

For all $c < \frac{3}{5}$ when $\frac{p}{p+2} < \alpha < 2$, there exists a negative travelling wave solution Q_c $\in H^{\frac{\alpha}{2}}(\mathbb{R})$ of (1.1), where $Q_c = -R_c$, $R_c > 0$ and $Q_c' \neq 0$. Furthermore $cF'(Q_c) + H'(Q_c) = 0$.

(A3) For all $c > 1$ when $\frac{p}{p+2} < \alpha < 2$, the linearized Hamiltonian operator around Q_c is defined by

$$\mathscr{L}_c^+ = \left(\frac{5}{4}c - \frac{3}{4}\right) D^\alpha + c - 1 - \frac{p+1}{2} Q_c^p.$$

For all $\frac{3}{5} < c$ when $\frac{p}{p+2} < \alpha < 2$, the linearized Hamiltonian operator around $Q_c = -R_c$ is defined by

$$\mathscr{L}_c^- = \left(\frac{3}{4} - \frac{5}{4}c\right) D^\alpha + 1 - c - \frac{p+1}{2} R_c^p.$$

The operators $\mathscr{L}_c^+$ and $\mathscr{L}_c^-$ have exactly one negative simple eigenvalue, their kernels are spanned by Q_c' and R_c' and the rest of the spectrum is positive and bounded away from zero, respectively.

Under the assumptions **(A1)-(A3)**, the GSS approach states that the solitary wave solution is orbitally stable if the scalar function $d(c) = H(Q_c) + cF(Q_c)$ is convex.

Now, we show that our problem falls into the framework of the GSS approach. First, we consider our problem in the form (2.9). Here, the operator J is given in the equation (2.10). J is skew-symmetric with respect to the inner product on $L^2(\mathbb{R})$, for all $u, v \in H^{s+\frac{\alpha}{2}}$. Moreover, the Hamiltonian $H(u)$ is obtained in the equation (2.11). As a conclusion, GSS approach is now applicable to the problem proposed in this study if the assumptions **(A1)-(A3)** are satisfied. In the next chapters, we will present the sufficient conditions which ensure the orbital stability of solitary wave solutions.

2.5.2 Proof of Stability for Positive Solitary Waves

Now, we state our main result:

Theorem 7. *Let $\alpha \in (0,2)$ and $c > 1$. The positive solitary wave solution Q_c of the equation (1.1) is orbitally stable for $c \in (1,\infty)$ when $\frac{p}{2} < \alpha < 2$ and for $c \in (c_{1,p},\infty)$ when $\frac{p}{p+2} < \alpha < \frac{p}{2}$, where*

$$c_{1,p} = \frac{6\alpha + 2p + 3\alpha p + \sqrt{2}p\sqrt{2\alpha - p + \alpha p}}{5(2\alpha + \alpha p)}.$$

Proof. In the first step, we show that the assumptions **(A1)-(A3)** hold.

- **(A1)** By the help of the chain rule we see that $H(u)$ is the conserved functional of the equation (1.1), i.e.,

  $$\frac{d}{dt}H(u) = (H'(u), u_t) = (H'(u), JH'(u)) = -(JH'(u), H'(u)) = 0.$$

  In addition to the Hamiltonian, the second conserved functional of the equation (1.1) is

  $$F(u) = \frac{1}{2}\int_{\mathbb{R}} (u^2 + \frac{5}{4}|D^{\frac{\alpha}{2}}u|^2)dx,$$

  as

  $$\frac{d}{dt}F(u) = \frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} (u^2 + \frac{5}{4}|D^{\frac{\alpha}{2}}u|^2)dx = 0.$$

- **(A2)** We recall that the equation (1.1) does not posses any nontrivial solutions unless $c < 3/5$ or $c > 1$ and $\alpha > \frac{p}{p+2}$ according to the theorem (6). In the previous section, it is proved that the equation (1.1) has unique positive ground state solution $Q_c \in$

$H^{\frac{\alpha}{2}}(\mathbb{R})$ for $c > 1$ and $\alpha > \frac{p}{p+2}$. The equation (2.47) can be written in terms of conserved functionals as

$$cF'(Q_c) + H'(Q_c) = 0, \tag{2.66}$$

where

$$H'(Q_c) = -Q_c - \frac{Q_c^{p+1}}{2} - \frac{3}{4}D^\alpha Q_c,$$

and

$$F'(Q_c) = Q_c + \frac{5}{4}D^\alpha Q_c.$$

Therefore, the equation (2.66) guarantees the **(A**2**)**.

- **(A**3**)** To show the validity of **(A**3**)**, we investigate the spectral properties of the operator

$$\mathscr{L}_c^+ = cF''(u) + H''(u).$$

Since the second variational derivatives of the F and H are computed as

$$F''(Q_c) = I + \frac{5}{4}D^\alpha,$$

and

$$H''(Q_c) = -I - \frac{p+1}{2}Q_c^p - \frac{3}{4}D^\alpha,$$

respectively, one gets

$$\mathscr{L}_c^+ = (\frac{5}{4}c - \frac{3}{4})D^\alpha + c - 1 - \frac{p+1}{2}Q_c^p.$$

In this part, we consider that the Q_c is a positive solitary wave solution. Although all calculations addressing the stability depends on solution Q_c, we do not need to know the explicit form of it. First, we consider the equation (2.56). The properties of ground state solutions of this fractional differential equation are discussed in [34]. They consider the operator associated with the solitary wave Q as

$$\mathscr{L}_p = D^\alpha + 1 - (p+1)Q^p.$$

We again use the elementary scaling given in (2.58)

$$Q_c = \big(2(c-1)\big)^{1/p} Q\big(\theta x\big), \tag{2.67}$$

with $\theta = \left(\frac{4(c-1)}{5c-3}\right)^{1/\alpha}$. We follow the idea in [14] to relate the operators $\mathscr{L}_c^+$ and $\mathscr{L}_p$. Defining the dilation operator T_θ as

$$(T_\theta f)(x) = f(\theta x),$$

with $\theta \neq 0$ and using the definitions above, one can write

$$\begin{aligned}
\mathscr{L}_c^+ T_\theta f &= (\frac{5}{4}c - \frac{3}{4})D^\alpha (T_\theta f)(x) + (c-1)(T_\theta f)(x) - \frac{p+1}{2}Q_c^p (T_\theta f)(x) \\
&= (\frac{5}{4}c - \frac{3}{4})\theta^\alpha D^\alpha f(\theta x) + (c-1)f(\theta x) - \frac{p+1}{2}Q_c^p f(\theta x) \\
&= (c-1)[D^\alpha f(\theta x) + f(\theta x) - (p+1)Q^p f(\theta x)] \\
&= (c-1)[T_\theta \mathscr{L}_p f].
\end{aligned}$$

This implies that $\mathscr{L}_c^+$ and $\mathscr{L}_p$ have the same spectral structure with

$$\mathscr{L}_c^+ = (c-1)T_\theta \mathscr{L}_p T_\theta^{-1}. \tag{2.68}$$

Let us consider that $\mathscr{L}_p$ has an eigenvalue λ with the corresponding eigenfunction ψ as

$$\mathscr{L}_p \psi = \lambda \psi.$$

Using the equation (2.68) we have

$$\frac{1}{c-1}T_\theta^{-1}\mathscr{L}_c^+ T_\theta \psi = \lambda \psi,$$

and

$$\mathscr{L}_c^+ T_\theta(\psi) = (c-1)\lambda T_\theta(\psi).$$

Then, $\mathscr{L}_c^+$ has the eigenvalue $(c-1)\lambda$ with corresponding eigenfunction $T_\theta(\psi)$.

The work [34] provides following properties of $\mathscr{L}_p$:

(i) $\mathscr{L}_p$ is a self-adjoint operator in $L^2(R)$,

(ii) The essential spectrum is $\sigma_{ess}(\mathscr{L}_p) = [1, \infty)$,

(iii) The number of strictly negative eigenvalues is $n(\mathscr{L}_p) = 1$,

(iv) Ker$(\mathscr{L}_p)$ =span$\{Q'\}$.

