ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

ON GEODESIC MAPPINGS OF RIEMANNIAN MANIFOLDS

M.Sc. THESIS

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Department of Mathematical Engineering

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

RIEMANN MANİFOLDLARINDA JEODEZİK DÖNÜŞÜMLER

YÜKSEK LİSANS TEZİ

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Ahmet Umut ÇORAPLI, a M.Sc. student of ITU Graduate School 509181210 successfully defended the thesis entitled "ON GEODESIC MAPPINGS OF RIEMANNIAN MANIFOLDS", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my family,

FOREWORD

I would like to express my sincere thanks to Prof. Dr. Elif Özkara CANFES, who has always shared her valuable knowledge since the beginning of Master's Degree. She has always motivated me to perform research of good quality and write an efficient thesis.

I am glad to be a student of Istanbul Technical University Department of Mathematical Engineering, where I had enjoyable and interesting courses.

Finally, I'd like to thank my family for their valuable support.

January 2022

Ahmet Umut ÇORAPLI
(Data Engineer)

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SYMBOLS

$C^{\infty}(M)$	: The set of smooth functions from the manifold M to $\mathbb{R}$
X_p	: Tangent vector at a point $p \in M$
δ	: Kronecker delta
$T_p(M)$	: Tangent space at a point $p \in M$
$\mathfrak{X}(M)$	: The set of smooth vector fields on M
$[\cdot,\cdot]$	: Lie bracket operator
$T_p^*(M)$	: Cotangent space at a point $p \in M$
$T_s^r(T_p(M))$	: The set of (r,s) tensors on M
$\otimes$	: Tensor product operator
Γ	: Christoffel symbol (connection coefficient)
(M,g)	: Riemannian manifold M with Riemannian metric g
∇	: Covariant derivative operator
$R(X,Y)$	: Curvature operator
$R(X,Y,Z,W)$	: Riemannian curvature tensor
$K(X,Y)$	: Sectional curvature
$Ric(X,Y)$	: Ricci tensor
R	: Scalar curvature
$(QE)_n$	: n-dimensional quasi Einstein manifold
$N(QE)_n$	: n-dimensional nearly quasi Einstein manifold
R_n	: n-dimensional Ricci recurrent manifold
GRK_n	: n-dimensional generalized Ricci recurrent manifold
$(PRS)_n$	: n-dimensional pseudo Ricci symmetric manifold
$(APRS)_n$	: n-dimensional almost pseudo Ricci symmetric manifold

ON GEODESIC MAPPINGS OF RIEMANNIAN MANIFOLDS

SUMMARY

Riemannian geometry is a branch of differential geometry that studies smooth manifolds with Riemannian metric, which is a non-degenerate and a positive definite tensor field. It was found by Bernhard Riemann in 1850's by the generalization of three dimensional surfaces. In this thesis, we will investigate geodesic mappings of Riemannian manifolds. These maps play a key role in mechanics and general theory of relativity. First, the literature on geodesic mappings is reviewed. Then, the definitions and formulas of smooth manifolds and Riemannian manifolds are given. Next, the general rules and the basic formulas of geodesic mappings are considered. Furthermore, geodesic mappings on some special Riemannian manifolds are investigated. The thesis ends with our conclusions and recommendations.

The thesis consists of five chapters.

In the introduction chapter, a brief description of geodesic mappings on Riemannian manifolds is given. This chapter also includes the literature review of geodesic mappings. We should remark that Josef Mikes, a mathematician from Palacky University Olomouc, Czech Republic, has important contributions to the investigation of geodesic mappings. Some of the manifolds that he has studied so far involve 2 Ricci symmetric manifolds, Einstein manifolds, 3 symmetric manifolds, manifolds with affine connection, generalized Ricci symmetric manifolds and Ricci flat manifolds.

The second chapter is the preliminaries part of the thesis. This chapter begins with the definition of a topological manifold. Extra structures that are needed to define smooth manifolds are given, namely C^∞-compatible charts and maximal atlases. Next, the vital concepts of differential geometry are explained. These involve differentiable maps, tangent vectors, tangent spaces, vector fields and Lie brackets. Furthermore, the conventions and the notations that will be used throughout the thesis are stated. Then, tensors on smooth manifolds are defined and their properties are given. Next, an affine connection (covariant derivative) and the connection coefficients (Christoffel symbols) are defined. In the last part, a Riemannian manifold and a Riemannian connection are defined. Moreover, the fundamental theorem of Riemannian geometry is proven. Important geometrical objects such as Riemannian curvature tensor, Ricci tensor, scalar curvature are defined. In addition, the definitions of the special Riemannian manifolds on which geodesic mappings will be investigated are given.

In the next chapter, the main subject of the thesis is explained in detail. This chapter begins with the definition of vector fields parallel along a curve. Next, geodesics are defined by using this concept. After giving these definitions, the differential equations of geodesics are derived. Next, geodesic mappings are defined and the necessary and sufficient conditions for the existence of such mappings are stated. Moreover, the important formulas under arbitrary geodesic mappings are given. This chapter

ends with the derivation of Sinyukov equations, which are the equivalent necessary and sufficient conditions for the existence of geodesic mappings. Sinyukov equations are very crucial for the investigation of special geodesic mappings of Riemannian manifolds.

In the fourth chapter, geodesic mappings of some special Riemannian manifolds are considered. First, the proof of Mikes' Theorem on geodesic mappings of Einstein manifolds is given. Then, the important results of Chepurna's PhD thesis (Einstein Tensor Preserving Geodesic Mappings) are stated. Using the results of Chepurna's thesis, a new result on quasi Einstein manifolds is obtained. In addition, we consider geodesic mappings of generalized Ricci recurrent manifolds. Next, we examine geodesic mappings of pseudo Ricci symmetric and almost pseudo Ricci symmetric manifolds.

In the last chapter, the results of the thesis are stated, which are the following:

(i) If there exists an Einstein tensor preserving geodesic mapping from a quasi Einstein manifold V_n onto a Riemannian manifold $\bar{V}_n$, then $\bar{V}_n$ is nearly quasi Einstein.

(ii) Let $V_n = (M, g, \nabla)$ and $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$ be two Riemannian manifolds. If V_n and $\bar{V}_n$ are in geodesic correspondence and V_n is generalized Ricci recurrent, then the following identity holds

$$\lambda_h \left(nR^h_k - \delta^h_k R \right) = 0,$$

where $R^h_k = g^{jh} R_{jk}$, λ_h is a gradient vector and δ^h_k is the Kronecker delta.

(iii) If $V_n = (M, g)$ is a pseudo Ricci symmetric manifold admitting geodesic mapping onto $\bar{V}_n = (\bar{M}, \bar{g})$ and $\nabla_k \psi_{ij} = 2A_k \psi_{ij} + A_i \psi_{kj} + A_j \psi_{ik}$, then $\bar{V}_n$ is pseudo Ricci symmetric.

(iv) If $V_n = (M, g)$ is an almost pseudo Ricci symmetric manifold admitting geodesic mapping onto $\bar{V}_n = (\bar{M}, \bar{g})$ and $\nabla_k \psi_{ij} = (A_k + B_k) \psi_{ij} + A_i \psi_{kj} + A_j \psi_{ik}$, then $\bar{V}_n$ is almost pseudo Ricci symmetric.

Moreover, in the same chapter, we discuss the special Riemannian manifolds for which further studies of geodesic mappings can be done. These manifolds involve Ricci solitons and quasi Einstein manifolds. For quasi Einstein manifolds, an investigation can be made by either discarding our assumption that the mapping is Einstein tensor preserving or having a new separate assumption.

RIEMANN MANİFOLDLARINDA JEODEZİK DÖNÜŞÜMLER

ÖZET

Riemann geometrisi, diferansiyel geometrinin en önemli alanlarından birisidir. Bernhard Riemann tarafından 1850'li yıllarda üç boyutlu yüzeylerin genelleştirilmesi ile keşfedilmiştir. Bu geometride kullanılan metriğe Riemann metriği, bu geometride incelenen uzaylara ise Riemann uzayları (Riemann manifoldları) adı verilir.

Sadece diferansiyel geometri alanında değil, aynı zamanda mekanik ve genel görelilik kuramında da önemli bir yere sahip olan jeodezik dönüşümler, jeodeziklerin, bir manifold üzerindeki iki nokta arasındaki en kısa yolun, korunmasıyla elde edilir. Tezin amacı, jeodezik dönüşümleri Riemann manifoldları üzerinde ayrıntılı olarak incelemek ve bu dönüşümlerin ele alınmadığı özel Riemann manifoldları ile ilgili gerek ve yeter koşullar elde etmektir.

Beş bölümden oluşan tezin giriş bölümünde jeodezik dönüşümlerin tanımı yapılmış ve bu dönüşümler altında geçerli olan temel formüller verilmiştir. Jeodezik dönüşümlerin literatür taraması yapılarak giriş bölümü tamamlanmıştır. Sonraki bölümde; düzgün manifoldlar, tensörler, afin koneksiyonlar ve Riemann manifoldlar ile ilgili tanımlar ve formüller verilmiştir. Üçüncü bölümde, tezin ana konusuna geçilmiştir. Jeodezik tanımı yapılmış ve jeodeziklerin diferansiyel denklemleri elde edilmiştir. Riemann manifoldları üzerinde jeodezik dönüşüm tanımı yapıımış ve bu dönüşümlerin genel kuralları açıklanmıştır. Dördüncü bölümde, özel Riemann manifoldları üzerindeki jeodezik dönüşümler incelenmiş ve bu manifoldlar arasında jeodezik dönüşümlerin var olabilmesi için gerek ve yeter koşullar elde edilmiştir. Öneriler ve sonuçlar bölümü ile tez tamamlanmıştır.

Tezin birinci bölümü olan giriş bölümünde jeodezik dönüşümlerin tanımı ve bu alandaki çalışmalara yer verildi. Çalışmalarının çoğunluğunu jeodezik dönüşümlere adayan Josef Mikes bu alandaki en önemli isimlerden biridir. Jeodezik dönüşümlerin incelendiği manifoldlara; sabit eğriliğe sahip manifoldlar, simetrik manifoldlar, Ricci m-simetrik manifoldar, Einstein manifoldlar, afin koneksiyona sahip manifoldlar, konformal düz manifoldlar, Ricci rekürent manifoldlar, Einstein olmayan Ricci 2-simetrik manifoldlar, 4 boyutlu ve sabit eğriliğe sahip olmayan Einstein manifoldlar, 3-simetrik Riemann manifoldlar örnek verilebilir. Ayrıca, Riemann manifoldlarındaki jeodezik dönüşümler altında geçerli olan temel formüllere de bu bölümde yer verildi.

Sonraki bölüm ise tezin ön hazırlığını oluşturan bölümdür. İlk olarak bir Hausdorff uzayının lokal olarak bir Öklid uzayına benzemesi tanımlanmıştır. Bu tanım kullanılarak topolojik manifold tanımı yapılmıştır. Daha sonrasında iki kartın C^∞ uyumlu olması için gereken koşullar belirtilmiştir. Atlas ve maksimal atlas tanımı yapıldıktan sonra düzgün manifoldlar tanımlanmıştır. Düzgün manifoldlara örnekler verilmiş ve bu manifoldlar üzerinde geometrik yapılar incelenmiştir. Bu yapılara diferansiyellenebilir fonksiyonlar, difeomorfizmalar, teğet vektörler, teğet uzaylar,

vektör alanlar ve Lie işlemcileri örnek verilebilir. Daha sonra ise, tezde kullanılan notasyonlar ve konvansiyonlar açıklanmıştır. Dual uzay tanımı verildikten sonra kotanjant uzayı ve 1-form tanımlanmıştır. $\mathbb{R}^n$ ve düzgün manifoldlar üzerinde tensör tanımı yapıldıktan sonra, diferansiyel geometri ve fizik için önemli bir kavram olan tensör alan tanımı verilmiştir. Simetrik ve asimetrik tensör tanımları da bu kısımda yapılmış olup, lokal koordinatlarda $(0,2)$ tensörlerin simetrik ve asimetrik olma durumları incelenmiştir. Afin koneksiyonun tanımı ve özellikleri açıklandıktan sonra Christoffel sembolleri (koneksiyon katsayıları) tanımlanmıştır. Bu bölümün sonunda Riemann geometrisinin temel tanımları ve formülleri verilmiştir. İlk önce, Riemann manifoldları ve Riemann metriği tanımlanmıştır. Ardından, Riemann koneksiyon tanımı verilip Riemann geometrisinin temel teoremi kanıtlanmıştır. Riemann eğrilik tensörü, Ricci tensörü, kesit eğriliği ve skaler eğrilik tanımları da bu bölümde verilmiştir. Schur teoremi ispatlandıktan sonra sabit eğrilik tanımı yapılmıştır. Jeodezik dönüşümlerin inceleneceği özel Riemann manifoldlarının tanımı verilerek bu bölüm tamamlanmıştır.

Üçüncü bölümde tezin ana konusuna giriş yapılmıştır. Riemann manifoldları üzerinde bir vektör alanın bir eğri boyunca paralel olmasının tanımı verilmiş ve bu tanım kullanılarak jeodezikler tanımlanmıştır. Jeodeziklere ait eşdeğer bir tanım verildikten sonra, jeodeziklerin diferansiyel denklemleri elde edilmiştir. İki Riemann manifoldu arasında tanımlanan bir difeomorfizma bütün jeodezikleri koruyorsa, bu difeomorfizmaya jeodezik dönüşüm denir. Jeodeziklerin diferansiyel denklemleri dikkate alınarak jeodezik dönüşümlerin gerek ve yeter koşulları elde edilmiştir. Bu koşullar sonucunda elde edilen denklem Levi-Civita denklemi olarak adlandırılır. Sonra, iki önemli sanı kanıtlanıp, iki manifold arasındaki Riemann eğrilik tensörü ve Ricci tensörü arasındaki ilişki bulunmuştur. Ayrıca, Riemann manifoldları üzerinde Weyl tensörü tanımlanmış ve jeodezik dönüşüm altında bu tensörün korunduğu kanıtlanmıştır. Daha sonra ise Beltrami'nin Teoremi ispatlanmıştır. Bu teoreme göre iki Riemann manifoldu arasında bir jeodezik dönüşüm var ise ve bir manifold sabit eğriliğe sahip ise, diğer manifold da sabit eğriliğe sahiptir. Jeodezik dönüşümlerin eşdeğer gerek ve yeter koşulu olan Sinyukov denklemleri elde edilerek bu bölüm tamamlanmıştır. Sinyukov denklemleri, özel Riemann manifoldları üzerinde jeodezik dönüşümleri araştırmak için çok önemli bir yere sahiptir.

Dördüncü bölümde jeodezik dönüşümler özel Riemann manifoldları üzerinde incelendi. Bu bölümdeki hesaplar lokal koordinatlar kullanarak yapıldı. İlk olarak daha önce yapılmış çalışmalara yer verildi. Mikes'in Einstein manifoldları üzerine olan teoreminin ispatı verildi. Bu teoreme göre iki Riemann manifoldu arasında bir jeodezik dönüşüm var ise ve bir manifold Einstein manifoldu ise diğer manifold da Einstein manifoldudur. Ayrıca, Einstein tensörünü koruyan jeodezik dönüşümleri ele alan Chepurna'nın doktora tezindeki önemli sonuçlar belirtildi. Chepurna'nın tezindeki sonuçlar kullanılarak bir yarı Einstein manifold üzerinden tanımlanan ve Einstein tensörünü koruyan jeodezik dönüşümler için özel bir şart bulundu. Daha sonra ise genelleştirilmiş Ricci rekürent manifodlar için yeni sonuçlar elde edildi. Sözde Ricci simetrik ve hemen hemen sözde Ricci simetrik manifoldların jeodezik dönüşümleri de incelendi.

Beşinci bölümde elde ettiğimiz yeni sonuçlara yer verilmiştir. Bu sonuçlar aşağıdaki gibidir:

(i) Bir $V_n = (M, g)$ yarı Einstein manifoldu üzerinden herhangi bir $\bar{V}_n = (\bar{M}, \bar{g})$ Riemann manifoldu üzerine tanımlı Einstein tensörünü koruyan jeodezik dönüşüm var ise $\bar{V}_n$ neredeyse yarı Einstein'dır.

(ii) Genelleştirilmiş Ricci rekürent manifoldu üzerinden herhangi bir Riemann manifolduna tanımlı jeodezik dönüşüm mevcut ise aşağıdaki koşul sağlanmaktadır:

$$\lambda_h \left(nR_k^h - \delta_k^h R \right) = 0,$$

bu denklemde ∇ kovaryant türev operatörü, R_{ij} Ricci tensörü ve ψ_{ij} ise bir $(0,2)$ simetrik tensördür.