Since $\mathscr{L}_c^+$ and $\mathscr{L}_p$ have the same spectral structure, we conclude that spec$\{\mathscr{L}_c^+\} = \{(c-1)\lambda : \lambda \in \text{spec}\{\mathscr{L}_p\}\}$, thus $n(\mathscr{L}_c^+) = 1$ and Ker$(\mathscr{L}_c^+) = [Q_c{}']$ such that

$$\mathscr{L}_c^+ Q_c' = (\frac{5}{4}c - \frac{3}{4})D^\alpha Q_c' + (c-1)Q_c' - \frac{1}{2}(Q_c^{p+1})' = 0.$$

Given the above properties, the theory in [8] implies that the stability of the solitary wave solutions of (1.1) can be determined by convexity of the scalar function

$$d(c) = H(Q_c) + cF(Q_c).$$

In the second step of proof, we will investigate if $d(c)$ is convex, i.e, $d''(c) > 0$. To this end, we first compute

$$d'(c) = F(Q_c) + (H'(Q_c) + cF'(Q_c), \partial_c Q_c) = F(Q_c),$$

where

$$F(Q_c) = \frac{1}{2}\int_{\mathbb{R}} (Q_c^2 + \frac{5}{4}|D^{\frac{\alpha}{2}} Q_c|^2)dx.$$

Thus we get

$$d''(c) = \frac{d}{dc}F(Q_c) = \frac{1}{2}\frac{d}{dc}\int_{\mathbb{R}} (Q_c^2 + \frac{5}{4}|D^{\frac{\alpha}{2}} Q_c|^2)d\xi. \tag{2.69}$$

Let us recall the Pohozaev type identity (2.54)

$$\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c(\xi)|^2 d\xi = \frac{4p(c-1)}{(5c-3)(\alpha(p+2)-p)}\int_{\mathbb{R}} Q_c(\xi)^2 d\xi.$$

Moreover using (2.67), we have

$$\int_{\mathbb{R}} |Q_c(\xi)|^2 d\xi = \big(2(c-1)\big)^{2/p}\int_{\mathbb{R}} |Q(\theta\xi)|^2 d\xi = \frac{\big(2(c-1)\big)^{2/p}}{\theta}\int_{\mathbb{R}} |Q(\overline{\xi})|^2 d\overline{\xi},$$

with $\theta\xi = \overline{\xi}$. Using the above relations, the equation (2.69) can be rewritten as

$$\begin{aligned} d''(c) &= \frac{1}{2}\frac{d}{dc}\int_{\mathbb{R}} (Q_c^2 + \frac{5}{4}|D^{\frac{\alpha}{2}} Q_c|^2)d\xi \\ &= \frac{d}{dc}\left[(c-1)^{2/p}\left(\frac{5c-3}{4(c-1)}\right)^{1/\alpha}\left(1 + \frac{5p(c-1)}{(5c-3)(\alpha(p+2)-p)}\right)\right]\|Q\|^2. \end{aligned} \tag{2.70}$$

Now, we rewrite $d''(c)$ in the following form:

$$d''(c) = M_p(c)\|Q\|^2, \tag{2.71}$$

where

$$\begin{aligned} M_p(c) &= \frac{d}{dc}\left[(c-1)^{2/p}\left(\frac{5c-3}{4(c-1)}\right)^{1/\alpha}\left(1 + \frac{5p(c-1)}{(5c-3)(\alpha(p+2)-p)}\right)\right] \\ &= \left(\frac{5c-3}{4(c-1)}\right)^{\frac{1}{\alpha}} \frac{2(c-1)^{\frac{2}{p}}\omega(c)}{\alpha p(5c-3)^2(c-1)(2\alpha - p + \alpha p)}. \end{aligned} \tag{2.72}$$

Here, $\omega(c)$ is a polynomial which is obtained as $\omega(c) = 25\alpha^2c^2p + 50\alpha^2c^2 - 30\alpha^2cp - 60\alpha^2c + 9\alpha^2p + 18\alpha^2 - 20\alpha cp - 2\alpha p^2 + 12\alpha p + 2p^2$. Since the term $\|Q\|^2$ in the equation (2.71) is always positive, the sign of $d''(c)$ depends on the sign of $M_p(c)$. Namely, positivity of (2.72) implies the convexity of $d(c)$ and orbital stability of solitary wave solutions. The function $M_p(c)$ in (2.72) is always positive for $c > 1$ when $\frac{p}{2} < \alpha < 2$. In the case $\frac{p}{p+2} < \alpha < \frac{p}{2}$, $M_p(c)$ has two roots:

$$c_{1,p} = \frac{6\alpha + 2p + 3\alpha p + \sqrt{2}p\sqrt{2\alpha - p + \alpha p}}{5(2\alpha + \alpha p)}, \tag{2.73}$$

$$c_{2,p} = \frac{6\alpha + 2p + 3\alpha p - \sqrt{2}p\sqrt{2\alpha - p + \alpha p}}{5(2\alpha + \alpha p)}. \tag{2.74}$$

For this case $M_p(c)$ is positive when $c > c_{1,p}$. Finally, we obtain orbital stability of solitary wave solutions of gfBBM equation for the following cases when $1 \leq p$:
1) $c > 1$ and $\alpha > p/2$,
2) $c > c_{1,p}$ and $\frac{p}{p+2} < \alpha < \frac{p}{2}$. □

To see the restrictions given in the theorem more clearly, we present some figures. For simplicity, we consider the quadratic nonlinearity, i.e. $p = 1$. Figure 2.2 presents the function $M_1(c)$ for several values of α. Regarding the sign of $M_1(c)$, we observe that the function $M_1(c)$ is positive if for all $c > 1$ if $\alpha > 1/2$. On the other hand $M_1(c)$ is positive for $c > c_{1,1} > 1$ if $\frac{1}{3} < \alpha < \frac{1}{2}$.

2.5.3 Proof of Stability for Negative Solitary Waves

First, we state our main result:

Theorem 8. *Let $\alpha \in (0,2)$. The negative solitary wave solution Q_c of the equation* (1.1) *is orbitally stable for $c \in (c_{2,p}, \frac{3}{5})$ when $1 < \alpha < 2$, where*

$$c_{2,p} = \frac{6\alpha + 2p + 3\alpha p - \sqrt{2}p\sqrt{2\alpha - p + \alpha p}}{5(2\alpha + \alpha p)}.$$

Proof. Similar with the previous section, we prove that the assumptions **(A**1**)**-**(A**3**)** hold.

- **(A**1**)** Since first assumption is same for both cases, it is already proved in the previous section.

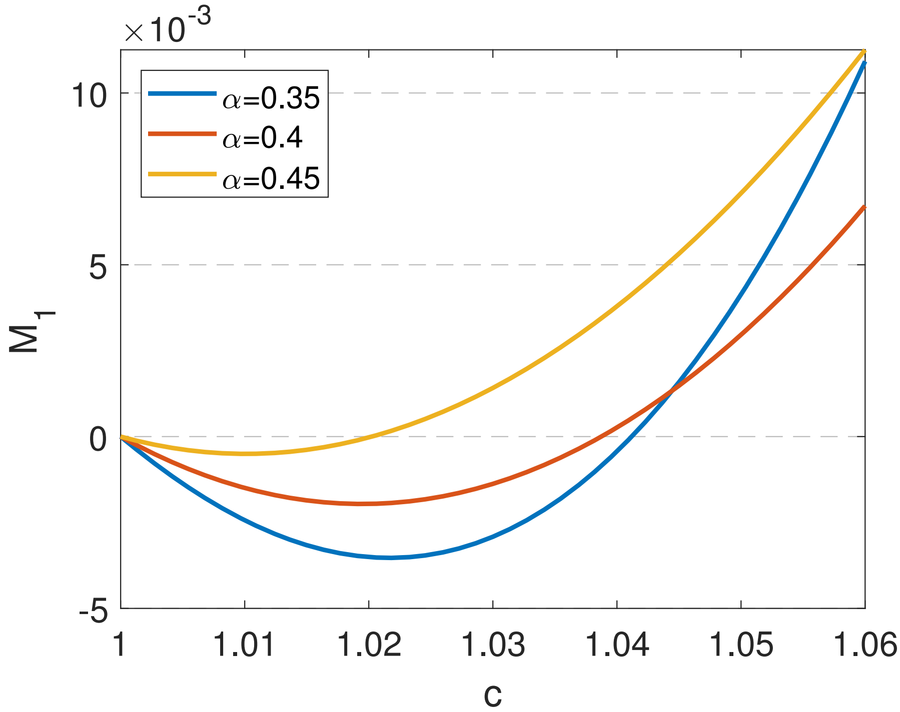

Figure 2.2 : Variation of M_1 with c, $\alpha = 0.35$, $\alpha = 0.4$ and $\alpha = 0.45$ for $p = 1$.

- **(A2)** By considering the negative solitary wave solution $Q_c = -R_c$ with $R_c > 0$, the equation (2.60) can be written in terms of conserved functionals as

$$-cF'(R_c) - H'(R_c) = 0, \tag{2.75}$$

where

$$H'(R_c) = -R_c + \frac{R_c^{p+1}}{2} - \frac{3}{4}D^{\alpha}R_c,$$

and

$$F'(R_c) = R_c + \frac{5}{4}D^{\alpha}R_c.$$

Thus, the equation (2.75) guarantees the assumption **(A2)** for negative solitary waves also.

- **(A3)** To show the validity of **(A3)**, we investigate the spectral properties of the operator

$$\mathscr{L}_c^- = -cF''(u) - H''(u).$$

Since the second variational derivatives of the F and H are computed as

$$F''(R_c) = I + \frac{5}{4}D^{\alpha},$$

and

$$H''(R_c) = -I + \frac{p+1}{2}R_c^p - \frac{3}{4}D^{\alpha},$$

respectively, one gets

$$\mathscr{L}_c^- = (\frac{3}{4} - \frac{5}{4}c)D^\alpha + 1 - c - \frac{p+1}{2}R_c^p.$$

Using the following form of elementary scaling given in (2.61)

$$R_c(\xi) = (2(1-c))^{1/p} Q(\theta\xi) \tag{2.76}$$

and by the help of the dilation operator we can relate $\mathscr{L}_c^-$ and $\mathscr{L}_p$ as:

$$\begin{aligned}
\mathscr{L}_c^- T_\theta f &= (\frac{3}{4} - \frac{5}{4}c)D^\alpha (T_\theta f)(x) + (1-c)(T_\theta f)(x) - \frac{p+1}{2}R_c^p (T_\theta f)(x) \\
&= (\frac{3}{4} - \frac{5}{4}c)\theta^\alpha D^\alpha f(\theta x) + (1-c) f(\theta x) - \frac{p+1}{2}R_c^p(\theta x) \\
&= (1-c)[D^\alpha f(\theta x) + f(\theta x) - (p+1)Q^p f(\theta x)] \\
&= (1-c)[T_\theta \mathscr{L}_p f],
\end{aligned}$$

which proves that $\mathscr{L}_c^-$ and $\mathscr{L}_p$ have the same spectral structure with

$$\mathscr{L}_c^- = (1-c) T_\theta \mathscr{L}_p T_\theta^{-1}. \tag{2.77}$$

Let us consider that $\mathscr{L}_p$ has eigenvalue λ with eigenfunction ψ as Then, similar calculation with previous section gives that $\mathscr{L}_c^-$ has eigenvalue $(1-c)\lambda$ with eigenfunction $T_\theta(\psi)$. Now, since $\mathscr{L}_c^-$ and $\mathscr{L}_p$ have the same spectral structure, we conclude that $\text{spec}\{\mathscr{L}_c^-\} = \{(1-c)\lambda : \lambda \in \text{spec}\{\mathscr{L}_p\}\}$, thus $n(\mathscr{L}_c^-) = 1$ and $\text{Ker}(\mathscr{L}_c^-) = [R_c{}']$ such that

$$\mathscr{L}_c^-(R_c)' = (\frac{3}{4} - \frac{5}{4}c)D^\alpha R_c' + (1-c)R_c' - \frac{1}{2}(R_c^{p+1})' = 0.$$

According to the above properties, we are also allowed to use GSS's approach for the negative solitary wave solutions to the gfBBM equation. For this aim we investigate the convexity of the scalar function

$$d(c) = H(R_c) + cF(R_c).$$

To consider the sign of $d''(c)$, we first compute

$$d'(c) = F(R_c) + (H'(R_c) + cF'(R_c), \partial_c(R_c)) = F(R_c),$$

and then second derivative as

$$d''(c) = \frac{d}{dc}F(R_c) = \frac{1}{2}\frac{d}{dc}\int_{\mathbb{R}} (R_c^2 + \frac{5}{4}|D^{\frac{\alpha}{2}} R_c|^2) d\xi. \tag{2.78}$$

Since we do not have negative solitary wave solution $-R_c$, explicitly, we now derive Pohozaev type identity for this solution.

Let us multiply the equation (2.60) by R_c and integrate on $\mathbb{R}$, we have

$$(\frac{3}{4}-\frac{5}{4}c)\int_{\mathbb{R}} R_c D^{\alpha} R_c d\xi + (1-c)\int_{\mathbb{R}} R_c^2 d\xi - \frac{1}{2}\int_{\mathbb{R}} R_c^{p+2} d\xi = 0. \tag{2.79}$$

From Plancherel's formula, the equation (2.79) becomes

$$(\frac{3}{4}-\frac{5}{4}c)\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} R_c|^2 d\xi + (1-c)\int_{\mathbb{R}} R_c^2 dx - \frac{1}{2}\int_{\mathbb{R}} R_c^{p+2} d\xi = 0. \tag{2.80}$$

Additionally we multiplying the equation (2.60) by $\xi R_c'$ and integrating on $\mathbb{R}$ we have

$$(\frac{3}{4}-\frac{5}{4}c)\int_{\mathbb{R}} \xi R_c' D^{\alpha} R_c d\xi + (1-c)\int_{\mathbb{R}} \xi R_c' R_c d\xi - \frac{1}{2}\int_{\mathbb{R}} \xi R_c' R_c^{p+1} d\xi = 0. \tag{2.81}$$

Integration by parts, Plancherel's formula and some properties of Fourier transform lead above equation to the Pohozaev type identity

$$(\frac{3}{4}-\frac{5}{4}c)\frac{\alpha-1}{2}\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} R_c|^2 d\xi - \frac{1-c}{2}\int_{\mathbb{R}} R_c^2 d\xi + \frac{1}{2(p+2)}\int_{\mathbb{R}} R_c^{p+2} d\xi = 0. \tag{2.82}$$

If we multiply (2.80) by $\frac{1}{p+2}$ and then add to (2.82), we gather

$$\left(\frac{3}{4}-\frac{5}{4}c\right)\left(\frac{1}{p+2}+\frac{\alpha-1}{2}\right)\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} R_c|^2 d\xi + \left(\frac{1-c}{p+2}-\frac{1-c}{2}\right)\int_{\mathbb{R}} R_c^2 d\xi = 0.$$

Rewriting above equation we have

$$\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} R_c|^2 d\xi = \frac{4p(1-c)}{(3-5c)\left(\alpha(p+2)-p\right)}\int_{\mathbb{R}} R_c^2 d\xi, \tag{2.83}$$

where $c<\frac{3}{5}$, $\alpha>\frac{p}{p+2}$ and p is odd. Furthermore, (2.61) gives

$$\int_{\mathbb{R}} |R_c(\xi)|^2 d\xi = \left(2(1-c)\right)^{2/p}\int_{\mathbb{R}} |R_c(\theta\xi)|^2 d\xi = \frac{\left(2(1-c)\right)^{2/p}}{\theta}\int_{\mathbb{R}} |R_c(\overline{\xi})|^2 d\overline{\xi},$$

with $\theta\xi=\overline{\xi}$. By using the above equations, the equation (2.78) turns into

$$\begin{aligned} d''(c) &= \frac{1}{2}\frac{d}{dc}\int_{\mathbb{R}} (R_c^2+\frac{5}{4}|D^{\frac{\alpha}{2}} R_c|^2) d\xi \\ &= \frac{d}{dc}\left[(1-c)^{2/p}\left(\frac{5c-3}{4(c-1)}\right)^{1/\alpha}\left(1+\frac{5p(c-1)}{(5c-3)(\alpha(p+2)-p)}\right)\right]\|Q\|^2. \end{aligned} \tag{2.84}$$

Now, we rewrite $d''(c)$ in the following form:

$$d''(c) = M_p^-(c)\|Q\|^2, \tag{2.85}$$

where

$$M_p^-(c) = \frac{d}{dc}\left[(1-c)^{2/p}\left(\frac{5c-3}{4(c-1)}\right)^{1/\alpha}\left(1+\frac{5p(c-1)}{(5c-3)(\alpha(p+2)-p)}\right)\right]$$

$$= \left(\frac{5c-3}{4(c-1)}\right)^{\frac{1}{\alpha}} \frac{2(1-c)^{\frac{2}{p}}\omega(c)}{\alpha p(5c-3)^2(c-1)(2\alpha-p+\alpha p)}. \quad (2.86)$$

The function $M_p^-(c)$ in (2.86) has also same roots with $M_p(c)$ as given in (2.73) and (2.74) and $M_p^-(c)$ is positive for $c_{2,p} < c < \frac{3}{5}$ when $1 < \alpha < 2$. Finally, we obtain orbital stability of negative solitary wave solutions of gfBBM equation for the following case when $p \geq 1$:

1) $c_{2,p} < c < \frac{3}{5}$ and $1 < \alpha < 2$ □

As a conclusion, the sign of $d''(c)$ varies with roots with M_p (2.73) and (2.74) for both cases $1 < c$ and $\frac{3}{5} < c$. First, we illustrate the variation of the roots with α for $p = 1$ and $p = 2$ in Figure 2.3 and Figure 2.4, respectively.