(iii) $V_n = (M, g, \nabla)$ ve $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$ iki Riemann manifoldu olsun. V_n'den $\bar{V}_n$'e tanımlı bir jeodezik dönüşüm olduğunu ve V_n'in sözde Ricci simetrik olduğunu varsayalım. $\nabla_k \psi_{ij} = 2A_k \psi_{ij} + A_i \psi_{kj} + A_j \psi_{ik}$ koşulu sağlanıyorsa, $\bar{V}_n$ manifoldu da sözde Ricci simetriktir.

(iv) $V_n = (M, g, \nabla)$ ve $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$ iki Riemann manifoldu olsun. V_n'den $\bar{V}_n$'e tanımlı bir jeodezik dönüşüm olduğunu ve V_n'in hemen hemen sözde Ricci simetrik olduğunu varsayalım. $\nabla_k \psi_{ij} = (A_k + B_k)\, \psi_{ij} + A_i \psi_{kj} + A_j \psi_{ik}$ koşulu sağlanıyorsa, $\bar{V}_n$ manifoldu da hemen hemen sözde Ricci simetriktir.

Ayrıca, bu bölümde jeodezik dönüşümler üzerine yapılabilecek çalışmalara yer verildi. Ricci solitonları üzerine yeni bir çalışma yapılabilir veya tezimizde quasi Einstein manifoldlar için kabul ettiğimiz için koşul değiştirilerek daha genel bir sonuç elde edilebilir.

1. INTRODUCTION

The thesis is devoted to the investigation of geodesic mappings on some special Riemannian manifolds. First, we give the definitions and formulas of Riemannian geometry. Next, we investigate the laws and the basic equations of geodesic mappings. After that, we examine geodesic mappings on some special Riemannian manifolds.

In this chapter and throughout the thesis, all the definitions and the properties of geodesic mappings will be given for Riemannian manifolds unless otherwise stated. Before defining geodesic mappings, we will first define geodesics:

Let (M,g) be a Riemannian manifold with Riemannian connection ∇. Let γ: $J \to M$ be a smooth curve on M, where J is an open interval. The curve γ is called a *geodesic* on M if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, [1].

A geodesic mapping from $V_n = (M,g)$ onto $\bar{V}_n = (\bar{M},\bar{g})$ is a diffeomorphism that preserves the geodesics, [2].

Formally, a geodesic mapping on Riemannian manifolds can be defined as follows:

Let V_n and $\bar{V}_n$ be two Riemannian manifolds with common coordinate system x, and let f be a diffeomorphism from V_n onto $\bar{V}_n$. f is a *geodesic mapping* if and only if the Levi-Civita equation

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X$$

is satisfied for any tangent vector fields X and Y, where ψ is some 1-form. If $\psi \not\equiv 0$, then the map is nontrivial. Otherwise, it is trivial, [2].

Due to the nature of the subject, we will carry out our studies in local coordinates. This will aid us to perform an efficient research and it will be advantageous for the calculations throughout the thesis.

In local coordinates, the necessary and sufficient condition for existence of a geodesic mapping from V_n onto $\bar{V}_n$ is

$$\bar{\Gamma}^l_{ij} = \Gamma^l_{ij} + \delta^l_i \psi_j + \delta^l_j \psi_i,$$

where Γ^l_{ij} and $\bar{\Gamma}^l_{ij}$ are the Christoffel symbols of V_n and $\bar{V}_n$, respectively, and ψ_i are the components of the 1-form ψ, [2].

Moreover, under geodesic mappings, the following conditions hold:

$$\bar{R}^h_{ijl} = R^h_{ijl} + \delta^h_l \psi_{ij} - \delta^h_j \psi_{il}$$

and

$$\bar{R}_{ij} = R_{ij} + (n-1)\, \psi_{ij},$$

where $\psi_{ij} = \nabla_j \psi_i - \psi_i \psi_j$, [3].

Sinyukov proved that below set of equations, known as the Sinyukov equations, are the equivalent necessary and sufficient conditions for the existence of geodesic mappings from a Riemannian manifold $V_n = (M, g)$ onto $\bar{V}_n = (\bar{M}, \bar{g})$:

$$\begin{aligned}
&\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ki},\\
&n\nabla_l \lambda_i = \mu g_{il} + a_{ih} g^{jh} R_{jl} - a_{hj} g^{jm} R^h_{ilm},\\
&(n-1)\nabla_k \mu = 2(n+1)\lambda_l R_{kj} g^{jl} + a_{ih} g^{jh} g^{il} \left(2\nabla_l R_{jk} - \nabla_k R_{jl}\right),
\end{aligned}$$

where $a_{ij} = e^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}$, $\lambda_i = -e^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha$ and $\mu = g^{\alpha\beta} \nabla_\beta \lambda_\alpha$. In this case, the map is nontrivial if $\lambda_i \not\equiv 0$, [4].

Geodesic mappings play a key role in mechanics and general theory of relativity. Moreover, it is an interesting subject of differential geometry and many mathematicians have dedicated their studies to the theory of geodesic mappings. In addition to the theory, mathematicians have been also investigating the geodesic mappings of special manifolds. Some of these manifolds involve Einstein manifolds, compact Riemannian manifolds, affine connected manifolds, manifolds with constant curvature, conformally flat manifolds, symmetric manifolds, Ricci symmetric manifolds. Before giving the important results, we should remark that geodesic mappings were first considered for manifolds of constant curvature.

Some of the important results on geodesic mappings are as follows:

In 1972; Rosenfeld and Gorbaty proved that if a conformally flat Riemannian manifold C_n, $(n \geq 3)$ admits a nontrivial geodesic mapping onto $\bar{V}_n$, then $\bar{V}_n$ is either subprojective or it is a hypersurface of a manifold with constant curvature, [3].

In 1976; Mikes proved that if there exits a geodesic mapping from a Riemannian manifold V_n of nonconstant curvature onto a semisymmetric equiaffine manifold $\bar{A}_n$, then $\psi_{ij} = Bg_{ij}$ holds in V_n and moreover, the following identity holds:

$$\nabla_m \nabla_l R_{hijk} - \nabla_l \nabla_m R_{hijk} = B \left(g_{mh} R_{lijk} + g_{mi} R_{hljk} + g_{mj} R_{hilk} + g_{mk} R_{hijl} - g_{lh} R_{mijk} - g_{li} R_{hmjk} - g_{lj} R_{himk} - g_{lk} R_{hijm} \right),$$

where B is a constant, [3].

In January 1978; Mikes proved that a non-Einstein Ricci 2-symmetric manifold V_n $(n \geq 3)$ does not admit nontrivial geodesic mappings, [5] .

In December 1978; Mikes also showed if there exists a geodesic mapping from an Einstein manifold V_n onto another Riemannian manifold $\bar{V}_n$, then $\bar{V}_n$ is also Einstein, [6].

In 1979; Sinyukov proved that if V_n is a manifold with covariantly nonconstant concircular vector fields, then it admits nontrivial geodesic mappings. Moreover, he also proved the vice versa, i.e., on the manifolds V_n which do not admit nontrivial geodesic mappings, there are no covariantly nonconstant concircular vector fields, [3].

In 1982; Mikes and Kiosak proved that if V_n is a four-dimensional Einstein manifold with nonconstant curvature, then it does not admit nontrivial geodesic mappings, [3].

In 1994; Mikes and Sobchuk proved that 3-symmetric Riemannian manifolds V_n $(n \geq 3)$ of nonconstant curvature and Ricci 3-symmetric manifolds V_n $(n \geq 5)$ which are not Einstein, do not admit nontrivial geodesic mappings, [7].

In 2018; Berezovski, Hinterleitner and Mikes investigated the geodesic mappings from manifolds with affine connection onto Ricci symmetric manifolds. They proved the following: A manifold A_n with affine connection ∇ admits a geodesic mapping onto a Ricci symmetric manifold $\bar{A}_n$ if and only if the below Cauchy type equations in covariant derivative are satisfied, [8]

$$\nabla_m \bar{R}_{ij} = 2\psi_m \bar{R}_{ij} + \psi_i \bar{R}_{mj} + \psi_j \bar{R}_{im}$$

and

$$\nabla_j \psi_i = \frac{1}{n^2 - 1} \left[n\bar{R}_{ij} + \bar{R}_{ji} - \left(nR_{ij} + R_{ji} \right) \right] + \psi_i \psi_j.$$

Moreover, in the same work, [8], the authors proved the following: A manifold A_n with affine connection ∇ admits a geodesic mapping onto a symmetric manifold $\bar{A}_n$ if and only if the below Cauchy type equations in covariant derivative are satisfied

$$\nabla_j \psi_i = \frac{1}{n^2-1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - \left(nR_{ij} + R_{ji}\right)\right] + \psi_i \psi_j,$$

$$\nabla_m \bar{R}^h_{ijk} = 2\psi_m \bar{R}^h_{ijk} + \psi_i \bar{R}^h_{mjk} + \psi_j \bar{R}^h_{imk} + \psi_k \bar{R}^h_{ijm} - \delta^h_m \psi_\alpha \bar{R}^\alpha_{ijk}$$

and

$$\bar{R}^h_{i(jk)} = 0, \ \bar{R}^h_{(ijk)} = 0,$$

where brackets imply symmetrization.

In 2020; Berezovski, Cherevko, Hinterleitner and Peska proved the following theorem: There exists a geodesic mapping from a manifold A_n with an affine connection onto a 2-symmetric manifold $\bar{A}_n$ if and only if the below Cauchy type equations in covariant derivative are satisfied

$$\nabla_j \psi_i = \psi_i \psi_j + \frac{2}{n^2-1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - \left(nR_{ij} + R_{ji}\right)\right],$$

$$\begin{aligned}\nabla_\rho \nabla_m \bar{R}^h_{ijk} = {} & 2\psi_m \nabla_\rho \bar{R}^h_{ijk} + \psi_j \nabla_\rho \bar{R}^h_{imk} + \psi_k \nabla_\rho \bar{R}^h_{ijm} + \psi_i \nabla_\rho \bar{R}^h_{mjk} \\ & - \delta^h_m \psi_\alpha \nabla_\rho \bar{R}^\alpha_{ijk} - B^h_{ijkm\rho} + C^h_{ijkm\rho}\end{aligned}$$

and

$$\bar{R}^h_{i(jk)} = \bar{R}^h_{(ijk)} = 0, \ \nabla_m \bar{R}^h_{i(jk)} = \nabla_m \bar{R}^h_{(ijk)} = 0,$$

where $B^h_{ijkm\rho}$ and $C^h_{ijkm\rho}$ are some $(1,5)$ tensors and brackets imply symmetrization, [9].

In this thesis, we will investigate the geodesic mappings of quasi Einstein, generalized Ricci recurrent, pseudo Ricci symmetric and almost pseudo Ricci symmetric manifolds.

2. PRELIMINARIES

In this chapter, some basic notations and definitions used in the thesis will be given.

2.1 Smooth Manifolds

Definition 2.1.1 Let M be a Hausdorff space. If every point in M has a neighborhood U such that there is a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$, the pair (U,φ) is called a *chart*. U is a *coordinate neighborhood* and φ is a *coordinate system* on U. In this case, M is called a *locally Euclidean space*.

Definition 2.1.2 A *topological manifold* is a Hausdorff, second countable and locally Euclidean space. It is said to be *n-dimensional* if every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Example 2.1.3 $\mathbb{R}^n$ is an n-dimensional topological manifold: $\mathbb{R}^n$ can be covered by a single chart $(\mathbb{R}^n, i)$, where $i\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

Remark 2.1.4 Suppose (U,φ) and (V,ψ) are two charts of a topological manifold M, where U and V are open subsets of $\mathbb{R}^n$. Since φ is a homeomorphism, and $U\cap V$ is open, it follows that $\varphi(U\cap V)$ is open.

Definition 2.1.5 Two charts (U,φ) and (V,ψ) of a topological manifold M are C^∞*-compatible* if the following two maps are C^∞:

i) $\varphi\circ\psi^{-1}\colon \psi(U\cap V)\to\varphi(U\cap V)$,

ii) $\psi\circ\varphi^{-1}\colon \varphi(U\cap V)\to\psi(U\cap V)$.

These two functions are called *transition functions* or *transition maps*.

We will give an example to emphasize that both maps have to be C^∞, as stated in the above definition.

Example 2.1.6 Let $A=\{(\mathbb{R},\varphi),(\mathbb{R},\psi)\}$ be a set consisting of two charts, where $\varphi\colon \mathbb{R}\to\mathbb{R}$, $a\mapsto\varphi(a)=a$ and $\psi\colon \mathbb{R}\to\mathbb{R}$, $a\mapsto\psi(a)=a^3$. Then, the set A is not C^∞-compatible.

Clearly, φ and ψ are homeomorphisms.

i) $\left(\psi\circ\varphi^{-1}\right)(a)=\psi\left(\varphi^{-1}(a)\right)=\psi(a)=a^3$ is a differentiable function. So, $\psi\circ\varphi^{-1}$ is C^∞.

ii) $\left(\varphi\circ\psi^{-1}\right)(a)=\varphi\left(\psi^{-1}(a)\right)=\varphi(a^{1/3})=a^{1/3}$ is not differentiable at 0. Thus, $\varphi\circ\psi^{-1}$ is not C^∞.

Therefore, the set A is not C^∞-compatible.

Definition 2.1.7 A C^∞*-atlas* or simply an *atlas* on a locally Euclidean space M is a collection $B=\{(U_\alpha,\varphi_\alpha)\}$ which are pairwise C^∞-compatible charts that cover M.

Definition 2.1.8 An atlas on a locally Euclidean space is *maximal*, if it is not contained in a larger atlas.

Definition 2.1.9 A *smooth* or C^∞ *manifold* is a topological manifold M which has a maximal atlas. In other words, a smooth manifold is a topological manifold for which all the transition maps are smooth.

Example 2.1.10 Some examples of smooth manifolds:

i) $\mathbb{R}^n$ is an n-dimensional smooth manifold with a single chart $\left(\mathbb{R}^n,u^1,\cdots,u^n\right)$, where u^i are standard coordinates in $\mathbb{R}^n$.

ii) Any open subset of a manifold is a smooth manifold.

iii) General linear group $GL(n,\mathbb{R}):=\{A\in\mathbb{R}^{n\times n}|\ \det(A)\neq 0\}$ is a smooth manifold.

iv) n-dimensional sphere $S^n:=\left\{(x_1,\cdots,x_{n+1})\in\mathbb{R}^{n+1}|\ x_1^2+\cdots+x_{n+1}^2=1\right\}$ is a smooth manifold.

Definition 2.1.11 Let M be an n-dimensional smooth manifold. Let $p=(x_1,..,x_n)\in M$ and (U_i,φ_i) be charts of M. Lef $f\colon U\subset M\to\mathbb{R}$ be a function. f is *differentiable at p*, if $f\circ\varphi^{-1}$ is differentiable at $\varphi(p)$. The set of differentiable functions from M to $\mathbb{R}$ is denoted by $C^\infty(M)$.

Definition 2.1.12 Suppose that $f\colon M\to N$ is a map from a C^∞ manifold M of dimension m to another C^∞ manifold N of dimension n. If there exists a chart (U,φ) at $p\in M$ and (V,ψ) at $f(p)\in N$ such that $\psi\circ\varphi^{-1}$ is differentiable at $\varphi(p)$, then f is called *differentiable at p*. If f is differentiable at every $p\in M$, then f is called a *differentiable map* from M to N.

In the Definition 2.1.12, let $M=(a,b)\subset\mathbb{R}$. Then $f\colon (a,b)\to N$ is called a *curve* on N. If f is smooth, then it is called a *smooth curve* on N.

Definition 2.1.13 Let M and N be two smooth manifolds. If $f\colon M\to N$ is a differentiable, bijective map and has a differentiable inverse, then f is called a *diffeomorphism*.

Let M be an n-dimensional smooth manifold. Our aim is to assign an n-dimensional vector space isomorphic to $\mathbb{R}^n$, which will be called tangent space, for each point p of the manifold. For this purpose, a tangent vector must be defined.

Definition 2.1.14 Let M be an n-dimensional smooth manifold and $p\in M$. Let a and b be real numbers and f and g be real-valued differentiable functions on M. A *tangent vector* on M at p is a function $X_p\colon C^\infty(M)\to\mathbb{R}$ which satisfies the following two conditions:

i) Linearity: $X_p(af+bg)=aX_p(f)+bX_p(g)$

ii) Leibniz's Rule: $X_p(fg)=g(p)X_p(f)+f(p)X_p(g)$.

Definition 2.1.15 Let U be a coordinate neighborhood on an n-dimensional manifold M with local coordinates $x^1,\cdots,x^n$. Let $\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^n}$ be the usual coordinate vector fields on U. The *tangent space* at p of M is

$$T_p(M)=\operatorname{span}\left\{\frac{\partial}{\partial x^1}(p),\frac{\partial}{\partial x^2}(p),\cdots,\frac{\partial}{\partial x^n}(p)\right\}.$$

The set of tangent vectors on M at p is the tangent space at p. It is a vector space and $\dim(T_p(M))=\dim(M)$.