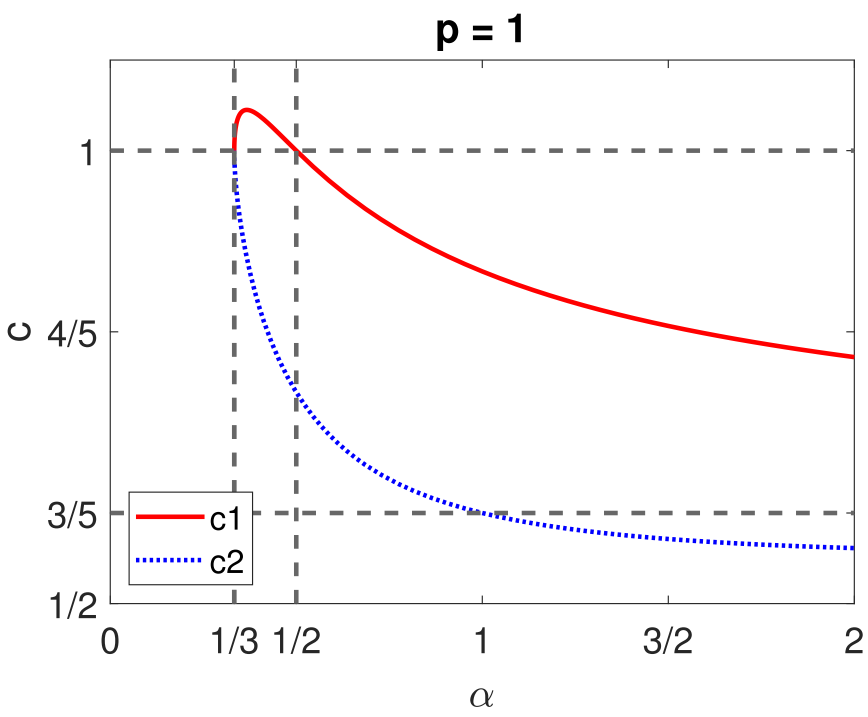

Figure 2.3 : Variations of the roots of M_p with α for $p = 1$.

To clear the conditions we summarize stability results for quadratic nonlinearity in Figure 2.5.

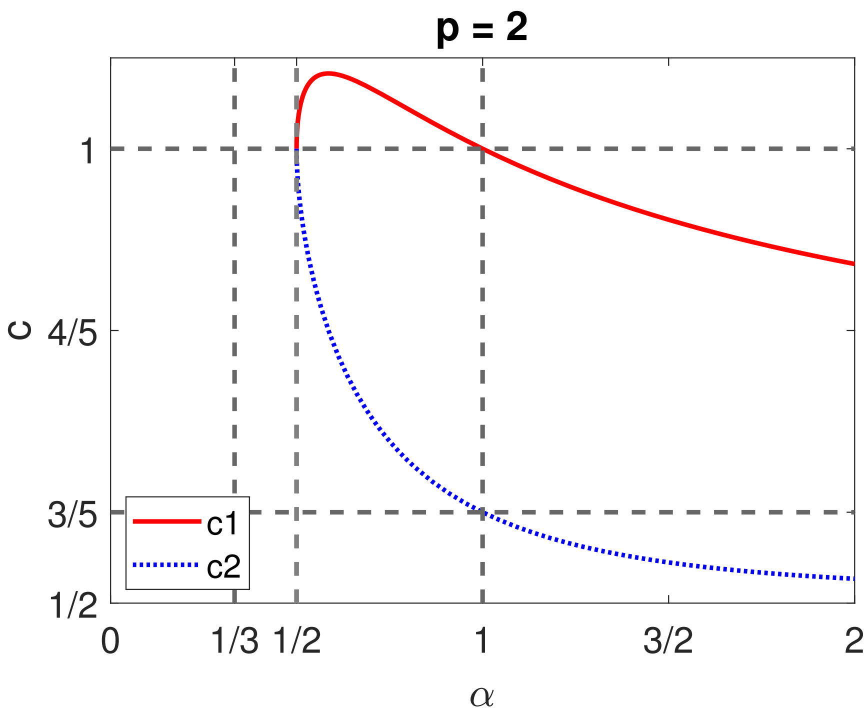

Figure 2.4 : Variations of the roots of M_p with α for $p = 2$.

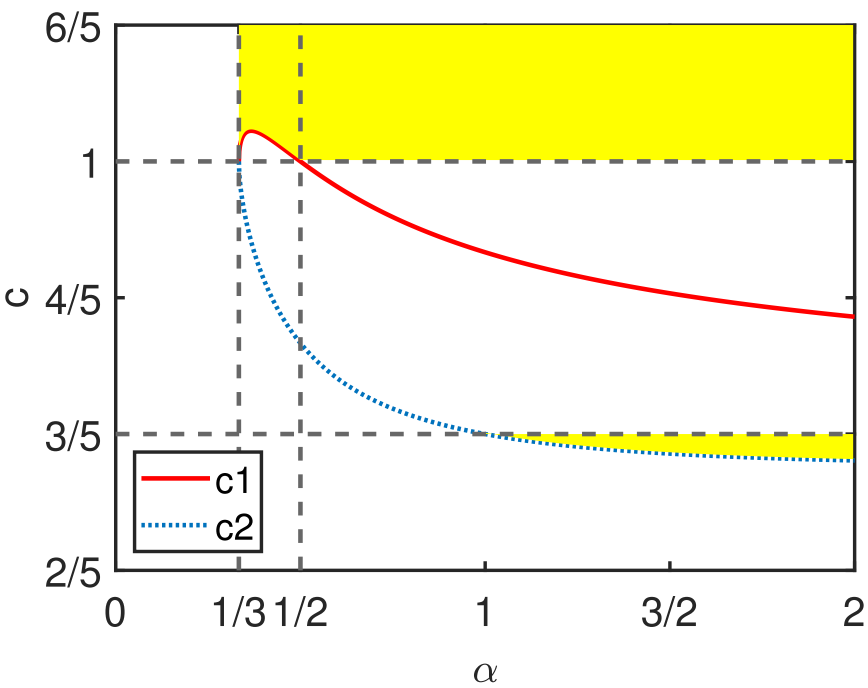

Figure 2.5 : The region for the orbital stability of solutions when $p = 1$ (shaded area).

3. NUMERICAL RESULTS FOR THE GENERALIZED FRACTIONAL BENJAMIN-BONA-MAHONY EQUATION

In this chapter, we first propose Petviashvili method to generate solitary wave profiles, numerically. Second we present Fourier pseudo-spectral method for the gfBBM equation to investigate the time evolution of these solutions. Finally, we carry out some simulations to understand the behaviour of the solutions and justify proposed schemes.

3.1 Petviashvili Method

In this section, we construct approximate solitary wave solutions of the equation (1.1). For this aim, we propose Petviashvili method. Since we do not know the analytical solitary wave solutions of the equation (1.1) for $\alpha \in (0,1)$, numerical wave generation plays an important role for the numerical experiments in this chapter. This method was first used by V. I. Petviashvili to generate solitary wave solution of the KP equation in [9]. Although this method seems like a basic fixed point iteration technique, it has been improved by introducing a stabilizing factor. In this way, Petviashvili method is easy for implementation and converges rapidly but only to the ground state solution of nonlinear wave equations.