Intuitively, tangent spaces and tangent vectors are a generalization of tangent vectors and spaces for smooth curves and two dimensional surfaces in $\mathbb{R}^3$.

Proposition 2.1.16 [1] Let M be an n-dimensional smooth manifold and (U,φ) be a coordinate chart about $p\in M$ associated with local coordinates $x^1,\cdots,x^n$. If $v\in T_p(M)$, then

i) $v=\displaystyle\sum_{i=1}^{n} v(x_i)\left(\frac{\partial}{\partial x^i}\right)_p$.

ii) $\left\{\frac{\partial}{\partial x^1}(p),\frac{\partial}{\partial x^2}(p),\cdots,\frac{\partial}{\partial x^n}(p)\right\}$ is a basis for $T_p(M)$.

Definition 2.1.17 A *vector field* on a smooth manifold M is an assignment of a tangent vector $X_p \in T_p(M)$ for each $p \in M$. That is, $V : M \to T_p(M)$, $p \mapsto X_p$.

Let M be a smooth manifold and let X be a vector field on M. Considering the vector field X in local coordinates, we have

$$X = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i},$$

where b^i's are functions defined in a coordinate neighborhood of M. X is called a *smooth vector field* if all b^i's are smooth functions. The set of smooth vector fields on M is denoted by $\mathfrak{X}(M)$, [10].

Definition 2.1.18 Let M be a smooth manifold. Let X and Y be smooth vector fields on M. The *Lie bracket* of X and Y is defined as

$$[X,Y] = XY - YX$$

and for a smooth function $f : M \to \mathbb{R}$,

$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

Proposition 2.1.19 (Properties of the Lie Bracket)
(i) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ and $[X, aY + bZ] = a[X,Y] + b[X,Z]$, where a and b are real constants (Linear in both arguments).

(ii) $[X,Y] = -[Y,X]$ (Anti symmetric).

(iii) $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$, where $f, g : M \to \mathbb{R}$ are smooth functions.

(iv) $[X,[Y,Z]] + [Y,[Z,X]] + [Z[X,Y]] = 0$ (Jacobi identity).
(v) $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$, where $x^1, \cdots, x^n$ are local coordinates.

Remark 2.1.20 When the same index appears as a subscript and a superscript in an expression, this index is summed until the desired value. This index is called a *dummy index* and it can be replaced by any letter which is not involved in the expression. In this case, the Σ symbol can be omitted. This convention is due to Albert Einstein and it is called the *Einstein summation convention*. Throughout this thesis, the Einstein summation convention will be used, unless otherwise indicated, summation is implied whenever repeated indices occur up and down in an expression.

Local coordinates will be used in several parts of this thesis. We will do this in such a way that our expressions in local coordinates will agree with those of Eisenhart, [11]. The notation $\frac{\partial}{\partial x^i} = \partial_i$ will be used in the remaining parts of the thesis, where ∂_i are the partial differential operators with respect to local parametrization.

2.2 Tensors on Smooth Manifolds

Throughout this section, first the definitions will be given for $\mathbb{R}^n$ and then will be generalized to smooth manifolds.

Suppose that V is an n-dimensional vector space over $\mathbb{R}$. The *dual space* of V, which is denoted by V^*, is the set of all linear maps from V to $\mathbb{R}$.

$$V^* = \{\sigma |\ \sigma \colon V \to \mathbb{R}\}$$
$$v \mapsto \sigma(v).$$

Note that $\dim(V) = \dim(V^*)$ and $\sigma(v)$ is also denoted as $\langle \sigma, v\rangle$.

Let V be a vector space of dimension n with a basis $\{e_1, e_2, ..., e_n\}$. Then, there exists a unique dual basis $\{w^1, w^2, ..., w^n\}$ for V^* such that $w^i(e_j) = \delta^i_j$. Let $v \in V$, then $v = v^i e_i$. By considering $w^j(v)$ we have,

$w^j(v) = w^j(v^i e_i) = \left\langle w^j, v^i e_i \right\rangle = v^i \left\langle w^j, e_i \right\rangle = v^i w^j e_i = v^i \delta^j_i = v^j.$

As $\{e_1, e_2, ..., e_n\}$ is a basis for V and $\{w^1, w^2, ..., w^n\}$ for V^*, it follows that $v = v^j e_j$ and $\sigma = \sigma_i w^i$. Hence, $\sigma(v) = \langle \sigma, v\rangle = \left\langle \sigma_i w^i, v^j e_j \right\rangle = \sigma_i v^j \left\langle w^i, e_j \right\rangle = \sigma_i v^j w^i e_j = \sigma_i v^j \delta^i_j = \sigma_i v^i$.

Let's apply the above definition for a smooth manifold M of dimension n.

Definition 2.2.1 Let M be a smooth manifold of dimension n with charts (U_i, φ_i) and coordinates x^i. Consider $\sigma \colon T_p(M) \to \mathbb{R}$.

It follows that, σ is an element of the dual space of $T_p(M)$. That space is denoted by $T^*_p(M)$ and called the *cotangent space* of the manifold M. Obviously, $\dim\left(T^*_p(M)\right) = \dim\left(T_p M\right) = \dim(M)$.

Moreover, σ is called a *1-form* with $dx^j\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial x^j}{\partial x^i} = \delta^j_i$. Therefore, $\left\{dx^1, ..., dx^n\right\}$ is a basis for $T^*_p(M)$. Hence, any 1-form $\sigma \in T^*_p(M)$ can be written as $\sigma = \sigma_i dx^i$.

Definition 2.2.2 Suppose that V is an n-dimensional vector space and V^* is the dual space of V over $\mathbb{R}$. Then, the multilinear map

$$\phi : V^* \times ... \times V^* \times V \times ... \times V \to \mathbb{R},$$

where V^* appears r-times and V appears s-times is called a *tensor of order (r,s)*. r is called the *contravariant order* and s is called the *covariant order* of the tensor. The set of (r,s) tensors is denoted by $T_s^r(V)$.

Consider the cases $r = 0$ and $s = 0$ in the Definition 2.2.2:

(i) Suppose $r = 0$. $T_s^0(V)$ is the set of covariant tensors of order s.

(ii) Suppose $s = 0$. $T_0^r(V)$ is the set of contravariant tensors of order r.

Now, let's generalize the definition of a tensor in $\mathbb{R}^n$ to smooth manifolds.

Definition 2.2.3 Let $V = T_p(M)$ and $V^* = T_p^*(M)$. Then, the multilinear map

$$\phi : T_p^*(M) \times ... \times T_p^*(M) \times T_p(M) \times ... \times T_p(M) \to \mathbb{R},$$

where $T_p^*(M)$ appears r-times and $T_p(M)$ appears s-times is called a *tensor of order (r,s)* on the manifold M. r is called the *contravariant order* and s is called the *covariant order* of the tensor. The set of (r,s) tensors on M is denoted by $T_s^r(T_p(M))$.

Special cases:

(i) $r = 0$, $s = 1$: $v\colon T_p(M) \to \mathbb{R}$ is called a *1-covariant tensor* or *1-form* and $v \in T_p^*(M)$.

(ii) $r = 1$, $s = 0$: $w\colon T_p^*(M) \to \mathbb{R}$ is called a *1-contravariant tensor* and $w \in T_p(M)$.

Definition 2.2.4 A C^∞ *covariant tensor field of order* r on a C^∞ manifold M is a function ϕ which assigns an $(0,r)$ tensor $\phi_p \in T_r^0(T_p(M))$ for each $p \in M$.

Definition 2.2.5 The *product* of two covariant tensors $\phi \in T_r^0(T_p(M))$ and $\psi \in T_s^0(T_p(M))$ is an $(0, r+s)$ tensor such that

$$(\phi \otimes \psi)(v_1, v_2, ..., v_r, v_{r+1}, ..., v_{r+s}) = \phi(v_1, v_2, ..., v_r)\ \psi(v_{r+1}, ..., v_{r+s}).$$

Remark 2.2.6 $\otimes$ is bilinear, associative but not commutative.

Definition 2.2.7 Let $S(r)$ be the permutation group of the set of numbers $\{1, 2, ..., n\}$. Let $\sigma \in S(r)$. $sign(\sigma) = 1$ if σ is even and $sign(\sigma) = -1$ if σ is odd. Let $\phi \in T_r^0(T_p(M))$. Using the notation $\phi^\sigma(x_1, x_2, ..., x_r) = \phi\left(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(r)}\right)$ antisymmetric and symmetric tensors are defined by,

(i) ϕ is an *antisymmetric tensor* if $\phi^{\sigma}(x_1,x_2,...,x_r) = sign(\sigma)\phi(x_1,x_2,...,x_r)$

(ii) ϕ is a *symmetric tensor* if $\phi^{\sigma}(x_1,x_2,...,x_r) = \phi(x_1,x_2,...,x_r)$.

Let T be a tensor of type $(0,2)$ with components T_{ij} in some basis. If $T_{ij} = T_{ji}$ we say that T is *symmetric*, while if $T_{ij} = -T_{ji}$ we say that T is *antisymmetric*.

The *symmetric part* $T_{(ij)}$ of T is the symmetric tensor with components

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)$$

while the *antisymmetric part* $T_{[ij]}$ of T is the antisymmetric tensor with components

$$T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right).$$

Evidently, for a $(0,2)$ tensor, T is the sum of its symmetric and antisymmetric parts.

Definition 2.2.8 Let M be an n-dimensional smooth manifold. Let ϕ be a $(0,2)$ symmetric tensor. Then, ϕ is called *positive definite* if $\phi(u,v) \geqslant 0$ $\forall u,v \in T_p(M)$ and $\phi(u,v) = 0$ if and only if $u = 0$ or $v = 0$.

2.3 Affine Connections on Manifolds

Definition 2.3.1 Let M be a C^∞ manifold. An *affine connection* or a *covariant derivative* on M, is an operator ∇ that assigns for each pair of C^∞ vector fields X and Y, a C^∞ vector field $\nabla_X Y$.

Let X, Y and Z be C^∞ vector fields and $\rho \in C^\infty(M)$. The following properties hold for an affine connection ∇:

$$\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z \tag{2.3.1}$$

$$\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z \tag{2.3.2}$$

$$\nabla_{(\rho X)} Y = \rho \nabla_X Y \tag{2.3.3}$$

$$\nabla_X \rho Y = \rho \nabla_X Y + X(\rho) Y. \tag{2.3.4}$$

Let $U \subset M$ and $x^1, \cdots, x^n$ be the local coordinates of M.

Let $X_p \in T_p(U)$ and $Y \in \mathfrak{X}(U)$. Assume $X_p = a^i \partial_i\big|_p$ and $Y = b^j \partial_j$. By using the above properties, we have

$$\begin{aligned} \nabla_{X_p} Y &= \nabla_{a^i(p)\partial_i} b^j \partial_j \\ &= a^i(p) \nabla_{\partial_i} b^j \partial_j \\ &= a^i(p) \left[\left(\partial_i \left(b^j \right) \partial_j + b^j \nabla_{\partial_i} \partial_j \right) \right]. \end{aligned}$$

Definition 2.3.2 The expression $\nabla_{\partial_i}\partial_j$ is determined by the formula

$$\nabla_{\partial_i}\partial_j = \Gamma^h_{ij}\partial_h \tag{2.3.5}$$

and the functions Γ^h_{ij} are called *connection coefficients* or *Christoffel symbols* of the affine connection ∇.

Definition 2.3.3 Let A^i_j be the components of a mixed tensor of order 2. Then, the covariant derivative of A^i_j with respect to the connection ∇ is

$$\nabla_k A^i_j = \frac{\partial A^i_j}{\partial x^k} + A^h_j \Gamma^i_{hk} - A^i_h \Gamma^h_{jk}.$$

2.4 Riemannian Manifolds

Definition 2.4.1 Let M be an n-dimensional smooth manifold. If at each point $p \in M$, there exists a bilinear, symmetric and positive definite tensor field $g = < \cdot, \cdot >$ defined on $T_p(M)$, then M is called a *Riemannian manifold* and g is called a *Riemannian metric* or *fundamental tensor*. The manifold is represented as (M, g).

$$\begin{aligned} g \colon\, & T_p(M) \times T_p(M) \to \mathbb{R} \\ & (X,Y) \mapsto g\left(X_p, Y_p\right) = \langle X, Y \rangle_p \in \mathbb{R}. \end{aligned}$$

Let $x^1, \cdots, x^n$ be the local coordinates of M. The components g_{ij} of the Riemannian metric g are given as

$$g_{ij} = g\left(\partial_i, \partial_j\right), \ i, j = 1, \cdots, n.$$

Remark 2.4.2 In three dimensional Euclidean space, the distance between the points (x, y, z) and $(x + dx, y + dy, z + dz)$ is given by the formula

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The above formula was generalized to n-dimensions by Bernhard Riemann and the distance between the points $(x^1,\cdots,x^n)$ and $(x^1+dx^1,\cdots,x^n+dx^n)$ is given by the following identity, [12]

$$ds^2 = g_{ij}dx^i dx^j. \tag{2.4.1}$$

Definition 2.4.3 The reciprocal of g_{ij} is denoted as g^{ij} and called the *fundamental contravariant tensor*, where $g_{hm}g^{mk} = \delta_h^k$, [12].

Definition 2.4.4 Let (M,g) be a Riemannian manifold and ∇ be an affine connection on (M,g). If ∇ satisfies the following conditions, then it is called a *Riemannian connection*.

$$\nabla_X Z - \nabla_Z X = [X,Z] \tag{2.4.2}$$

$$Zg(X,Y) = g(\nabla_Z X,Y) + g(X,\nabla_Z Y). \tag{2.4.3}$$

(2.4.2) is called torsion free property and (2.4.3) is called metric compatibility condition.

Considering the equation (2.4.2) for $X=\partial_i$ and $Z=\partial_j$ gives,

$$\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = \left[\partial_i,\partial_j\right].$$

By using Proposition 2.1.19 (v), we have $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$. Hence, it follows that $\Gamma_{ij}^h = \Gamma_{ji}^h$. Moreover, metric compatibility condition of a Riemannian connection yields $\nabla_k g_{ij} = 0$ and similarly $\nabla_k g^{ij} = 0$.

Theorem 2.4.5 (Fundamental Theorem of Riemannian Geometry) [10] There exists a unique Riemannian connection ∇ on a Riemannian manifold (M,g).

Proof: Let (M,g) be a Riemannian manifold and X, Y and Z be smooth vector fields. Permuting X, Y and Z in (2.4.3) yields two more equations, namely

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z) \tag{2.4.4}$$

and

$$Yg(Z,X) = g(\nabla_Y Z,X) + g(Z,\nabla_Y X). \tag{2.4.5}$$

Adding (2.4.5) and (2.4.4), then subtracting (2.4.3) from the resulting equation, it follows that

$$\begin{aligned}Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) &= g(Y,\nabla_X Z - \nabla_Z X) \\ &\quad + g(X,\nabla_Y Z - \nabla_Z Y) + g(Z,\nabla_X Y + \nabla_Y X).\end{aligned}$$

Adding and subtracting $\nabla_X Y$ from the last term of the right hand side and using the equation (2.4.2), the above identity can be written as

$$Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) = g(Y,[X,Z]) + g(X,[Y,Z]) \\ +2g(Z,\nabla_X Y) + g(Z,[Y,X]).$$

Rearranging the above equation leads to the Koszul formula:

$$2g(Z,\nabla_X Y) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(Y,[X,Z]) \\ -g(X,[Y,Z]) - g(Z,[Y,X]). \tag{2.4.6}$$

Given any smooth vector field Z, the right hand side of the equation (2.4.6) can be uniquely determined. Assume that there exists a smooth vector field $U \neq \nabla_X Y$ such that $g(\nabla_X Y, Z) = g(U,Z)$. It follows that $g(\nabla_X Y, Z) - g(U,Z) = g(\nabla_X Y - U, Z) = 0$. As g is a non-degenerate metric, this implies $\nabla_X Y = U$, contrary to the assumption that $U \neq \nabla_X Y$. Therefore, ∇ is unique.

Defining ∇ as in the equation (2.4.6) proves the existence of the connection. To prove that ∇ is a Riemannian connection, it must be shown that ∇ satisfies (2.3.1) - (2.3.4), (2.4.2) and (2.4.3). □

Considering (2.4.6) for $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$, the following holds:

$$2g\left(\partial_k, \nabla_{\partial_i}\partial_j\right) = \partial_i g\left(\partial_j, \partial_k\right) + \partial_j g\left(\partial_k, \partial_i\right) - \partial_k g\left(\partial_i, \partial_j\right).$$

Using the equation (2.3.5), the above identity can be written as

$$\Gamma^h_{ij} g_{kh} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right). \tag{2.4.7}$$

Multiplying both sides of (2.4.7) with g^{mk} gives,

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right). \tag{2.4.8}$$

Contracting the equation (2.4.8) for m and j yields, [11, 12]

$$\Gamma^j_{ij} = \frac{\partial}{\partial x^i}\left(\ln\sqrt{g}\right). \tag{2.4.9}$$

Definition 2.4.6 Let X, Y and Z be smooth vector fields. The operator $R(X,Y)$ which is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is called the *curvature operator*, [1].

If we write $X = \partial_j$, $Y = \partial_k$ and $Z = \partial_i$, then we have

$$\begin{aligned} R(X,Y)Z &= \nabla_{\partial_j}\left(\nabla_{\partial_k}\partial_i\right) - \nabla_{\partial_k}\left(\nabla_{\partial_j}\partial_i\right) \\ &= \nabla_{\partial_j}\left(\Gamma^l_{ki}\partial_l\right) - \nabla_{\partial_k}\left(\Gamma^l_{ji}\partial_l\right) \\ &= \left(\partial_j\Gamma^l_{ik} - \partial_k\Gamma^l_{ij} + \Gamma^l_{hj}\Gamma^h_{ik} - \Gamma^l_{hk}\Gamma^h_{ij}\right)\partial_l \\ &= R^l_{ijk}\partial_l. \end{aligned}$$

Definition 2.4.7 Let X, Y, Z and W be smooth vector fields.

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$$

is called the *Riemannian curvature tensor of type (0,4).*

In local coordinates, the Riemannian curvature tensor is determined by either of the functions R^l_{ijk} and R_{lijk}, which are defined as

$$R^l_{ijk} = \partial_j\Gamma^l_{ik} - \partial_k\Gamma^l_{ij} + \Gamma^l_{hj}\Gamma^h_{ik} - \Gamma^l_{hk}\Gamma^h_{ij} \tag{2.4.10}$$

and

$$R_{lijk} = g_{lm}R^m_{ijk}. \tag{2.4.11}$$

From the equation (2.4.10), it follows that $R^l_{ijk} = -R^l_{ikj}$.

For covariant tensors λ_i and b_{im}, we have the following identities, [11]:

$$\nabla_k\nabla_j\lambda_i - \nabla_j\nabla_k\lambda_i = \lambda_l R^l_{ijk} \tag{2.4.12}$$

and

$$\nabla_k\nabla_j b_{im} - \nabla_j\nabla_k b_{im} = b_{ih}R^h_{mjk} + b_{hm}R^h_{ijk}. \tag{2.4.13}$$

Proposition 2.4.8 (Properties of the Riemannian Curvature Tensor)

(i) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

(ii) $R(X,Y,Z,W) = -R(X,Y,Z,W)$

(iii) $R(X,Y,Z,W) = R(Z,W,X,Y)$

(iv) $R(X,Y,Z,W) + R(X,W,Y,Z) + R(X,Z,W,Y) = 0$ (first Bianchi identity)

(v) $\nabla_T R(X,Y,Z,W) + \nabla_Z R(X,Y,W,T) + \nabla_W R(X,Y,T,Z) = 0$ (second Bianchi identity)

Definition 2.4.9 Let $X,Y \in T_p(M)$ and $\Pi \subseteq T_p(M)$. The *sectional curvature* $K(\Pi)$ is defined as

$$K(\Pi) = K(X,Y) = \frac{R_p(X,Y,X,Y)}{\langle X,X\rangle_p \langle Y,Y\rangle_p - \langle X,Y\rangle_p^2}$$

where (X,Y) is any basis of Π.

Letting $X = \alpha^m \partial_m\big|_p$ and $Y = \beta^l \partial_l\big|_p$ and considering the sectional curvature in local coordinates, we have, [12]

$$K = \frac{R_{hijk}\alpha^h\beta^i\alpha^j\beta^k}{\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right)\alpha^h\beta^i\alpha^j\beta^k}. \qquad (2.4.14)$$

Theorem 2.4.10 (Schur's Theorem) [12] If the sectional curvature of a Riemannian manifold is independent of the orientation chosen at each point, then it is constant throughout the manifold.

Proof: Suppose that K is independent of the orientation chosen. Then, by using (2.4.14), we have

$$R_{hijk} = K\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right). \qquad (2.4.15)$$

Covariantly differentiating the equation (2.4.15) with respect to the connection ∇ gives,

$$\nabla_l R_{hijk} = \left(g_{hj}g_{ik} - g_{hk}g_{ij}\right)\nabla_l K. \qquad (2.4.16)$$

Permuting j, k and l in (2.4.16) yields two more equations namely,

$$\nabla_j R_{hikl} = \left(g_{hk}g_{il} - g_{hl}g_{ik}\right)\nabla_j K \qquad (2.4.17)$$

and

$$\nabla_k R_{hilj} = \left(g_{hl}g_{ij} - g_{hj}g_{il}\right)\nabla_k K. \qquad (2.4.18)$$

Adding (2.4.16), (2.4.17), (2.4.18) and using the second Bianchi identity, we obtain

$$\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right)\nabla_l K + \left(g_{hk}g_{il} - g_{hl}g_{ik}\right)\nabla_j K + \left(g_{hl}g_{ij} - g_{hj}g_{il}\right)\nabla_k K = 0.$$

If $n = 2$, then the manifold has only one orientation at each point. So, assume that $n \geq 3$. Multiplying both sides with g^{hj} gives,

$$(n-2)\left(g_{ik}\nabla_l K - g_{il}\nabla_k K\right) = 0.$$

As the above identity holds for all $i = 1, \cdots, n$, it follows that, [12]

$$\nabla_l K = \nabla_k K = 0.$$

Since the covariant derivatives of K are zero, the manifold has constant sectional curvature. Considering (2.4.15) for constant K, it follows that the Riemannian curvature tensor is also constant. □

Definition 2.4.11 Let $V_n = (M, g)$ be a Riemannian manifold.

$$R(X,Y,Z,W) = K\left(g\left(X,Z\right)g\left(Y,W\right) - g\left(X,W\right)g\left(Y,Z\right)\right)$$

if and only if V_n is of *constant curvature* K. In local coordinates, the above condition takes the form (2.4.15).

Definition 2.4.12 Let (M, g) be a Riemannian manifold. Let X, Y and Z be smooth vector fields. The *Ricci tensor* $Ric(X,Y)$ is defined as the trace

$$Ric(X,Y) = tr\left(Z \mapsto R\left(Z,X\right)Y\right).$$

In local coordinates, the Ricci tensor R_{ij} is obtained by the contraction of the Riemannian curvature tensor R^l_{ijk} for l and k, which is given by the formula

$$R_{ij} = \frac{\partial}{\partial x^j}\Gamma^h_{ih} - \frac{\partial}{\partial x^h}\Gamma^h_{ij} + \Gamma^h_{mj}\Gamma^m_{ih} - \Gamma^h_{mh}\Gamma^m_{ij}.$$

Equation (2.4.9) gives,

$$R_{ij} = \frac{\partial^2 ln\sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^h}\Gamma^h_{ij} + \Gamma^h_{mj}\Gamma^m_{ih} - \Gamma^h_{ij}\frac{\partial ln\sqrt{g}}{\partial x^h}. \tag{2.4.19}$$

Definition 2.4.13 The *scalar curvature* R is given by the formula, [12]

$$R = R_{ij}g^{ij}. \tag{2.4.20}$$

Definition 2.4.14 A Riemannian manifold (M, g) is called an *Einstein manifold* if its Ricci tensor $Ric(X,Y)$ satisfies

$$Ric(X,Y) = \lambda g(X,Y),$$

where λ is a real constant, [1].

In local coordinates, an Einstein manifold (M, g) is characterized by the equation

$$R_{ij} = \lambda g_{ij}. \tag{2.4.21}$$

Lemma 2.4.15 Let (M,g) be a Riemannian manifold of dimension $n \geq 3$. If M has constant curvature, then it is an Einstein manifold.

Proof: Multiplying both sides of (2.4.15) with g^{lh} gives,

$$R^l_{ijk} = K\left(\delta^l_j g_{ik} - \delta^l_k g_{ij}\right).$$

Contracting the above equation for l and k, it follows that

$$R_{ij} = -K(n-1)g_{ij}. \tag{2.4.22}$$

□

For lower dimensions, the following special cases hold, [12]:

i) $n = 2$: (M,g) is an Einstein manifold.

ii) $n = 3$: If (M,g) is an Einstein manifold, then it is of constant curvature.

Definition 2.4.16 A non-flat n-dimensional $(n > 2)$ Riemannian manifold (M,g) is called a *quasi Einstein manifold* if its Ricci tensor $Ric(X,Y)$ is not identically zero and satisfies

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{2.4.23}$$

where a and b are scalars such that $b \neq 0$ and A is a non-zero 1-form satisfying $g(X,U) = A(X)$ for any vector field X and any unit vector field U. Moreover, a and b are called associated scalars, A is called associated 1-form and U is called the generator of the manifold. An n-dimensional quasi Einstein manifold is denoted by $(QE)_n$, [13].

If in the above definition $b = 0$, then the manifold is Einstein.

Definition 2.4.17 A non-flat n-dimensional $(n \geq 3)$ Riemannian manifold (M,g) is called a *nearly quasi Einstein manifold* if its Ricci tensor $Ric(X,Y)$ is not identically zero and satisfies

$$Ric(X,Y) = ag(X,Y) + bA(X,Y), \tag{2.4.24}$$

where a and b are non-zero scalars and A is a non-zero $(0,2)$ symmetric tensor. An n-dimensional nearly quasi Einstein manifold is denoted by $N(QE)_n$, [14].

Any quasi Einstein manifold is a nearly quasi Einstein manifold, but the converse is not necessarily true.

Definition 2.4.18 A non-flat n-dimensional $(n \geq 3)$ Riemannian manifold (M,g) is called *Ricci recurrent* if its Ricci tensor satisfies

$$\nabla_X R(Y,Z) = \phi(X)R(Y,Z) \tag{2.4.25}$$

for some non-zero 1-form ϕ. An n-dimensional Ricci recurrent manifold is denoted by R_n, [15].

Definition 2.4.19 A non-flat n-dimensional $(n \geq 3)$ Riemannian manifold (M,g) is called *generalized Ricci recurrent* if its Ricci tensor satisfies

$$\nabla_X R(Y,Z) = \phi(X)R(Y,Z) + \alpha(X)g(Y,Z) \tag{2.4.26}$$

for some non-zero 1-forms ϕ and α. An n-dimensional generalized Ricci recurrent manifold is denoted by GRK_n, [16].

Definition 2.4.20 An n-dimensional Riemannian $(n \geq 3)$ manifold (M,g) is called *Ricci symmetric* if its Ricci tensor $Ric(X,Y)$ satisfies, [17]

$$\nabla_Z Ric(X,Y) = 0. \tag{2.4.27}$$

Definition 2.4.21 A non-flat n-dimensional $(n \geq 3)$ Riemannian manifold (M,g) is called *pseudo Ricci symmetric* if its Ricci tensor $Ric(X,Y)$ is not identically zero and satisfies

$$(\nabla_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + A(Y)Ric(X,Z) + A(Z)Ric(Y,X), \tag{2.4.28}$$

where A is a non-vanishing 1-form. An n-dimensional pseudo Ricci symmetric manifold is denoted by $(PRS)_n$, [17].

Definition 2.4.22 An n-dimensional $(n \geq 3)$ Riemannian manifold (M,g) is called *almost pseudo Ricci symmetric* if its Ricci tensor $Ric(X,Y)$ is not identically zero and satisfies

$$(\nabla_X Ric)(Y,Z) = [A(X) + B(X)]\,Ric(Y,Z) + A(Y)Ric(X,Z) + A(Z)Ric(Y,X), \tag{2.4.29}$$

where A and B are non-vanishing 1-forms associated with the unique vector fields U and V respectively such that $g(X,U) = A(X)$ and $g(X,V) = B(X)$ for all $X \in \mathfrak{X}(M)$. The vector fields U and V are called the generators of the manifold and the 1-forms A and B are called associated 1-forms. An n-dimensional almost pseudo Ricci symmetric manifold is denoted by $A(PRS)_n$, [18].

3. GEODESIC MAPPINGS OF RIEMANNIAN MANIFOLDS

In this chapter, geodesics and the geodesic mappings of Riemannian manifolds will be defined. Moreover, necessary and sufficient conditions for the existence of geodesic mappings will be derived and the important formulas will be stated. Furthermore, Sinyukov equations, equivalent conditions for the existence of geodesic mappings, will be proven.

3.1 Geodesics

Definition 3.1.1 Let (M,g) be a Riemannian manifold with Riemannian connection ∇. Let γ: $J \to M$ be a smooth curve on M, where J is an open interval. A vector field X is *parallel along* γ, if $\nabla_{\dot{\gamma}}X = 0$, [1].

Definition 3.1.2 Let (M,g) be a Riemannian manifold with Riemannian connection ∇. Let γ: $J \to M$ be a smooth curve on M, where J is an open interval. The curve γ is called a *geodesic* on M, if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Explicitly, the curve γ is *geodesic* if $\dot{\gamma}$ is parallel along γ, [1].

Suppose that $X = X^i\partial_i$ and $Y = Y^j\partial_j$. Then,

$$\begin{aligned}
\nabla_X Y &= \nabla_X \left(Y^j\partial_j\right) \\
&= X(Y^j)\partial_j + Y^j\nabla_X\partial_j \\
&= X(Y^j)\partial_j + Y^jX^i\nabla_{\partial_i}\partial_j \\
&= \left[X(Y^k) + \Gamma^k_{ij}X^iY^j\right]\partial_k.
\end{aligned}$$

Hence $\nabla_X Y = 0$, provided that

$$X(Y^k) + \Gamma^k_{ij}X^iY^j = 0.$$

As,

$$X^i = X(x^i) = \gamma_*\left(\frac{d}{dt}\right)(x^i) = \left(\frac{d}{dt}\right)\left(x^i \circ \gamma\right) = \frac{d\gamma^i(t)}{dt},$$

it follows that

$$\frac{d}{dt}Y^k + \Gamma^k_{ij}\frac{d\gamma^i}{dt}Y^j = 0.$$

Another way to describe geodesics is to use the property that they are a path of minimum (or maximum) length, [11, 12].

To see this, let $V_n = (M, g)$ be an n-dimensional Riemannian manifold and γ be a curve in V_n. Let P be an arbitrary point on the curve whose coordinates are given by the functions $x^i = x^i(t)$. Assume that A and B are two fixed points on the curve with coordinate functions $x^i = x^i(t_0)$ and $x^i = x^i(t_1)$ respectively.

Consider an infinitesimal deformation of the curve γ to $\bar{\gamma}$, where the points A and B remain fixed but the point P is displaced to $\bar{P}$. The coordinates of the point $\bar{P}$ are given by the formula

$$\bar{x}^i(t) = x^i(t) + z^i(t), \tag{3.1.1}$$

where z^i are infinitesimal functions of t.

As the points A and B remain fixed under this deformation, we have

$$z^i(t_0) = z^i(t_1) = 0. \tag{3.1.2}$$

The length of the original curve from A to B is

$$\int_{t_0}^{t_1} \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\, dt = \int_{t_0}^{t_1} \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, dt \tag{3.1.3}$$

and the length of the deformed curve from A to B is

$$\int_{t_0}^{t_1} \sqrt{g_{ij}\frac{dx^i + dz^i}{dt}\frac{dx^j + dz^j}{dt}}\, dt = \int_{t_0}^{t_1} \sqrt{g_{ij}(\dot{x}^i + \dot{z}^i)(\dot{x}^j + \dot{z}^j)}\, dt, \tag{3.1.4}$$

where $\dot{x}^j = \dfrac{dx^j}{dt}$ and $\dot{z}^j = \dfrac{\partial z^j}{\partial x^k}\dfrac{dx^k}{dt}$.

Suppose that the Taylor series of $\sqrt{g_{ij}(\dot{x}^i + \dot{z}^i)(\dot{x}^j + \dot{z}^j)}$ is expanded to the first order. If the integral (3.1.3) and the integral of this expanded function from t_0 to t_1 are the same, then the curve γ is called a *geodesic*, [11, 12].