Now, we implement the Petviashvili algorithm for the equation (1.1). In order to solve (2.47) we rewrite it in Fourier space as

$$\left(c-1+(\frac{5}{4}c-\frac{3}{4})|k|^{\alpha}\right)\widehat{Q}_c(k)=\frac{1}{2}\widehat{Q_c^{p+1}}(k). \tag{3.1}$$

Here $\widehat{Q}_c$ is the Fourier transform of Q_c. Hereafter, we use the notation Q instead of Q_c for simplicity. $\widehat{Q}_n(k)$ denotes n^{th} iteration of numerical solution in Fourier space. We propose the standard iterative algorithm to (3.1) in the form

$$\widehat{Q}_{n+1}(k)=\frac{1}{2[c-1+(\frac{5}{4}c-\frac{3}{4})|k|^{\alpha}]}\widehat{Q_n^{p+1}}(k). \tag{3.2}$$

Unfortunately, the fixed point iteration diverges or converges to trivial solution $\widehat{Q}=0$, although one may start with exact solution. To handle with this problem, (3.2) is multiplied by a stabilizing factor. Now, Petviashvili method for the gfBBM equation can be formulated as

$$\widehat{Q}_{n+1}(k) = \frac{M_n^{\nu}}{2[c-1+(\frac{5}{4}c-\frac{3}{4})|k|^{\alpha}]}\widehat{Q_n^{p+1}}(k),$$

with stabilizing factor

$$M_n = \frac{\int_{\mathbb{R}}[c-1+(\frac{5}{4}c-\frac{3}{4})|k|^{\alpha}][\widehat{Q}_n(k)]^2 dk}{\int_{\mathbb{R}}\frac{1}{2}\widehat{Q}_n^{p+1}(k)\widehat{Q}_n(k)dk},$$

for some parameter ν. According to the study [10], the fastest convergence occurs when $\nu = (p+1)/p$. Therefore to reduce the CPU time, we use $\nu = (p+1)/p$ in the numerical simulations.

3.2 The Fourier Pseudo-Spectral Method

In this section, we investigate time evolution of the numerically generated solitary waves by using a scheme combining a Fourier pseudo-spectral method for space and a fourth order Runge-Kutta method for the time integration. Since the fractional derivative in the gfBBM equation is defined by a Fourier multiplier, the Fourier spectral method will be the most appropriate method for investigating the evolution of the solution in time. In order to explain the numerical algorithms $u(x,t)$ is assumed as a solution of (1.1) in a truncated domain $[-L,L]\times[0,T]$. We assume that $u(x,t)$ has periodic boundary condition $u(-L,t) = u(L,t)$ for $(x,t) \in \mathbb{R}\times[0,T]$.

The MATLAB functions "*fft*" and "*ifft*" computes the discrete Fourier transform and its inverse for any function $f(x)$ by using efficient Fast Fourier Transform (FFT) for N equally spaced discrete points on $x \in [0,2\pi]$. Therefore, the spatial period must be normalized from finite interval $x \in [-L,L]$ to $X \in [0,2\pi]$ using the transformation $X = 2\pi(x+L)/2L$. In this case, the equation (1.1) becomes

$$\left[I+\frac{5}{4}(\frac{\pi}{L})^{\alpha}D^{\alpha}\right]u_t = -\frac{\pi}{L}u_X - \frac{\pi}{2L}(u^{p+1})_X - \frac{3}{4}(\frac{\pi}{L})^{\alpha+1}D^{\alpha}u_X. \tag{3.3}$$

The interval $[0,2\pi]$ is divided into N equal subintervals with grid spacing $\Delta X = 2\pi/N$, where the integer N is even. The spatial grid points are given by $X_j = 2\pi j/N$, $j = 0,1,2,...,N$. The approximate solutions at (X_j,t) are denoted by $U_j(t)$. The discrete Fourier transform of the sequence $\{U_j\}$, i.e.

$$\widetilde{U}_k = \mathscr{F}_k[U_j] = \frac{1}{N}\sum_{j=0}^{N-1} U_j e^{-ikX_j}, \quad -N/2 \le k \le N/2-1 \tag{3.4}$$

gives the corresponding Fourier coefficients. Likewise, $\{U_j\}$ can be recovered from the Fourier coefficients by the inversion formula for the discrete Fourier transform (3.4), as follows:

$$U_j = \mathscr{F}_j^{-1}[\widetilde{U}_k] = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \widetilde{U}_k e^{ikX_j}, \quad j = 0,1,2,...,N-1 \; . \tag{3.5}$$

Here $\mathscr{F}$ denotes the discrete Fourier transform and $\mathscr{F}^{-1}$ its inverse. Derivatives with respect to X may also be calculated approximately by using *fft* algorithm, for example, the first-order derivative at (X_j,t) is given by $\mathscr{F}_j^{-1}[ik\mathscr{F}_k[U_j]]$ and so on.

Applying the fft MATLAB function to the equation (3.3), we get the ordinary differential equation given by

$$(\widetilde{U}_k)_t = \frac{-\frac{\pi}{L}ik\widetilde{U}_k - \frac{\pi}{2L}ik(\widetilde{U^{p+1}})_k - \frac{3}{4}(\frac{\pi}{L})^{\alpha+1}ik|k|^{\alpha}\widetilde{U}_k}{1+\frac{5}{4}(\frac{\pi}{L})^{\alpha}|k|^{\alpha}}. \tag{3.6}$$

We then use the fourth order Runge-Kutta (RK4) method to solve the resulting equation (3.6) in time. The time interval $[0,T]$ is divided into M equal subintervals with time step Δt. The temporal grid points are shown by $t_m = \frac{mT}{M}, \; m = 0,...,M$.

By the help of RK4 method we can compute approximate solution at time t_{m+1} if we know the solution at any time t_m. This method applies to the equations of the form.

$$(\widetilde{U}_k)_t = G(\widetilde{U}_k).$$

The structure of the scheme can be written as

$$\begin{aligned}
U_1 &= \widetilde{U}_k^m, \quad g_1 = G(U_1) \\
U_2 &= U_1 + \frac{\Delta t}{2}g_1, \quad g_2 = G(U_2) \\
U_3 &= U_2 + \frac{\Delta t}{2}g_2, \quad g_3 = G(U_3) \\
U_4 &= U_3 + \Delta t g_3, \quad g_4 = G(U_4).
\end{aligned}$$

Now, combining the functions we obtain the solution in Fourier space as follows

$$\widetilde{U}_k^{m+1} = \widetilde{U}_k^m + \frac{\Delta t}{6}(g_1 + 2g_2 + 2g_3 + g_4).$$

Finally, the approximate solution can be obtained by applying the inversion formula of the discrete Fourier transform.

3.3 Numerical Experiments

In this section, we perform the numerical experiments to investigate the effects of dispersion and nonlinearity on the solutions and test the analytical results obtained in the previous sections. For general values of α the exact solution is unknown. Therefore, we propose the numerical control parameters such as

$$Error(n) = \|Q_n - Q_{n-1}\|, \quad n = 0, 1, ...$$

between two consecutive iterations defined with the number of iterations, the stabilization factor error

$$|1 - M_n|, \quad n = 0, 1, ...$$

and the residual error

$$RES(n) = \|\mathscr{S}Q_n\|_\infty, \quad n = 0, 1, ...$$

where

$$\mathscr{S}Q = (\frac{3}{4} - \frac{5c}{4})D^\alpha Q - (1-c)Q - \frac{1}{2}Q^{p+1}. \tag{3.7}$$

Iterative process is stopped when the all given numerical errors are less than 10^{-12}, simultaneously.

In the first numerical experiment, we test our scheme by comparing the numerical result with the exact solution. To this end, we use the analytical solitary wave solution given in (2.64) at the initial time

$$u(x,0) = \frac{4(c-1)}{1 + (\frac{c-1}{\tilde{c}})^2 x^2}.$$

The space interval and number of grid points are chosen as $x \in [-2048, 2048]$ and $N = 2^{18}$, respectively.

In Figure 3.1, we present the difference between the obtained numerical and exact solitary wave solution and the variation of three different errors with the number of iterations in semi-log scale. As it is seen from the Figure 3.1, Petviashvili method captures the solution remarkably well.