Now, we obtain the differential equations of geodesics. First, consider the integral

$$I = \int_{t_0}^{t_1} \phi\left(x^1, .., x^n; \dot{x}^1, ..\dot{x}^n\right) dt, \tag{3.1.5}$$

where ϕ is an analytic function of the $2n$ arguments x^i and $\dot{x}^i$. Assume that I' is the integral where x^i's are replaced with x^i+z^i in (3.1.5). That is,

$$I' = \int_{t_0}^{t_1} \phi\left(x^1+z^1,..,x^n+z^n;\dot{x}^1+\dot{z}^1,..,\dot{x}^n+\dot{z}^n\right)dt. \tag{3.1.6}$$

Considering the integral I' up to the first order gives us, [11,12]

$$I'-I \cong \int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial x^i}z^i+\frac{\partial\phi}{\partial\dot{x}^i}\dot{z}^i\right]dt. \tag{3.1.7}$$

Let

$$\delta I = \int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial x^i}z^i+\frac{\partial\phi}{\partial\dot{x}^i}\dot{z}^i\right]dt. \tag{3.1.8}$$

Applying integration by parts and using the equation (3.1.2) yields,

$$\begin{aligned}\int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial\dot{x}^i}\dot{z}^i\right]dt &= \frac{\partial\phi}{\partial\dot{x}^i}z^i\Big|_{t_0}^{t_1}-\int_{t_0}^{t_1}z^i\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{x}^i}\right)dt\\ &= -\int_{t_0}^{t_1}z^i\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{x}^i}\right)dt.\end{aligned}$$

Therefore, δI can be expressed as

$$\begin{aligned}\delta I &= \int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial x^i}z^i\right]dt-\int_{t_0}^{t_1}z^i\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{x}^i}\right)dt\\ &= \int_{t_0}^{t_1}\left[\frac{\partial\phi}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{x}^i}\right)\right]z^i dt.\end{aligned} \tag{3.1.9}$$

Definition 3.1.3 If δI is equal to zero for all infinitesimal functions $z^i(t)$, which vanish at t_0 and t_1, the integral I is called *stationary*, [11,12].

The equation $\dfrac{\partial\phi}{\partial x^i}-\dfrac{d}{dt}\left(\dfrac{\partial\phi}{\partial\dot{x}^i}\right)=0$ is known as the Euler-Lagrange equation.

The differential equations of geodesics are obtained by applying the Euler-Lagrange equation to the integral (3.1.3).

For this purpose, let

$$\phi = \sqrt{g_{jk}\dot{x}^j\dot{x}^k} = \frac{ds}{dt} = \dot{s} = \left(g_{jk}\dot{x}^j\dot{x}^k\right)^{\frac{1}{2}}.$$

First, consider

$$\frac{\partial\phi}{\partial x^i} = \frac{1}{2}\left(g_{jk}\dot{x}^j\dot{x}^k\right)^{-\frac{1}{2}}\frac{\partial}{\partial x^i}\left(g_{jk}\dot{x}^j\dot{x}^k\right).$$

As $\left(g_{jk}\dot{x}^j\dot{x}^k\right)^{-\frac{1}{2}} = \dfrac{1}{\dot{s}} = \dfrac{1}{\phi}$ and $\dfrac{\partial}{\partial x^i}\left(g_{jk}\dot{x}^j\dot{x}^k\right) = \left(\dfrac{\partial g_{jk}}{\partial x^i}\right)\dot{x}^j\dot{x}^k$, it follows

$$\frac{\partial\phi}{\partial x^i} = \frac{1}{2\dot{s}}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k. \tag{3.1.10}$$

Now, consider

$$\frac{\partial\phi}{\partial\dot{x}^i} = \frac{1}{2}\left(g_{jk}\dot{x}^j\dot{x}^k\right)^{-\frac{1}{2}}\frac{\partial}{\partial\dot{x}^i}\left(g_{jk}\dot{x}^j\dot{x}^k\right).$$

As $\frac{\partial}{\partial\dot{x}^i}\left(g_{jk}\dot{x}^j\dot{x}^k\right) = g_{jk}\frac{\partial\dot{x}^j}{\partial\dot{x}^i}\dot{x}^k + g_{jk}\frac{\partial\dot{x}^k}{\partial\dot{x}^i}\dot{x}^j$, we have

$$\begin{aligned}\frac{\partial\phi}{\partial\dot{x}^i} &= \frac{1}{2\dot{s}}\left(g_{jk}\frac{\partial\dot{x}^j}{\partial\dot{x}^i}\dot{x}^k + g_{jk}\frac{\partial\dot{x}^k}{\partial\dot{x}^i}\dot{x}^j\right)\\ &= \frac{1}{\dot{s}}g_{ij}\dot{x}^j.\end{aligned} \tag{3.1.11}$$

From the equations (3.1.10) and (3.1.11), $\left[\frac{\partial\phi}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{x}^i}\right)\right] = 0$ is equivalent to

$$\frac{1}{2\dot{s}}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k - \frac{d}{dt}\left(\frac{1}{\dot{s}}g_{ij}\dot{x}^j\right) = 0. \tag{3.1.12}$$

Using $\frac{d}{dt}(g_{ij}) = \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k$ and the product rule yields,

$$\frac{d}{dt}\left(\frac{1}{\dot{s}}g_{ij}\dot{x}^j\right) = \frac{d}{dt}\left(\frac{1}{\dot{s}}\right)g_{ij}\dot{x}^j + \left[\frac{d}{dt}(g_{ij})\right]\frac{1}{\dot{s}}\dot{x}^j + \left[\frac{d}{dt}(\dot{x}^j)\right]\frac{1}{\dot{s}}g_{ij}.$$

$$= \frac{(-1)\ddot{s}}{(\dot{s})^2}g_{ij}\dot{x}^j + \frac{\partial g_{ij}}{\partial x^k}\frac{1}{\dot{s}}\dot{x}^k\dot{x}^j + \ddot{x}^j\frac{1}{\dot{s}}g_{ij}.$$

Hence, (3.1.12) can be written in the following way:

$$\frac{1}{2\dot{s}}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k + \frac{\ddot{s}}{(\dot{s})^2}g_{ij}\dot{x}^j - \frac{\partial g_{ij}}{\partial x^k}\frac{1}{\dot{s}}\dot{x}^k\dot{x}^j - \ddot{x}^j\frac{1}{\dot{s}}g_{ij} = 0. \tag{3.1.13}$$

Multiplying (3.1.13) with $-\dot{s}$ gives,

$$g_{ij}\ddot{x}^j + \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k - g_{ij}\dot{x}^j\frac{\ddot{s}}{\dot{s}} = 0.$$

Using $[k,ij] + [i,jk] = \frac{\partial g_{ik}}{\partial x^j}$, we get

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^j\dot{x}^k - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k &= \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\right)\dot{x}^j\dot{x}^k\\ &= \left(\frac{1}{2}[j,ik] + [i,kj] - \frac{1}{2}[k,ji]\right)\dot{x}^j\dot{x}^k\\ &= [i,kj]\dot{x}^j\dot{x}^k.\end{aligned}$$

Therefore, (3.1.13) is equivalent to

$$g_{ij}\ddot{x}^j + [i,kj]\dot{x}^j\dot{x}^k - g_{ij}\dot{x}^j\frac{\ddot{s}}{\dot{s}} = 0. \tag{3.1.14}$$

Multiplying (3.1.14) with g^{il} gives,

$$\ddot{x}^l + \Gamma^l_{kj}\dot{x}^j\dot{x}^k - \dot{x}^l\frac{\ddot{s}}{\dot{s}} = 0.$$

The above equation can be written as

$$\frac{d^2x^l}{dt^2}+\Gamma^l_{kj}\frac{dx^j}{dt}\frac{dx^k}{dt}-\frac{dx^l}{dt}\frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}}=0. \tag{3.1.15}$$

Choosing the arc-length s for the parameter t in (3.1.15) gives the differential equations of geodesics, [11, 12]

$$\frac{d^2x^l}{ds^2}+\Gamma^l_{kj}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{3.1.16}$$

3.2 Geodesic Mappings of Riemannian Manifolds

Definition 3.2.1 Let $V_n=(M,g)$ and $\bar{V}_n=(\bar{M},\bar{g})$ be two Riemannian manifolds with the Riemannian connections ∇ and $\bar{\nabla}$, respectively. The diffeomorphism $f\colon V_n\to\bar{V}_n$ is called a *geodesic mapping* or *geodesic correspondence* if f maps any geodesic of V_n onto a geodesic of $\bar{V}_n$, [2].

By considering (3.1.15) for V_n and using a general parameter yields, [11]

$$\frac{dx^j}{dt}\frac{d^2x^i}{dt^2}-\frac{dx^i}{dt}\frac{d^2x^j}{dt^2}+\left(\Gamma^i_{lm}\frac{dx^j}{dt}-\Gamma^j_{lm}\frac{dx^i}{dt}\right)\frac{dx^l}{dt}\frac{dx^m}{dt}=0. \tag{3.2.1}$$

Similarly, we have

$$\frac{dx^j}{dt}\frac{d^2x^i}{dt^2}-\frac{dx^i}{dt}\frac{d^2x^j}{dt^2}+\left(\bar{\Gamma}^i_{lm}\frac{dx^j}{dt}-\bar{\Gamma}^j_{lm}\frac{dx^i}{dt}\right)\frac{dx^l}{dt}\frac{dx^m}{dt}=0. \tag{3.2.2}$$

If any solution of (3.2.1) is a geodesic in $\bar{V}_n$, then from (3.2.1) and (3.2.2) we get

$$\left(\left(\bar{\Gamma}^i_{lm}-\Gamma^i_{lm}\right)\frac{dx^j}{dt}-\left(\bar{\Gamma}^j_{lm}-\Gamma^j_{lm}\right)\frac{dx^i}{dt}\right)\frac{dx^l}{dt}\frac{dx^m}{dt}=0. \tag{3.2.3}$$

Considering the law of transformation of Christoffel symbols for both manifolds and subtracting the resulting equations gives,

$$\left(\bar{\Gamma}'^{\lambda}_{\mu\sigma}-\Gamma'^{\lambda}_{\mu\sigma}\right)\frac{\partial x'^{\mu}}{\partial x^i}\frac{\partial x'^{\sigma}}{\partial x^j}\frac{\partial x^l}{\partial x'^{\lambda}}=\bar{\Gamma}^l_{ij}-\Gamma^l_{ij}. \tag{3.2.4}$$

By defining a tensor a^l_{ij} as

$$a^l_{ij}:=\left(\bar{\Gamma}'^{\lambda}_{\mu\sigma}-\Gamma'^{\lambda}_{\mu\sigma}\right)\frac{\partial x'^{\mu}}{\partial x^i}\frac{\partial x'^{\sigma}}{\partial x^j}\frac{\partial x^l}{\partial x'^{\lambda}},$$

(3.2.4) can be written as

$$\bar{\Gamma}^l_{ij}=\Gamma^l_{ij}+a^l_{ij}. \tag{3.2.5}$$

As Christoffel symbols are symmetric in the subscripts, it follows from the equation (3.2.5) that a^l_{ij} is symmetric in i and j.

Substituting (3.2.5) into (3.2.3) results,

$$\left(a^i_{lm}\frac{dx^j}{dt}-a^j_{lm}\frac{dx^i}{dt}\right)\frac{dx^l}{dt}\frac{dx^m}{dt}=0.$$

Using a dummy index k, the above equation takes the form

$$\left(\delta^j_k a^i_{lm}-\delta^i_k a^j_{lm}\right)\frac{dx^k}{dt}\frac{dx^l}{dt}\frac{dx^m}{dt}=0. \tag{3.2.6}$$

Permuting the dummy indices k, l, and m in (3.2.6), adding all the resulting three equations, we obtain

$$\left(\delta^j_k a^i_{lm}-\delta^i_k a^j_{lm}\right)+\left(\delta^j_l a^i_{mk}-\delta^i_l a^j_{mk}\right)+\left(\delta^j_m a^i_{kl}-\delta^i_m a^j_{kl}\right)=0. \tag{3.2.7}$$

Contracting the equation (3.2.7) for j and m yields,

$$a^i_{lk}=\delta^i_k\left(\frac{1}{n+1}\right)a^m_{lm}+\delta^i_l\left(\frac{1}{n+1}\right)a^m_{mk}. \tag{3.2.8}$$

By defining a tensor ψ_l as

$$\psi_l:=\left(\frac{1}{n+1}\right)a^m_{lm},$$

(3.2.8) is equivalent to

$$a^i_{lk}=\delta^i_k\psi_l+\delta^i_l\psi_k. \tag{3.2.9}$$

Using (3.2.9), (3.2.5) can be written as

$$\bar{\Gamma}^l_{ij}=\Gamma^l_{ij}+\delta^l_i\psi_j+\delta^l_j\psi_i. \tag{3.2.10}$$

Remark 3.2.2 Equation (3.2.10) is the necessary and sufficient condition for the existence of a geodesic mapping from the Riemannian manifold V_n onto another Riemannian manifold $\bar{V}_n$ with a common coordinate system x, $x=\left(x^1,x^2,\cdots,x^n\right)$. If $\psi_i\neq 0$, then the map is nontrivial.

Contracting (3.2.10) for l and j, and using the equation (2.4.9), it follows that

$$\frac{\partial\ln\bar{g}}{\partial x^i}=\frac{\partial\ln g}{\partial x^i}+2(n+1)\psi_i. \tag{3.2.11}$$

Thus, it follows that $\psi_i=\dfrac{1}{n+1}\dfrac{\partial}{\partial x^i}\left(\ln\sqrt{\dfrac{\bar{g}}{g}}\right)$ and ψ_i is the gradient of a function.

Lemma 3.2.3 [11] $\nabla_j\bar{g}_{ik}=2\bar{g}_{ik}\nabla_j\psi+\bar{g}_{jk}\nabla_i\psi+\bar{g}_{ij}\nabla_k\psi.$

Proof: Substituting (3.2.10) into $\bar{\nabla}_j\bar{g}_{ik}=\partial_j\bar{g}_{ik}-\bar{g}_{ih}\bar{\Gamma}^h_{kj}-\bar{g}_{hk}\bar{\Gamma}^h_{ij}=0$ gives us,

$$\partial_j\bar{g}^{ik}-\bar{g}_{ih}\Gamma^h_{kj}-2\bar{g}_{ik}\psi_j-\bar{g}_{ij}\psi_k-\bar{g}_{hk}\Gamma^h_{ij}-\bar{g}_{jk}\psi_i=0.$$

Moreover, as $\partial_j \bar{g}^{ik} - \bar{g}_{ih}\Gamma^h_{kj} - \bar{g}_{hk}\Gamma^h_{ij} = \nabla_j \bar{g}_{ik}$, and ψ_i is a gradient, the above expression can be written as

$$\nabla_j \bar{g}_{ik} = 2\bar{g}_{ik}\nabla_j \psi + \bar{g}_{jk}\nabla_i \psi + \bar{g}_{ij}\nabla_k \psi. \tag{3.2.12}$$

□

Lemma 3.2.4 [11] The integrability condition of

$$\nabla_j \bar{g}_{ik} = 2\bar{g}_{ik}\nabla_j \psi + \bar{g}_{jk}\nabla_i \psi + \bar{g}_{ij}\nabla_k \psi$$

is reducible to

$$\bar{g}_{mk}R^m_{ijl} + \bar{g}_{im}R^m_{kjl} = \bar{g}_{ij}\psi_{kl} - \bar{g}_{il}\psi_{kj} + \bar{g}_{kj}\psi_{il} - \bar{g}_{kl}\psi_{ij},$$

where $\psi_{ij} = \nabla_j \nabla_i \psi - \nabla_i \psi \nabla_j \psi$.