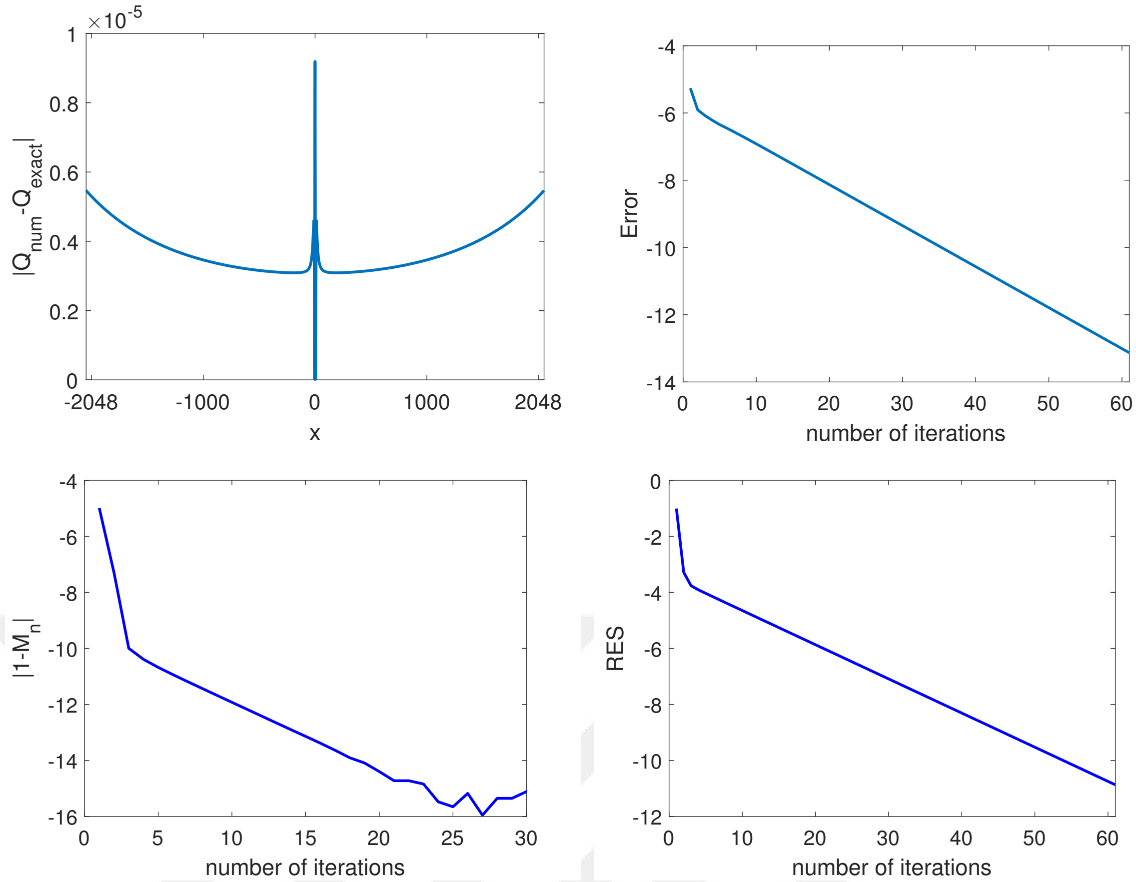

Figure 3.1 : Difference between the exact and the numerical solutions for $\alpha = 1$, $p = 1$ and the variation of the $Error(n)$, $|1 - M_n|$ and RES with the number of iterations in semi-log scale.

Since we do not know the exact solitary wave solution for different values of α, we cannot compare the numerical solution with the exact solution. Therefore, the iteration, stabilization factor and the residual errors are depicted in Figure 3.2, respectively. In this experiment we choose $0.4 \leq \alpha \leq 0.9$. These results show that the solitary wave solution generated by Petviashvili's method converges rapidly to the accurate solution. After justifying our numerical scheme we can continue with the next experiments.

In order to understand the effects of the fractional dispersion, we illustrate the solitary wave profiles generated by Petviashvili method for various values of $0 < \alpha \leq 1$ and $1 \leq \alpha \leq 2$ in Figure 3.3 and Figure 3.4, respectively. In these experiments, solitary waves generated by choosing $c = 1.1$ and $p = 1$. We observe that the solutions steepen with decreasing values of α for both cases since the nonlinearity effect becomes more visible.

In order to understand the effects of the nonlinearity, we present the solitary wave profiles generated by Petviashvili's iteration method for various nonlinearities with

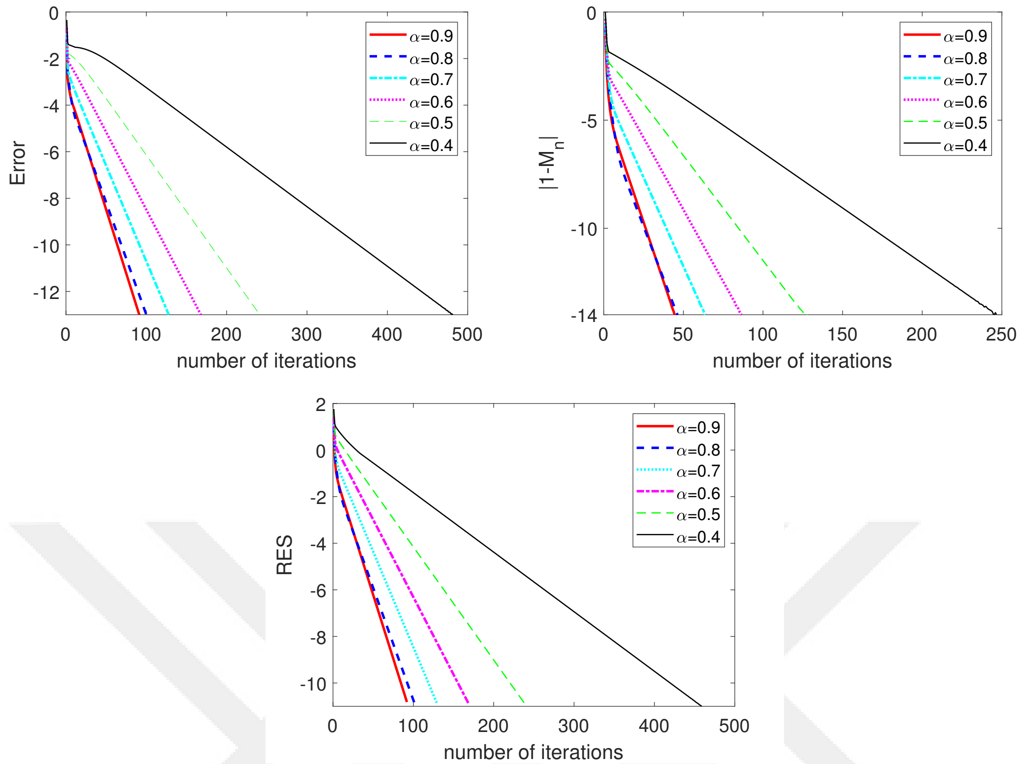

Figure 3.2 : The variation of the iteration, stabilization factor and the residual errors with the number of iterations in the semi-log scale ($c = 1.1$, $p = 1$).

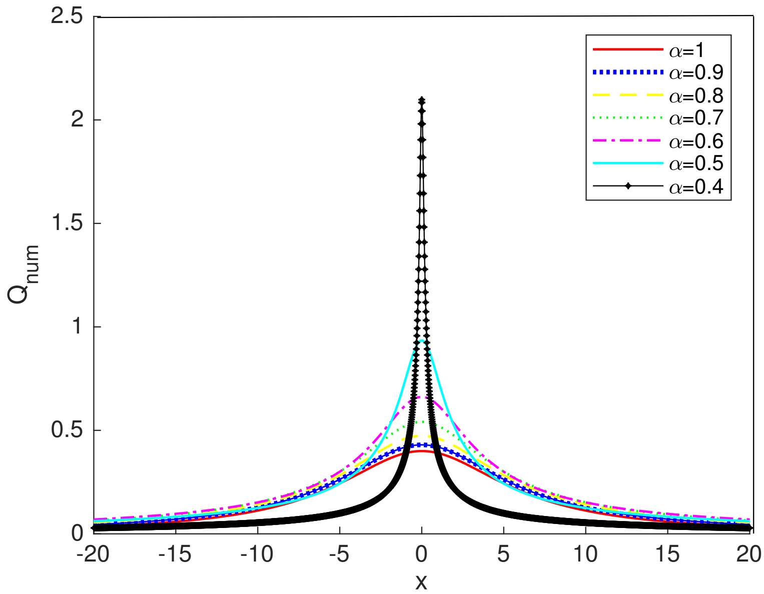

Figure 3.3 : Solitary wave profiles for various values of α $(0 < \alpha \leq 1)$ ($p = 1$, $c = 1.1$).