Proof: Replacing j with l in (3.2.12) gives,

$$\nabla_l \bar{g}_{ik} = 2\bar{g}_{ik}\nabla_l \psi + \bar{g}_{lk}\nabla_i \psi + \bar{g}_{il}\nabla_k \psi. \tag{3.2.13}$$

Considering (2.4.13) for $\bar{g}$ and using the connection ∇ yields,

$$\nabla_l \nabla_j \bar{g}_{ik} - \nabla_j \nabla_l \bar{g}_{ik} = \bar{g}_{im}R^m_{kjl} + \bar{g}_{mk}R^m_{ijl}. \tag{3.2.14}$$

Taking the covariant derivative of (3.2.12) with respect to the connection ∇ gives,

$$\begin{aligned}\nabla_l \nabla_j \bar{g}_{ik} = {} & (2\nabla_l \bar{g}_{ik})\left(\nabla_j \psi\right) + \left(\nabla_l \nabla_j \psi\right)(2\bar{g}_{ik}) + \left(\nabla_l \bar{g}_{jk}\right)(\nabla_i \psi) \\ & + \left(\nabla_l \nabla_i \psi)(\bar{g}_{jk}\right) + \left(\nabla_l \bar{g}_{ij}\right)(\nabla_k \psi) + \left(\nabla_l \nabla_k \psi)(\bar{g}_{ij}\right)\end{aligned} \tag{3.2.15}$$

and of (3.2.13) with respect to the connection ∇ yields,

$$\begin{aligned}\nabla_j \nabla_l \bar{g}_{ik} = {} & \left(2\nabla_j \bar{g}_{ik})(\nabla_l \psi\right) + \left(\nabla_j \nabla_l \psi)(2\bar{g}_{ik}\right) + \left(\nabla_j \bar{g}_{lk})(\nabla_i \psi\right) \\ & + \left(\nabla_j \nabla_i \psi)(\bar{g}_{lk}\right) + \left(\nabla_j \bar{g}_{il})(\nabla_k \psi\right) + \left(\nabla_j \nabla_k \psi\right)(\bar{g}_{il}).\end{aligned} \tag{3.2.16}$$

As ψ_i is a gradient, the first covariant derivative is symmetric. Subtracting (3.2.16) from (3.2.15), the following expression is obtained:

$$\begin{aligned}\nabla_l \nabla_j \bar{g}_{ik} - \nabla_j \nabla_l \bar{g}_{ik} = {} & (2\nabla_l \bar{g}_{ik})(\nabla_j \psi) + (\nabla_l \bar{g}_{jk})(\nabla_i \psi) + (\nabla_l \nabla_i \psi)(\bar{g}_{jk}) \\ & + (\nabla_l \bar{g}_{ij})(\nabla_k \psi) + (\nabla_l \nabla_k \psi)(\bar{g}_{ij}) - (2\nabla_j \bar{g}_{ik})(\nabla_l \psi) \\ & - (\nabla_j \bar{g}_{lk})(\nabla_i \psi) - (\nabla_j \nabla_i \psi)(\bar{g}_{lk}) \\ & - (\nabla_j \bar{g}_{il})(\nabla_k \psi) - (\nabla_j \nabla_k \psi)(\bar{g}_{il}).\end{aligned}$$

Using (3.2.12), the above expression can be written as

$$\begin{aligned}\nabla_l\nabla_j\bar{g}_{ik}-\nabla_j\nabla_l\bar{g}_{ik} &= \bar{g}_{kj}(\nabla_l\nabla_i\psi-\nabla_l\psi\nabla_i\psi)+\bar{g}_{ij}(\nabla_l\nabla_k\psi-\nabla_l\psi\nabla_k\psi)\\ &\quad -\bar{g}_{il}(\nabla_j\nabla_k\psi-\nabla_j\psi\nabla_k\psi)+\bar{g}_{lk}(\nabla_j\nabla_i\psi-\nabla_j\psi\nabla_i\psi)\\ &= \bar{g}_{ij}\psi_{kl}-\bar{g}_{il}\psi_{kj}+\bar{g}_{kj}\psi_{il}-\bar{g}_{kl}\psi_{ij}.\end{aligned}$$

Therefore, from (3.2.14), it follows that

$$\bar{g}_{im}R^m_{kjl}+\bar{g}_{mk}R^m_{ijl}=\bar{g}_{ij}\psi_{kl}-\bar{g}_{il}\psi_{kj}+\bar{g}_{kj}\psi_{il}-\bar{g}_{kl}\psi_{ij}. \tag{3.2.17}$$

□

Lemma 3.2.5 [11] Suppose that for the fundamental tensor $\bar{g}_{ij}$ of $\bar{V}_n$, the Riemannian curvature tensor is denoted by $\bar{R}^m_{ijl}$. Then, the following identity holds:

$$\bar{R}^m_{ijl}=R^m_{ijl}+\delta^m_l\psi_{ij}-\delta^m_j\psi_{il}.$$

Proof: Considering (2.4.10) and subtracting R^m_{ijl} from $\bar{R}^m_{ijl}$ gives,

$$\begin{aligned}\bar{R}^m_{ijl}-R^m_{ijl} &= \partial_j\left(\bar{\Gamma}^m_{il}-\Gamma^m_{il}\right)-\partial_l\left(\bar{\Gamma}^m_{ij}-\Gamma^m_{ij}\right)\\ &\quad+\left(\bar{\Gamma}^h_{il}\,\bar{\Gamma}^m_{hj}-\Gamma^h_{il}\Gamma^m_{hj}\right)-\left(\bar{\Gamma}^h_{ij}\bar{\Gamma}^m_{hl}-\Gamma^h_{ij}\Gamma^m_{hl}\right).\end{aligned}$$

Substituting (3.2.10) into the above equation yields,

$$\bar{R}^m_{ijl}-R^m_{ijl}=\delta^m_l\left(\frac{\partial}{\partial x^j}\psi_i-\Gamma^h_{ij}\psi_h-\psi_j\psi_i\right)-\delta^m_j\left(\frac{\partial}{\partial x^l}\psi_i-\Gamma^h_{il}\psi_h-\psi_l\psi_i\right).$$

As $\nabla_j\psi_i=\partial_j\psi_i-\psi_h\Gamma^h_{ij}$ and $\nabla_i\psi=\psi_i$, it follows that

$$\bar{R}^m_{ijl}-R^m_{ijl}=\delta^m_l\left(\nabla_j\nabla_i\psi-\nabla_j\psi\nabla_i\psi\right)-\delta^m_j\left(\nabla_l\nabla_i\psi-\nabla_l\psi\nabla_i\psi\right).$$

Hence, we get

$$\bar{R}^m_{ijl}=R^m_{ijl}+\delta^m_l\psi_{ij}-\delta^m_j\psi_{il}. \tag{3.2.18}$$

□

Contracting (3.2.18) for m and l gives,

$$\bar{R}_{ij}=R_{ij}+(n-1)\psi_{ij}. \tag{3.2.19}$$

Substituting the expression ψ_{ij} of (3.2.19) into (3.2.18) yields,

$$\bar{R}^m_{ijl}=R^m_{ijl}-\frac{1}{n-1}\left(\delta^m_j\left[\bar{R}_{il}-R_{il}\right]-\delta^m_l\left[\bar{R}_{ij}-R_{ij}\right]\right).$$

Hence, it follows that

$$\bar{R}^m_{ijl} - \frac{1}{n-1}\left(\delta^m_l \bar{R}_{ij} - \delta^m_j \bar{R}_{il}\right) = R^m_{ijl} - \frac{1}{n-1}\left(\delta^m_l R_{ij} - \delta^m_j R_{il}\right). \tag{3.2.20}$$

Definition 3.2.6 $W^m_{ijl} = R^m_{ijl} - \frac{1}{n-1}\left(\delta^m_l R_{ij} - \delta^m_j R_{il}\right)$ is called the *projective curvature tensor* or *Weyl tensor*, [11].

From equation (3.2.20), it follows that projective curvature tensor is preserved under geodesic mappings.

Theorem 3.2.7 (Beltrami's Theorem) [11] Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be two Riemannian manifolds with the Riemannian connections ∇ and $\bar{\nabla}$, respectively. Assume that V_n and $\bar{V}_n$ are in geodesic correspondence. If V_n is of constant curvature, then $\bar{V}_n$ is also of constant curvature.

Proof: Multiplying both sides of (2.4.15) with g^{mh} gives,

$$R^m_{ijl} = K\left(\delta^m_j g_{il} - \delta^m_l g_{ij}\right). \tag{3.2.21}$$

Substituting (3.2.21) into (3.2.17) results,

$$\bar{g}_{jk}\left(Kg_{il} - \psi_{il}\right) - \bar{g}_{kl}\left(Kg_{ij} - \psi_{ij}\right) + \bar{g}_{ij}\left(Kg_{kl} - \psi_{kl}\right) - \bar{g}_{il}\left(Kg_{kj} - \psi_{kj}\right) = 0.$$

By defining a tensor A_{ij} as

$$A_{ij} := Kg_{ij} - \psi_{ij},$$

the above equation can be written as

$$\bar{g}_{jk}A_{il} - \bar{g}_{kl}A_{ij} + \bar{g}_{ij}A_{kl} - \bar{g}_{il}A_{jk} = 0. \tag{3.2.22}$$

Multiplying both sides of (3.2.22) with $\bar{g}^{jk}$ gives,

$$A_{il} = \frac{1}{n}\bar{g}_{il}\bar{g}^{jk}A_{jk}.$$

By defining a scalar invariant ρ as

$$\rho := \frac{1}{n}\bar{g}^{jk}A_{jk},$$

the above equation can be written as

$$A_{il} = \rho\,\bar{g}_{il}. \tag{3.2.23}$$

Substituting (3.2.21) into (3.2.18) yields,

$$\begin{aligned}\bar{R}^m_{ijl} &= K\left(\delta^m_j g_{il} - \delta^m_l g_{ij}\right) + \delta^m_l \psi_{ij} - \delta^m_j \psi_{il} \\ &= \delta^m_j A_{il} - \delta^m_l A_{ij}.\end{aligned}$$

Multiplying both sides of the above equation with $\bar{g}_{mh}$ gives,

$$\bar{R}_{hijl} = \bar{g}_{hj} A_{il} - \bar{g}_{hl} A_{ij}. \tag{3.2.24}$$

Substituting (3.2.23) into (3.2.24), it follows that

$$\bar{R}_{hijl} = \rho\left(\bar{g}_{jh}\bar{g}_{il} - \bar{g}_{hl}\bar{g}_{ij}\right). \tag{3.2.25}$$

As ρ is a scalar invariant, $\bar{V}_n$ is a manifold of constant curvature. □

3.3 Sinyukov Equations

In the previous section, we have obtained the necessary and sufficient condition for the existence of a geodesic mapping between two Riemannian manifolds. However, we have not found any direct relationship between the metrics of these manifolds. Sinyukov introduced the tensors $a_{ij} = e^{2\Psi}\bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}$ and $\lambda_i = -e^{2\Psi}\bar{g}^{\alpha\beta} g_{\beta i}\psi_\alpha$, [4]. Using these objects, he obtained the necessary and sufficient conditions for V_n admitting a geodesic mapping onto $\bar{V}_n$, which were named as "Sinyukov equations". Before deriving the Sinyukov equations, we need to prove the following lemmas:

Lemma 3.3.1 $\nabla_l \bar{g}^{im} = -2\bar{g}^{im}\nabla_l\psi - \delta^m_l \bar{g}^{ih}\nabla_h\psi - \delta^i_l \bar{g}^{hm}\nabla_h\psi.$

Proof: Covariant differentiation of $\bar{g}^{ik}\bar{g}_{kj} = \delta^i_j$ with respect to the connection ∇ yields,

$$\nabla_l\left(\bar{g}^{ik}\bar{g}_{kj}\right) = \bar{g}_{kj}\left(\nabla_l\bar{g}^{ik}\right) + \bar{g}^{ik}\left(\nabla_l\bar{g}_{kj}\right) = \nabla_l\delta^i_j = 0.$$

Hence, it follows that $\bar{g}_{kj}\left(\nabla_l\bar{g}^{ik}\right) = -\bar{g}^{ik}\left(\nabla_l\bar{g}_{kj}\right)$. Using Lemma 3.2.3, this can be expressed in the following way:

$$\bar{g}_{kj}\left(\nabla_l\bar{g}^{ik}\right) = -\bar{g}^{ik}\left(2\bar{g}_{kj}\nabla_l\psi + \bar{g}_{lj}\nabla_k\psi + \bar{g}_{kl}\nabla_j\psi\right).$$

Multiplying both sides with $\bar{g}^{jm}$ gives,

$$\nabla_l\bar{g}^{im} = -2\bar{g}^{im}\nabla_l\psi - \delta^m_l\bar{g}^{ih}\nabla_h\psi - \delta^i_l\bar{g}^{hm}\nabla_h\psi. \tag{3.3.1}$$

□

Lemma 3.3.2 Contraction of the $(1,3)$ Riemannian curvature tensor gives the following identities:

$$g^{mj}R^s_{mjk} = R_{hk}g^{hs}, \qquad (i)$$
$$g^{mj}R^s_{jkm} = -R_{hk}g^{hs}, \qquad (ii)$$
$$g^{mj}R^s_{kmj} = 0. \qquad (iii)$$

Proof:

(i) Multiplying both sides of $g^{lm}R_{mhjk} = R^l_{hjk}$ with δ^j_l and using the skew symmetry property of the $(1,3)$ Riemannian curvature tensor, we have $g^{jm}R_{mhjk} = -R_{hk}$. Now, multiplying both sides with g^{hs} and using Proposition 2.4.8 (i), it follows that $g^{jm}g^{hs}R_{hmjk} = R_{hk}g^{hs}$. Therefore, we get $g^{mj}R^s_{mjk} = R_{hk}g^{hs}$.

(ii) $g^{jm}R^s_{jkm} = -g^{jm}R^s_{jmk} = -g^{mj}R^s_{jmk}$. Thus, by (i), we have $g^{mj}R^s_{jkm} = -R_{hk}g^{hs}$.

(iii) By using the first Bianchi identity, we have $g^{mj}\left(R^s_{kmj} + R^s_{jkm} + R^s_{mjk}\right) = 0$. Considering (i) and (ii), it follows that $g^{mj}R^s_{kmj} = 0$. □

Proposition 3.3.3 (Sinyukov Equations) [4] The condition, for which an n-dimensional $(n \geq 2)$ Riemannian manifold (M,g) admits a geodesic mapping onto another n-dimensional Riemannian manifold $(\bar{M},\bar{g})$, has the following form of differential equations of Cauchy type in covariant derivatives;

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ki}, \qquad (i)$$
$$n\nabla_l \lambda_i = \mu g_{il} + a_{ih}g^{jh}R_{jl} - a_{hj}g^{jm}R^h_{ilm}, \qquad (ii)$$
$$(n-1)\nabla_k \mu = 2(n+1)\lambda_l R_{kj}g^{jl} + a_{ih}g^{jh}g^{il}\left(2\nabla_l R_{jk} - \nabla_k R_{jl}\right), \qquad (iii)$$

where $\mu = g^{\alpha\beta}\nabla_\beta \lambda_\alpha$.

Proof:

(i) Covariant differentiation of a_{ij} with respect to the connection ∇ gives,

$$\nabla_k a_{ij} = \left(\nabla_k e^{2\Psi}\right)\bar{g}^{\alpha\beta}g_{\alpha i}g_{\beta j} + \left(\nabla_k \bar{g}^{\alpha\beta}\right)e^{2\Psi}g_{\alpha i}g_{\beta j}.$$

Using Lemma 3.3.1, it follows that

$$\begin{aligned}\nabla_k a_{ij} &= -e^{2\Psi}\bar{g}^{\alpha h}g_{\alpha i}g_{kj}(\nabla_h \psi) - e^{2\Psi}\bar{g}^{h\beta}g_{ki}g_{\beta j}(\nabla_h \psi)\\ &= \left(-e^{2\Psi}\bar{g}^{\alpha h}g_{\alpha i}\psi_h\right)g_{kj} + \left(-e^{2\Psi}\bar{g}^{h\beta}g_{\beta j}\psi_h\right)g_{ki} \\ &= \lambda_i g_{jk} + \lambda_j g_{ki}.\end{aligned} \qquad (3.3.2)$$

Note: Before proving the second Sinyukov equation, we need to show that the integrability condition of the first Sinyukov equation is

$$a_{ih}R^h_{jkl}+a_{jh}R^h_{ikl}=g_{ik}\nabla_l\lambda_j+g_{jk}\nabla_l\lambda_i-g_{il}\nabla_k\lambda_j-g_{jl}\nabla_k\lambda_i.$$

Considering (2.4.13) for a_{ij} gives,

$$\nabla_l\nabla_k a_{ij}-\nabla_k\nabla_l a_{ij}=a_{ih}R^h_{jkl}+a_{hj}R^h_{ikl}.$$

Taking the covariant derivative of the equation (3.3.2) with respect to the connection ∇ yields,

$$\nabla_l\nabla_k a_{ij}=g_{jk}\nabla_l\lambda_i+g_{ik}\nabla_l\lambda_j. \tag{3.3.3}$$

Interchanging l and k in the equation (3.3.3) gives,

$$\nabla_k\nabla_l a_{ij}=g_{jl}\nabla_k\lambda_i+g_{il}\nabla_k\lambda_j. \tag{3.3.4}$$

Subtracting the equation (3.3.4) from the equation (3.3.3), it follows that

$$a_{ih}R^h_{jkl}+a_{jh}R^h_{ikl}=g_{ik}\nabla_l\lambda_j+g_{jk}\nabla_l\lambda_i-g_{il}\nabla_k\lambda_j-g_{jl}\nabla_k\lambda_i. \tag{3.3.5}$$

(ii) Multiplying both sides of the equation (3.3.5) with g^{jk} yields,

$$\begin{aligned} a_{ih}R^h_{jkl}g^{jk}+a_{jh}R^h_{ikl}g^{jk} &= \delta^j_i\nabla_l\lambda_j+n\nabla_l\lambda_i-g^{jk}g_{il}\nabla_k\lambda_j-\delta^k_l\nabla_k\lambda_i \\ &= n\nabla_l\lambda_i-\mu g_{il}. \end{aligned}$$

Using the skew-symmetry property of the $(1,3)$ Riemannian curvature tensor and Lemma 3.3.2, it follows that

$$n\nabla_l\lambda_i=\mu g_{il}+a_{ih}g^{jh}R_{jl}-a_{hj}g^{jm}R^h_{ilm}. \tag{3.3.6}$$

(iii) Covariantly differentiating (3.3.6) with respect to the connection ∇ yields,

$$n\nabla_k\nabla_l\lambda_i=g_{il}\nabla_k(\mu)+g^{jh}\nabla_k\left(a_{ih}R_{jl}\right)-g^{jm}\nabla_k\left(a_{hj}R^h_{ilm}\right). \tag{3.3.7}$$