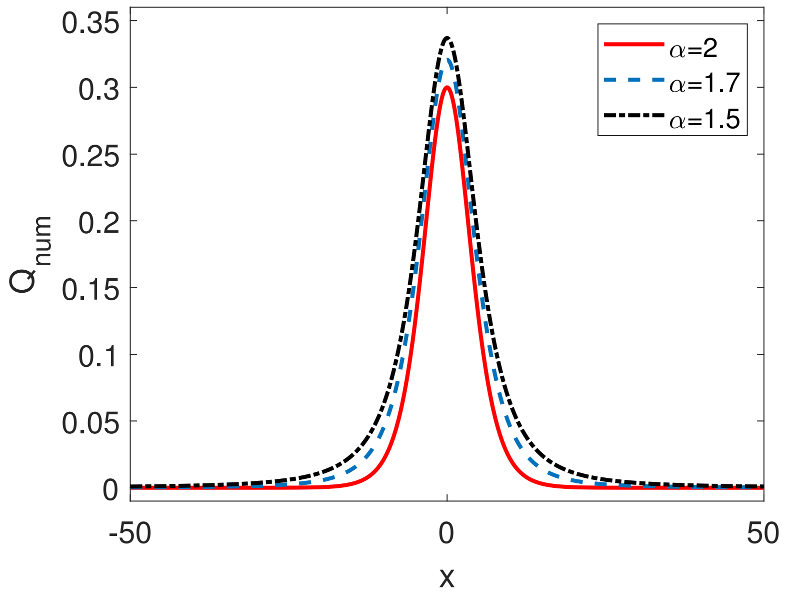

Figure 3.4 : Solitary wave profiles for various values of α $(1 \leq \alpha \leq 2)$ $(p = 1,$ $c = 1.1)$.

fixed $\alpha = 0.8$ in Figure 3.5. This numerical result agrees well with the fact that the wave becomes more peaked with increasing nonlinear effects. In the next experiment we investigate the speed-amplitude relation. We illustrate the variation of the amplitude with the speed parameter for various values of p and the fixed $\alpha = 0.8$ and various values of α and fixed nonlinearity $p = 1$ in Figure 3.6. We observe that the amplitude is increasing with the increasing values of speed c. This situation is expected for the solitary wave solutions. As it is seen from the left panel of Figure 3.6, there is a critical speed c_s near 1.5. For a fixed value of α and speed $(c < c_s)$, the amplitude increases with increasing nonlinearity. However, for a fixed value of α and speed $(c > c_s)$, the growing rate of amplitude decreases with increasing nonlinearity. The right panel of Figure 3.6 illustrates that the amplitude increases with decreasing values of α for a fixed value of nonlinearity $p = 1$ and speed.

We next present the negative solitary wave profiles for various values of α when p is fixed, and for various values of p when α is fixed in Figure 3.7. In both cases we choose $c = 0.5$. In Figure 3.3 and 3.5 we observe that, in case of positive solutions, decreasing the order of fractional derivative and increasing the order of nonlinearity have the same effect on the solutions. However, we do not observe the same behaviour for the negative solitary waves. The amplitude of the negative solitary wave solution increases with the decreasing values of α whereas the amplitude decreases with the

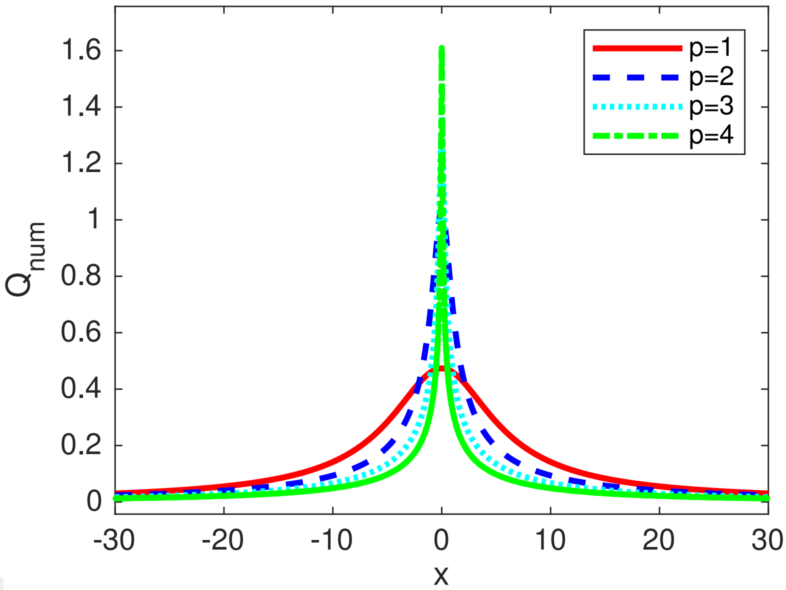

Figure 3.5 : Solitary wave profiles for various nonlinearities ($\alpha = 0.8$, $c = 1.1$).

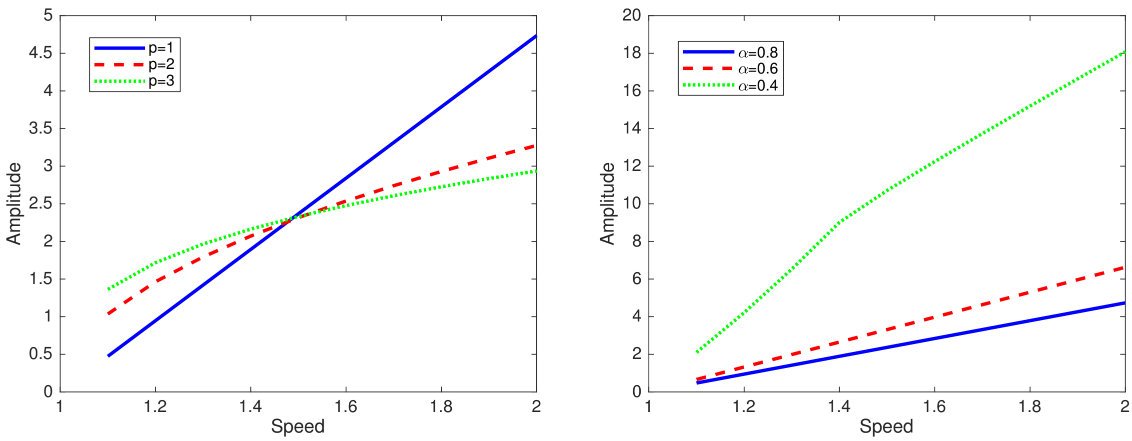

Figure 3.6 : Speed and amplitude relation for various values of p and the fixed $\alpha = 0.8$ (left panel) and for various values of α and fixed nonlinearity $p = 1$ (right panel).

increasing values of p. Furthermore, the speed-amplitude relation for negative solitary

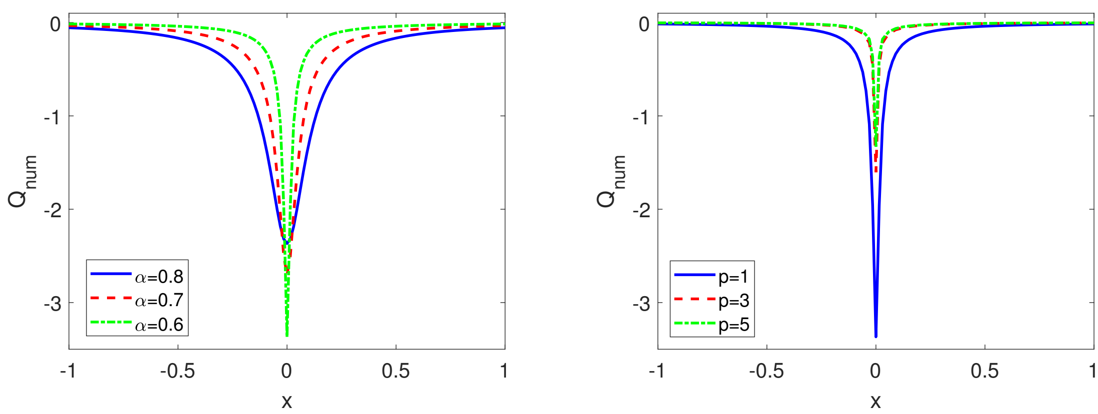

Figure 3.7 : Variation of negative solitary wave profiles with α (left panel) and with p (right panel) when $c = 0.5$.