Interchanging l and k in (3.3.7) gives,

$$n\nabla_l\nabla_k\lambda_i=g_{ik}\nabla_l(\mu)+g^{jh}\nabla_l\left(a_{ih}R_{jk}\right)-g^{jm}\nabla_l\left(a_{hj}R^h_{ikm}\right). \tag{3.3.8}$$

Subtracting (3.3.7) from (3.3.8) and by using (2.4.13), it follows that

$$\begin{aligned} n\lambda_h R^h_{ikl} &= g_{ik}\nabla_l\mu-g_{il}\nabla_k\mu+g^{jh}\left(R_{jk}\nabla_l a_{ih}-R_{jl}\nabla_k a_{ih}\right) \\ &\quad +a_{ih}g^{jh}\left(\nabla_l R_{jk}-\nabla_k R_{jl}\right)-a_{hj}g^{jm}\left(\nabla_l R^h_{ikm}-\nabla_k R^h_{ilm}\right) \\ &\quad -g^{jm}\left(R^h_{ikm}\nabla_l a_{hj}-R^h_{ilm}\nabla_k a_{hj}\right). \end{aligned}$$

By using the equation (3.3.2), second Bianchi identity, and multiplying both sides with g^{il}, we obtain the following identity:

$$(n-1)\nabla_k\mu = 2(n+1)\lambda_l R_{kj}g^{jl} + a_{ih}g^{jh}g^{il}\left(\nabla_l R_{jk} - \nabla_k R_{jl}\right)$$
$$+a_{hj}g^{jm}g^{il}\left(\nabla_m R^h_{ilk}\right).$$

Considering Lemma 3.3.2, the above equation can be written as

$$(n-1)\nabla_k\mu = 2(n+1)\lambda_l R_{kj}g^{jl} + a_{ih}g^{jh}g^{il}\left(\nabla_l R_{jk} - \nabla_k R_{jl}\right)$$
$$+a_{hj}g^{jm}g^{hs}\nabla_m\left(R_{ks}\right).$$

Letting $s = j$, $j = i$ and $m = l$ in the last term of the right hand side, it follows that

$$(n-1)\nabla_k\mu = 2(n+1)\lambda_l R_{kj}g^{jl} + a_{ih}g^{jh}g^{il}\left(2\nabla_l R_{jk} - \nabla_k R_{jl}\right). \tag{3.3.9}$$

□

4. GEODESIC MAPPINGS OF SOME SPECIAL RIEMANNIAN MANIFOLDS

In this chapter, first we will give the proof of Mikes' Theorem on geodesic mappings of Einstein manifolds, [6]. Furthermore, we will give the proofs of the important theorems of Chepurna's PhD Thesis, [19] considering the Einstein tensor preserving geodesic mappings. Then, we will investigate the geodesic mappings of some special Riemannian manifolds.

4.1 Geodesic Mappings of Einstein Manifolds

In this section, we will give the proof of Mikes' Theorem on geodesic mappings of Einstein manifolds, [6].

Theorem 4.1.1 [6] Let $V_n = (M, g, \nabla)$ and $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$ be two Riemannian manifolds of dimension greater than or equal to three. If V_n and $\bar{V}_n$ are in geodesic correspondence and V_n is an Einstein manifold, then $\bar{V}_n$ is Einstein.

Proof: Let V_n be an Einstein manifold with the Ricci tensor

$$R_{ij} = -K(n-1)g_{ij}. \tag{4.1.1}$$

When V_n is a Riemannian manifold of constant curvature, by Lemma 2.4.15, the Ricci tensor is given as above. However, if the equation (4.1.1) holds, this does not imply that the manifold is of constant curvature.

Covariantly differentiating (3.3.5) with respect to the connection ∇ and using (3.3.2) gives us,

$$\begin{aligned}&(\lambda_i g_{hm} + \lambda_h g_{im}) R^h_{jkl} + a_{ih} \nabla_m R^h_{jkl} + \left(\lambda_h g_{jm} + \lambda_j g_{hm}\right) R^h_{ikl} + a_{hj} \nabla_m R^h_{ikl} \\ &= g_{jk} \nabla_m \nabla_l \lambda_i + g_{ik} \nabla_m \nabla_l \lambda_j - g_{jl} \nabla_m \nabla_k \lambda_i - g_{il} \nabla_m \nabla_k \lambda_j.\end{aligned} \tag{4.1.2}$$

Multiplying both sides of the equation (4.1.2) with g^{lm}, the following equation is obtained:

$$\lambda_i R_{jk} + \lambda_h \left(R^h_{jki} + R^h_{ikj} \right) + \lambda_j R_{ik} + a_{ih} g^{lm} \nabla_m R^h_{jkl} + a_{hj} g^{lm} \nabla_m R^h_{ikl}$$
$$= g_{jk} g^{lm} \nabla_m \nabla_l \lambda_i + g_{ik} g^{lm} \nabla_m \nabla_l \lambda_j - \nabla_j \nabla_k \lambda_i - \nabla_i \nabla_k \lambda_j. \tag{4.1.3}$$

Alternating i and k in (4.1.3) gives us,

$$\lambda_k R_{ji} + \lambda_h \left(R^h_{jik} + R^h_{kij} \right) + \lambda_j R_{ki} + a_{kh} g^{lm} \nabla_m R^h_{jil} + a_{hj} g^{lm} \nabla_m R^h_{kil}$$
$$= g_{ji} g^{lm} \nabla_m \nabla_l \lambda_k + g_{ki} g^{lm} \nabla_m \nabla_l \lambda_j - \nabla_j \nabla_i \lambda_k - \nabla_k \nabla_i \lambda_j. \tag{4.1.4}$$

Subtracting (4.1.4) from (4.1.3), using the first Bianchi identity, the equation (2.4.12) and noting that λ_i is gradient, [4], we get

$$\lambda_i R_{jk} - \lambda_k R_{ij} + 4\lambda_h R^h_{jki} + a_{hj} g^{lm} \nabla_m R^h_{lki} + a_{ih} g^{lm} \nabla_m R^h_{jkl} - a_{kh} g^{lm} \nabla_m R^h_{jil}$$
$$= g_{jk} g^{lm} \nabla_m \nabla_l \lambda_i - g_{ij} g^{lm} \nabla_m \nabla_l \lambda_k. \tag{4.1.5}$$

Considering the equation (4.1.1), second and the third Sinyukov equations become

$$n \nabla_l \lambda_i = \mu g_{il} - K(n-1) a_{il} - a_{hs} g^{st} R^h_{ilt} \tag{4.1.6}$$

and

$$\nabla_k \mu = -2(n+1) K \lambda_k \tag{4.1.7}$$

respectively.

Covariantly differentiating (4.1.6) with respect to the connection ∇ and using (4.1.7), we obtain

$$n \nabla_k \nabla_l \lambda_i = g_{il} \nabla_k \mu - K(n-1) \left[\lambda_i g_{lk} + \lambda_l g_{ik} \right] - \lambda_h R^h_{ilk}$$
$$- \lambda_s g_{hk} g^{st} R^h_{ilt} - a_{hs} g^{st} \nabla_k R^h_{ilt}. \tag{4.1.8}$$

Multiplying both sides of (4.1.5) with n and substituting (4.1.1) into the resulting equation, it follows that

$$Kn(n-1)[-\lambda_i g_{jk} + \lambda_k g_{ij}] + 4n\lambda_h R^h_{jki} + n a_{hj} g^{lm} \nabla_m R^h_{lki} + n a_{ih} g^{lm} \nabla_m R^h_{jkl}$$
$$- n a_{kh} g^{lm} \nabla_m R^h_{jil} = n g_{jk} g^{lm} \nabla_m \nabla_l \lambda_i - n g_{ij} g^{lm} \nabla_m \nabla_l \lambda_k.$$

Using (4.1.8), the above equation can be written as

$$
\begin{aligned}
4n\lambda_i K g_{jk} - 4n\lambda_k K g_{ij} + 4n\lambda_h R^h_{jki} + na_{hj} g^{lm} \nabla_m R^h_{lki} + na_{ih} g^{lm} \nabla_m R^h_{jkl} \\
+ na_{kh} g^{lm} \nabla_m R^h_{jli} = a_{hs} g^{st} g_{lm} \left(g_{jk} \nabla_m R^h_{itl} + g_{ij} \nabla_m R^h_{klt} \right). \qquad (4.1.9)
\end{aligned}
$$

Simplifying the equation (4.1.9) gives us, [6]

$$\lambda_h R^h_{ijk} = K \left(g_{ij} \lambda_k - g_{ik} \lambda_j \right). \qquad (4.1.10)$$

Contracting the equation (3.3.5) with λ^l and considering (4.1.10) leads to

$$g_{ki} \theta_{jh} \lambda^h + g_{kj} \theta_{ih} \lambda^h - \lambda_i \theta_{jk} - \lambda_j \theta_{ik} = 0, \qquad (4.1.11)$$

where $\theta_{ij} = \nabla_j \lambda_i - K a_{ij}$.

Defining a scalar invariant μ satisfying $\lambda^h \theta_{hi} = \mu \lambda_i$ and using the equation (4.1.11), we have

$$\nabla_j \lambda_i = \mu g_{ij} + K a_{ij}. \qquad (4.1.12)$$

Taking the covariant derivative of λ_i with respect to the connection ∇ and using the equations (3.2.12) and (3.3.2), we obtain

$$\psi_{ij} = \bar{K} \bar{g}_{ij} - K g_{ij}, \qquad (4.1.13)$$

where $\bar{K}$ is a scalar invariant.

By using the equations (3.2.19), (4.1.1) and (4.1.13), it follows that

$$\bar{R}_{ij} = (n-1) \bar{K} \bar{g}_{ij}. \qquad (4.1.14)$$

Therefore, $\bar{V}_n$ is an Einstein manifold. □

4.2 Einstein Tensor Preserving Geodesic Mappings

In this section, we will give the important definitions and theorems of Chepurna's PhD thesis, [19].

Definition 4.2.1 Let (M, g) be a Riemannian manifold. The *Einstein tensor* E_{ij} is defined by

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij}.$$

Definition 4.2.2 A map between two Riemannian manifolds (M,g) and $(\bar{M},\bar{g})$ is called *Einstein tensor-preserving* if it satisfies

$$\bar{E}_{ij} = E_{ij}. \tag{4.2.1}$$

Definition 4.2.3 Let (M,g) be a Riemannian manifold. The *concircular curvature tensor* Y^h_{ijk} is given by the formula

$$Y^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right). \tag{4.2.2}$$

Theorem 4.2.4 [19] Let $V_n = (M,g)$ and $\bar{V}_n = (\bar{M},\bar{g})$ be two Riemannian manifolds. If there exists an Einstein tensor preserving geodesic mapping from V_n onto $\bar{V}_n$, then the following identity holds:

$$\nabla_j \lambda_i = e^{2\Psi}\left(\bar{g}^{\alpha\beta}\psi_\alpha\psi_\beta - \frac{\bar{R}}{n(n-1)}\right)g_{ij} + \frac{R}{n(n-1)}a_{ij}.$$

Proof: Taking the covariant derivative of $\lambda_i = -e^{2\Psi}\bar{g}^{\alpha\beta}g_{\beta i}\psi_\alpha$ with respect to the connection ∇ and using Lemma 3.3.1 gives us,

$$\nabla_j \lambda_i = e^{2\Psi}\bar{g}^{\alpha\beta}g_{ij}\psi_\alpha\psi_\beta - e^{2\Psi}\bar{g}^{\alpha\beta}g_{\beta i}\psi_{\alpha j}. \tag{4.2.3}$$

Since the map is Einstein tensor preserving, we have

$$R_{ij} - \frac{R}{n}g_{ij} = \bar{R}_{ij} - \frac{\bar{R}}{n}\bar{g}_{ij}. \tag{4.2.4}$$

Substituting the equation (4.2.4) into the equation (3.2.19), it follows that

$$\psi_{ij} = \frac{\bar{R}}{n(n-1)}\bar{g}_{ij} - \frac{R}{n(n-1)}g_{ij}. \tag{4.2.5}$$

Now, substituting the equation (4.2.5) into the equation (4.2.3), we have

$$\begin{aligned}
\nabla_j \lambda_i &= \left(e^{2\Psi}\bar{g}^{\alpha\beta}\psi_\alpha\psi_\beta\right)g_{ij} - \left(\frac{\bar{R}}{n(n-1)}\bar{g}_{\alpha j} - \frac{R}{n(n-1)}g_{\alpha j}\right)e^{2\Psi}\bar{g}^{\alpha\beta}g_{\beta i} \\
&= e^{2\Psi}\left(\bar{g}^{\alpha\beta}\psi_\alpha\psi_\beta - \frac{\bar{R}}{n(n-1)}\right)g_{ij} + \frac{R}{n(n-1)}\left(e^{2\Psi}\bar{g}^{\alpha\beta}g_{\alpha j}g_{\beta i}\right) \\
&= e^{2\Psi}\left(\bar{g}^{\alpha\beta}\psi_\alpha\psi_\beta - \frac{\bar{R}}{n(n-1)}\right)g_{ij} + \frac{R}{n(n-1)}a_{ij}. \qquad (4.2.6)
\end{aligned}$$

□

Theorem 4.2.5 [19] Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be two Riemannian manifolds. If there exists an Einstein tensor preserving geodesic mapping from V_n onto $\bar{V}_n$, then the following identity holds:

$$a_{ih}Y^h_{jkl} + a_{jh}Y^h_{ikl} = 0.$$

Proof: Substituting (4.2.6) into (3.3.5) we obtain,

$$a_{ih}R^h_{jkl} + a_{jh}R^h_{ikl} - \frac{R}{n(n-1)}\left(a_{il}g_{jk} + a_{jl}g_{ik} - a_{ik}g_{jl} - a_{jk}g_{il}\right) = 0. \tag{4.2.7}$$

By using the formula of concircular curvature tensor Y^h_{ijk}, the above equation can be written as

$$a_{ih}\left(Y^h_{jkl} - \frac{R}{n(n-1)}\left(\delta^h_l g_{jk} - \delta^h_k g_{jl}\right)\right) + a_{jh}\left(Y^h_{ikl} - \frac{R}{n(n-1)}\left(\delta^h_l g_{ik} - \delta^h_k g_{il}\right)\right)$$
$$- \frac{R}{n(n-1)}\left(a_{il}g_{jk} + a_{jl}g_{ik} - a_{ik}g_{jl} - a_{jk}g_{il}\right) = 0.$$

Simplifying the above equation gives us

$$a_{ih}Y^h_{jkl} + a_{jh}Y^h_{ikl} = 0. \tag{4.2.8}$$

□

Theorem 4.2.6 [19] Concircular curvature tensor is preserved under Einstein tensor preserving geodesic mappings.

Proof: As projective curvature tensor is preserved under geodesic mappings, we have

$$W^h_{ijk} = \bar{W}^h_{ijk}, \tag{4.2.9}$$

where $W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right)$.

Assume that the map is Einstein tensor preserving. Substituting (4.2.1) into (4.2.9), we obtain

$$R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k\left[\frac{R}{n}g_{ij} + \bar{R}_{ij} - \frac{\bar{R}}{n}\bar{g}_{ij}\right] - \delta^h_j\left[\frac{R}{n}g_{ik} + \bar{R}_{ik} - \frac{\bar{R}}{n}\bar{g}_{ik}\right]\right)$$
$$= \bar{R}^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k \bar{R}_{ij} - \delta^h_j \bar{R}_{ik}\right).$$

Simplifying the above equation, we get

$$R^h_{ijk} - \frac{R}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) = \bar{R}^h_{ijk} - \frac{\bar{R}}{n(n-1)}\left(\delta^h_k \bar{g}_{ij} - \delta^h_j \bar{g}_{ik}\right).$$

Hence, for Einstein tensor preserving geodesic mappings, it follows that

$$Y^h_{ijk} = \bar{Y}^h_{ijk}. \tag{4.2.10}$$

□

4.3 Geodesic Mappings of Quasi Einstein Manifolds

In this section, we will consider the Einstein tensor preserving geodesic mappings from a quasi Einstein manifold $V_n = (M, g)$ onto an arbitrary Riemannian manifold $\bar{V}_n = (\bar{M}, \bar{g})$ and we obtain new results.

From Chepurna's PhD thesis, [19] and from the paper of Chepurna, Kiosak and Mikes, [20], under Einstein tensor preserving geodesic mappings the concircular curvature tensor Y^h_{ijk} is preserved (Theorem 4.2.6). That is,

$$\bar{Y}^h_{ijk} = Y^h_{ijk}, \tag{4.3.1}$$

where $Y^h_{ijk} = R^h_{ijk} - \dfrac{R}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right)$.

Theorem 4.3.1 Let $V_n = (M, g, \nabla)$ and $\bar{V}_n = \left(\bar{M}, \bar{g}, \bar{\nabla}\right)$ be two n-dimensional Riemannian manifolds. If there is a geodesic map between V_n and $\bar{V}_n$ preserving the Einstein tensor and V_n is a quasi Einstein manifold, then $\bar{V}_n$ is nearly quasi Einstein.