waves is also studied and results are shown in Figure 3.8. The left panel of Figure 3.8 illustrates that the variation in the amplitude with the speed parameter for various odd values of p and the fixed $\alpha = 0.8$ when $c < \frac{3}{5}$. For a fixed value of α and speed $(c < \frac{3}{5})$, the decrease rate for the magnitude of amplitude goes down with increasing nonlinearity. The right panel of Figure 3.8 shows that the amplitude changes with the speed parameter for various values of α and fixed nonlinearity $p = 1$ when $c < \frac{3}{5}$. The decrease rate for the magnitude of amplitude increases with decreasing values of α for a fixed value of nonlinearity $p = 1$. Now we investigate the time evolution of

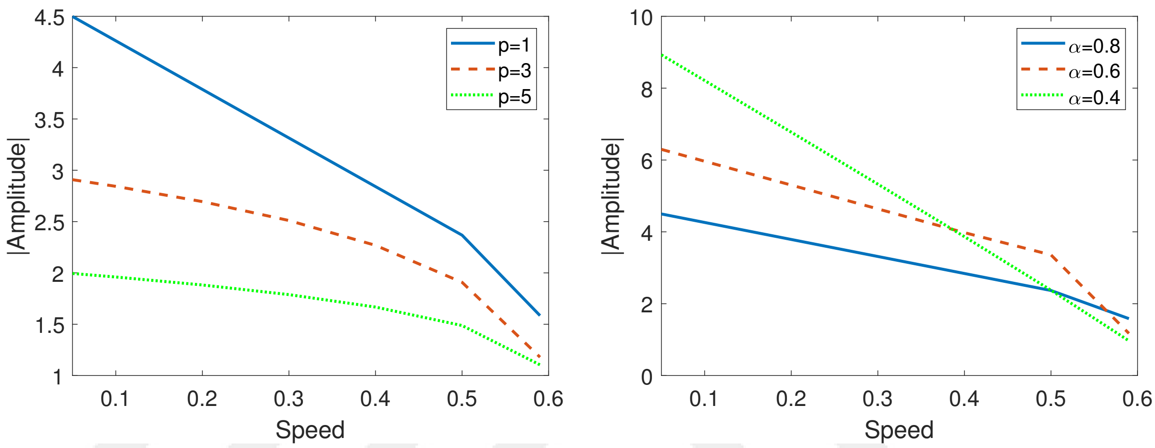

Figure 3.8 : Speed and amplitude relation for various values of p and the fixed $\alpha = 0.8$ (left panel) and for various values of α and fixed nonlinearity $p = 1$ (right panel).

the generated solutions by using Fourier pseudo-spectral method. First we show that the numerical scheme captures the exact solution (2.64) for $\alpha = 1$ well enough. We use the initial data (2.64) with $c = 1.1$, $\alpha = 1$, $p = 1$. The problem is solved in the space interval $-2048 \leq x \leq 2048$ up to $T = 20$. We set the number of grid points as $N = 2^{18}, M = 4000$. Figure 3.9 illustrates variation of change in the conserved quantity (2.8) with time and shows that it is preserved by the numerical scheme. Here we note that as the conserved quantity (2.4) is linear it is automatically preserved by the numerical scheme [35]. Figure 3.10 shows the wave profile calculated by the Fourier pseudo-spectral method with time step $\Delta t = 0.005$ at $t = 10$ and $t = 20$. To observe the wave profile more clear, we focus on the space interval $-150 \leq x \leq 150$. In Figure 3.10 we also show the change for the conserved quantity I_1 in time. It is seen from the figure, the proposed scheme conserves I_1 very well.

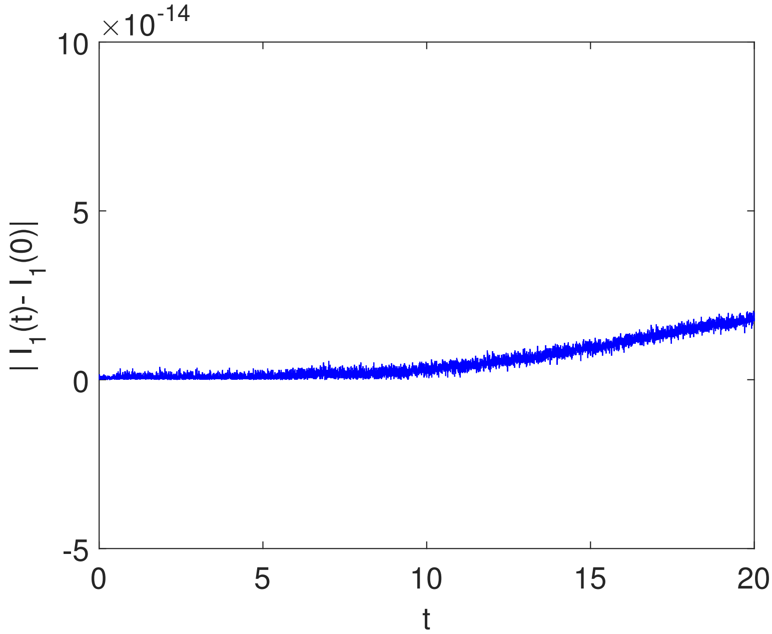

Figure 3.9 : Variation of the change in the conserved quantity I_1 with time.

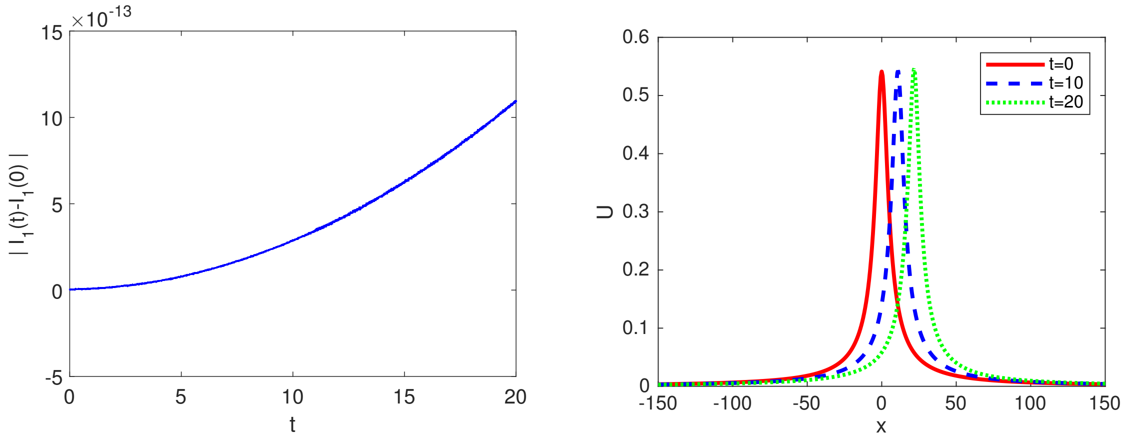

Figure 3.10 : The change in the conserved quantity I_1 with time (left panel) and time evolution of the wave profile ($c = 1.1$, $p = 1$, $\alpha = 0.6$) at several times (right panel).

4. CONCLUSIONS

This dissertation is devoted to the mathematical and the numerical analysis of the gfBBM equation.

In the first chapter, we introduced the gfBBM equation and presented literature overview for the fKdV, fBBM and gfBBM equations.

In the second chapter, we have proved the local well-posedness of solutions to the Cauchy problem for the gfBBM equation when $0 < \alpha \leq 2$. Then, the existence and non-existence results of the solitary wave solutions to the gfBBM equation have been presented. Moreover, the exact solitary wave solutions derived for $\alpha = 1$ and $\alpha = 2$ when $p = 1$. Additionally, orbital stability properties of the solitary waves to the gfBBM equation have been discussed and sufficient conditions have been obtained.

In the last section, we have presented the numerical methods and results. Petviashvili method has been proposed for construction of the solitary wave profiles to the gfBBM equation. The method is quite successful to generate ground states solutions, numerically. The Fourier pseudo-spectral method has been presented for the time evolution of generated solitary wave solutions. Validation of the method has been controlled by the help of the conserved quantities. The method has been provided highly accurate results.

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