Proof: Assume there is a geodesic mapping between V_n and $\bar{V}_n$ which preserves the Einstein tensor. By contracting the equation (4.3.1) for h and k, we obtain

$$\bar{R}_{ij} - \frac{\bar{R}}{n}\bar{g}_{ij} = R_{ij} - \frac{R}{n}g_{ij}. \tag{4.3.2}$$

If V_n is quasi Einstein, then we have

$$R_{ij} = a g_{ij} + b u_i u_j, \tag{4.3.3}$$

where a and b are nonzero scalars and $u_i u_j g^{ij} = 1$. Multiplying both sides of (4.3.3) with g^{ij} gives,

$$R = na + b. \tag{4.3.4}$$

Substituting the equations (4.3.3) and (4.3.4) into the equation (4.3.2), we get

$$\begin{aligned}\bar{R}_{ij} - \frac{\bar{R}}{n}\bar{g}_{ij} &= a g_{ij} + b u_i u_j - \frac{na+b}{n} g_{ij} \\ &= b\left(u_i u_j - \frac{1}{n} g_{ij}\right).\end{aligned}$$

Hence, we get

$$\bar{R}_{ij} = \frac{\bar{R}}{n}\bar{g}_{ij} + b\left(u_i u_j - \frac{1}{n}g_{ij}\right).$$

Put $\bar{a} := \frac{\bar{R}}{n}$, $\bar{b} := b$ and $\bar{A}_{ij} := u_i u_j - \frac{1}{n}g_{ij}$. Then, it follows that

$$\bar{R}_{ij} = \bar{a}\bar{g}_{ij} + \bar{b}\bar{A}_{ij}. \tag{4.3.5}$$

Therefore, $\bar{V}_n$ is a nearly quasi Einstein manifold. □

4.4 Geodesic Mappings of Ricci Recurrent and Ricci Symmetric Manifolds

Sinyukov proved that a recurrent manifold V_n with nonconstant curvature only admits a trivial geodesic mapping, [21].

Furhermore, Mikes proved that a non-Einstein Ricci symmetric manifold V_n does not admit a nontrivial geodesic mapping, [5].

In the next section, we will examine geodesic mappings of generalized Ricci recurrent manifolds.

4.5 Geodesic Mappings of Generalized Ricci Recurrent Manifolds

In this section, we will consider geodesic mappings of generalized Ricci recurrent manifolds and we obtain new results.

Before stating our theorem, we must prove the following lemmas:

Lemma 4.5.1 [4] $a_{ih}R^h_{jkl} + a_{kh}R^h_{jli} + a_{lh}R^h_{jik} = 0.$

Proof: Recall the integrability condition of the first Sinyukov equation:

$$a_{ih}R^h_{jkl} + a_{jh}R^h_{ikl} = g_{jk}\nabla_l\lambda_i + g_{ik}\nabla_l\lambda_j - g_{jl}\nabla_k\lambda_i - g_{il}\nabla_k\lambda_j. \tag{4.5.1}$$

Permuting the indices i, k and l we obtain two more equations, namely

$$a_{kh}R^h_{jli} + a_{jh}R^h_{kli} = g_{kl}\nabla_i\lambda_j + g_{jl}\nabla_i\lambda_k - g_{ki}\nabla_l\lambda_j - g_{ji}\nabla_l\lambda_k \tag{4.5.2}$$

and

$$a_{lh}R^h_{jik} + a_{jh}R^h_{lik} = g_{li}\nabla_k\lambda_j + g_{ji}\nabla_k\lambda_l - g_{lk}\nabla_i\lambda_j - g_{jk}\nabla_i\lambda_l. \tag{4.5.3}$$

Adding the equations (4.5.1), (4.5.2) and (4.5.3), using the first Bianchi identity, it follows that

$$a_{ih}R^h_{jkl} + a_{kh}R^h_{jli} + a_{lh}R^h_{jik} = 0. \tag{4.5.4}$$

□

Lemma 4.5.2 [4] $a_{hl}R^h_k - a_{hk}R^h_l = 0$, where $R^h_k = g^{hm}R_{mk}$.

Proof: Contracting the equation (4.5.4) with g^{ij} and using Lemma 3.3.2, we get

$$a_{ih}g^{ij}R^h_{jkl} + a_{lh}g^{hm}R_{mk} - a_{kh}g^{hm}R_{ml} = 0.$$

Considering the term $a_{ih}g^{ij}R^h_{jkl}$ and by using the first Bianchi identity, we get

$$\begin{aligned} a_{ih}g^{ij}R^h_{jkl} &= e^{2\Psi}\bar{g}^{\alpha\beta}g_{\alpha h}\delta^j_\beta\left(-R^h_{ljk} - R^h_{klj}\right) \\ &= e^{2\Psi}\bar{g}^{\alpha\beta}g_{\alpha h}\delta^j_\beta\left(R^h_{lkj} - R^h_{klj}\right) \\ &= e^{2\Psi}\bar{g}^{\alpha\beta}\left(R_{\alpha lk\beta} - R_{\alpha kl\beta}\right) \\ &= e^{2\Psi}\bar{g}^{\alpha\beta}R_{\alpha lk\beta} - e^{2\Psi}\bar{g}^{\beta\alpha}R_{\beta kl\alpha} \\ &= 0. \end{aligned}$$

Therefore, we obtain

$$a_{lh}R^h_k - a_{kh}R^h_l = 0. \tag{4.5.5}$$

□

Theorem 4.5.3 Let $V_n = (M, g, \nabla)$ and $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$ be two Riemannian manifolds. If V_n and $\bar{V}_n$ are in geodesic correspondence and V_n is a generalized Ricci recurrent manifold, then the following identity holds:

$$\lambda_h\left(nR^h_k - \delta^h_k R\right) = 0.$$

Proof: If V_n is generalized Ricci recurrent, then we have

$$\nabla_k R_{ij} = \phi_k R_{ij} + \alpha_k g_{ij} \tag{4.5.6}$$

for some 1-forms ϕ_k and α_k.

Taking the covariant derivative of the equation (4.5.5) with respect to the connection ∇ and using (4.5.6) gives,

$$\phi_m R^h_k a_{hl} + \alpha_m a_{hl}\delta^h_k - \phi_m R^h_l a_{hk} - \alpha_m a_{hk}\delta^h_l + R^h_k\nabla_m a_{hl} - R^h_l\nabla_m a_{hk} = 0.$$

Since a_{ij} is symmetric, equation (3.3.2) gives us,

$$R_k^h(\lambda_h g_{lm} + \lambda_l g_{hm}) - R_l^h(\lambda_h g_{km} + \lambda_k g_{hm}) = 0. \tag{4.5.7}$$

Multiplying both sides of (4.5.7) with g^{lm}, we get

$$(n+1)R_k^h \lambda_h - R_l^h \left(\lambda_h \delta_k^l + \lambda_k \delta_h^l\right) = 0.$$

Simplifying the above equation gives,

$$nR_k^h \lambda_h - R\lambda_k = 0.$$

As $\lambda_k = \delta_k^h \lambda_h$, it follows that

$$\lambda_h \left(nR_k^h - \delta_k^h R\right) = 0. \tag{4.5.8}$$

□

4.6 Geodesic Mappings of Pseudo Ricci Symmetric and Almost Pseudo Ricci Symmetric Manifolds

In this section, we will consider the geodesic mappings of pseudo Ricci symmetric and almost pseudo Ricci symmetric manifolds and we obtain new results.

Before stating our theorems, we must prove the following lemma:

Lemma 4.6.1 Let $V_n = (M, g, \nabla)$ and $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$ be two Riemannian manifolds. If V_n and $\bar{V}_n$ are in geodesic correspondence, then we have

$$\bar{\nabla}_k \bar{R}_{ij} = \nabla_k R_{ij} + (n-1)\nabla_k \psi_{ij} - 2\psi_k \bar{R}_{ij} - \psi_i \bar{R}_{kj} - \psi_j \bar{R}_{ik}.$$

Proof: Covariantly differentiating the Ricci tensor $\bar{R}_{ij}$ with respect to the connection $\bar{\nabla}$ gives,

$$\bar{\nabla}_k \bar{R}_{ij} = \partial_k \bar{R}_{ij} - \bar{\Gamma}_{ki}^h \bar{R}_{hj} - \bar{\Gamma}_{kj}^h \bar{R}_{ih}. \tag{4.6.1}$$

Using the equation (3.2.10), the equation (4.6.1) can be written as

$$\bar{\nabla}_k \bar{R}_{ij} = \partial_k \bar{R}_{ij} - \left(\Gamma_{ki}^h + \delta_k^h \psi_i + \delta_i^h \psi_k\right) \bar{R}_{hj} - \left(\Gamma_{kj}^h + \delta_k^h \psi_j + \delta_j^h \psi_k\right) \bar{R}_{ih}.$$

Simplifying the above equation gives,

$$\bar{\nabla}_k \bar{R}_{ij} = \partial_k \bar{R}_{ij} - \Gamma_{ki}^h \bar{R}_{hj} - 2\psi_k \bar{R}_{ij} - \psi_i \bar{R}_{kj} - \psi_j \bar{R}_{ik} - \Gamma_{kj}^h \bar{R}_{ih}.$$

As $\partial_k\bar{R}_{ij} - \Gamma^h_{ki}\bar{R}_{hj} - \Gamma^h_{kj}\bar{R}_{ih} = \nabla_k\bar{R}_{ij}$, it follows that

$$\bar{\nabla}_k\bar{R}_{ij} = \nabla_k\bar{R}_{ij} - 2\psi_k\bar{R}_{ij} - \psi_i\bar{R}_{kj} - \psi_j\bar{R}_{ik}.$$

Using (3.2.19), we get

$$\bar{\nabla}_k\bar{R}_{ij} = \nabla_k R_{ij} + (n-1)\nabla_k\psi_{ij} - 2\psi_k\bar{R}_{ij} - \psi_i\bar{R}_{kj} - \psi_j\bar{R}_{ik}. \tag{4.6.2}$$

□

Theorem 4.6.2 If $V_n = (M,g)$ is a pseudo Ricci symmetric manifold admitting geodesic mapping onto $\bar{V}_n = (\bar{M},\bar{g})$ and $\nabla_k\psi_{ij} = 2A_k\psi_{ij} + A_i\psi_{kj} + A_j\psi_{ik}$, then $\bar{V}_n$ is pseudo Ricci symmetric.

Proof: Assume that V_n is pseudo Ricci symmetric. Then, we have

$$\nabla_k R_{ij} = 2A_k R_{ij} + A_i R_{kj} + A_j R_{ik}.$$

Using the equation (3.2.19), the above equation can be written as

$$\nabla_k R_{ij} = 2A_k\left(\bar{R}_{ij} - (n-1)\psi_{ij}\right) + A_i\left(\bar{R}_{kj} - (n-1)\psi_{kj}\right) + A_j\left(\bar{R}_{ik} - (n-1)\psi_{ik}\right). \tag{4.6.3}$$

By using the equations (4.6.3) and (4.6.2), we get

$$\begin{aligned}\nabla_k R_{ij} = {} & 2A_k\bar{R}_{ij} + A_i\bar{R}_{kj} + A_j\bar{R}_{ik} - 2\psi_k\bar{R}_{ij} - \psi_i\bar{R}_{kj} - \psi_j\bar{R}_{ik} \\ & - (n-1)\left[2A_k\psi_{ij} + A_i\psi_{kj} + A_j\psi_{ik} - \nabla_k\psi_{ij}\right].\end{aligned}$$

Now, assume that $\nabla_k\psi_{ij} = 2A_k\psi_{ij} + A_i\psi_{kj} + A_j\psi_{ik}$. Then, we have

$$\bar{\nabla}_k\bar{R}_{ij} = 2\left(A_k - \psi_k\right)\bar{R}_{ij} + \left(A_i - \psi_i\right)\bar{R}_{kj} + \left(A_j - \psi_j\right)\bar{R}_{ik}. \tag{4.6.4}$$

Hence, $\bar{V}_n$ is pseudo Ricci symmetric associated to 1-form $\bar{A}_k = A_k - \psi_k$. □

Theorem 4.6.3 If $V_n = (M,g)$ is an almost pseudo Ricci symmetric manifold admitting geodesic mapping onto $\bar{V}_n = (\bar{M},\bar{g})$ and $\nabla_k\psi_{ij} = (A_k + B_k)\psi_{ij} + A_i\psi_{kj} + A_j\psi_{ik}$, then $\bar{V}_n$ is almost pseudo Ricci symmetric.

Proof: Assume that V_n is almost pseudo Ricci symmetric. Then, we have

$$\nabla_k R_{ij} = (A_k + B_k)R_{ij} + A_i R_{kj} + A_j R_{ik}.$$

Using the equation (3.2.19), the above equation can be written as

$$\nabla_k R_{ij} = (A_k + B_k)\left(\bar{R}_{ij} - (n-1)\,\psi_{ij}\right) + A_i\left(\bar{R}_{kj} - (n-1)\psi_{kj}\right) + A_j\left(\bar{R}_{ik} - (n-1)\psi_{ik}\right). \tag{4.6.5}$$

By using the equations (4.6.5) and (4.6.2), we get

$$\begin{aligned} \nabla_k R_{ij} &= (A_k + B_k)\,\bar{R}_{ij} + A_i\bar{R}_{kj} + A_j\bar{R}_{ik} - 2\psi_k\bar{R}_{ij} - \psi_i\bar{R}_{kj} - \psi_j\bar{R}_{ik} \\ &\quad - (n-1)\left[(A_k + B_k)\,\psi_{ij} + A_i\psi_{kj} + A_j\psi_{ik} - \nabla_k\psi_{ij}\right]. \end{aligned}$$

Now, assume that $\nabla_k\psi_{ij} = (A_k + B_k)\,\psi_{ij} + A_i\psi_{kj} + A_j\psi_{ik}$. Then, we have

$$\bar{\nabla}_k\bar{R}_{ij} = ((A_k - \psi_k) + (B_k - \psi_k))\,\bar{R}_{ij} + (A_i - \psi_i)\,\bar{R}_{kj} + \left(A_j - \psi_j\right)\bar{R}_{ik}. \tag{4.6.6}$$

Hence, $\bar{V}_n$ is almost pseudo Ricci symmetric, with $\bar{A}_k = A_k - \psi_k$ and $\bar{B}_k = B_k - \psi_k$. □

5. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, we investigated the geodesic mappings of some special Riemannian manifolds. First, we gave the properties of smooth and Riemannian manifolds. After the preliminaries chapter, we obtained the differential equations of geodesics. Next, using these equations, we found the necessary and sufficient conditions for the existence of geodesic mappings of Riemannian manifolds. Next, we derived the Sinyukov equations, which are the equivalent conditions for the existence of geodesic mappings.

In the fourth chapter, we obtained our own results. We began this chapter by giving the proof of Mikes' Theorem on geodesic mappings of Einstein manifolds. Next, we examined Chepurna's PhD thesis considering the Einstein tensor preserving geodesic mappings.

Using the results of Chepurna's work, we proved the following theorem:

If there exists an Einstein tensor preserving geodesic mapping from a quasi Einstein manifold $V_n = (M, g)$ onto a Riemannian manifold $\bar{V}_n$, then $\bar{V}_n = (\bar{M}, \bar{g})$ is nearly quasi Einstein.

In the next section, we investigated the geodesic mappings of generalized Ricci recurrent manifolds and proved:

Let $V_n = (M, g, \nabla)$ and $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$ be two Riemannian manifolds. If V_n and $\bar{V}_n$ are in geodesic correspondence and V_n is a generalized Ricci recurrent manifold, then the following identity holds:

$$\lambda_h \left(nR^h_k - \delta^h_k R \right) = 0.$$

In the final section, we considered the geodesic mappings of pseudo Ricci symmetric and almost pseudo Ricci symmetric manifolds and proved the following theorems:

(i) If $V_n = (M, g)$ is a pseudo Ricci symmetric manifold admitting geodesic mapping onto $\bar{V}_n = (\bar{M}, \bar{g})$ and $\nabla_k \psi_{ij} = 2A_k \psi_{ij} + A_i \psi_{kj} + A_j \psi_{ik}$, then $\bar{V}_n$ is pseudo Ricci symmetric.

(ii) If $V_n = (M, g)$ is an almost pseudo Ricci symmetric manifold admitting geodesic mapping onto $\bar{V}_n = (\bar{M}, \bar{g})$ and $\nabla_k \psi_{ij} = (A_k + B_k)\, \psi_{ij} + A_i \psi_{kj} + A_j \psi_{ik}$, then $\bar{V}_n$ is almost pseudo Ricci symmetric.

We should remark that the investigation of geodesic mappings is a broad subject of differential geometry. There are manifolds for which a detailed research has not been made yet. For example, geodesic mappings of Ricci solitons can be studied. Moreover, by tightening the conditions which we have assumed for quasi Einstein manifolds, a general result can be obtained.

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