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BIHARMONIC AND BICONSERVATIVE SUBMANIFOLDS OF LORENTZIAN SPACE FORMS

Ph.D. THESIS

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To me in the future,

FOREWORD

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ABBREVIATIONS

CMC	**:** Constant mean curvature
MCGL	**:** Mean curvature whose gradient is light-like
PNMCV	**:** Parallel normalized mean curvature vector

SYMBOLS

M	: Submanifold
$N_1,\ N_2, N$	: Normal vector field of submanifold
$\mathbb{S}_s^{m-1}(c)$	: $m-1$ dimensional pseudo-sphere with index s, curvature c, centered at the origin
$\mathbb{H}_s^{m-1}(-c)$	: $m-1$ dimensional pseudo-hyperbolic sphere with index s, curvature $-c$, centered at the origin
$\tilde{g},\ g$	: Metric tensor of the ambient space and the submanifold, respectively.
$\tilde{\nabla}$	: Levi-Civita connection of the ambient space
∇	: Levi-Civita connection of the submanifold
$\overline{\nabla}$	: Van Der Waerden-Bortolotti Connection
$\nabla^{\perp}$	: Normal connection
A	: Shape operator
h	: Second fundamental form
H	: Mean curvature vector
$\tilde{R}$	: Curvature tensor of the ambient space
R	: Curvature tensor of the submanifold
$R^{\perp}$	: Normal curvature tensor
S	: Scalar curvature
Δ	: Laplace operator

BIHARMONIC AND BICONSERVATIVE SUBMANIFOLDS OF LORENTZIAN SPACE FORMS

SUMMARY

In 1964, Eells and Sampson gave the definition of biharmonic maps as a generalization of harmonic maps during they were studying on the energy functional E between Riemannian manifolds which has geometrical and physical interest. Later, many geometers interested in biharmonic maps.

By the definition, a biharmonic map $\phi : M \to N$ between two Semi-Riemannian manifolds is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M \|\tau(\phi)\|^2 v_g,$$

where

$$\tau(\phi) = \operatorname{trace} \nabla d\phi$$

is the tension field of ϕ that vanishes for harmonic maps. If ϕ is a biharmonic isometric immersion into N then M is said to be biharmonic submanifold of N.

In the middle of 1980's, Chen studied *biharmonic submanifolds* in Euclidean spaces as a part of his program of understanding finite type submanifolds in Euclidean spaces. He gave an alternative definition of biharmonic submanifolds in Euclidean spaces. That definition is also same for pseudo-Euclidean spaces: If the position vector field $x : M \to \mathbb{E}^n$ satisfies

$$\Delta^2 x = 0$$

then M is called biharmonic submanifold, where Δ denote the Laplacian of M. By the well known Laplace-Beltrami identity this equation is equivalent to

$$\Delta H = 0,$$

where H is the mean curvature vector of M. In the mean time, independently, Jiang showed that a smooth map ϕ is biharmonic if and only if its bitension field $\tau_2(\phi)$ (which corresponds the Euler-Lagrange equation of bienergy functional) vanishes identically, i.e.,

$$\tau_2(\phi) = 0.$$

Jiang also showed that $\tau_2(\phi) = 0$ if and only if $\Delta H = 0$ for an isometric immersion $\phi : M \to \mathbb{E}^n$. As a result, definitions given by Chen and Jiang coincide for the class of Euclidean and pseudo-Euclidean submanifolds.

Biconservative submanifolds arose from the theory of biharmonic submanifolds. Stress-energy tensor for the energy function described by Hilbert was expanded for the bienergy function as follows

$$\begin{aligned} S_2(X,Y) = {} & \frac{1}{2}\|\tau(\phi)\|^2\langle X,Y\rangle + \langle d\phi, \nabla\tau(\phi)\rangle\langle X,Y\rangle \\ & -\langle d\phi(X), \nabla_Y\tau(\phi)\rangle - \langle d\phi(Y), \nabla_X\tau(\phi)\rangle \end{aligned}$$

satisfying $\operatorname{div} S_2 = -\langle \tau_2(\phi), d\phi \rangle$. In general, a submanifold is called *biconservative* if $\operatorname{div} S_2 = 0$. It means $(\tau_2(\phi))^T = 0$. Indeed, this is equivalent to $(\Delta H)^T = 0$ when the ambient space is pseudo-Euclidean. Because, for the isometric immersion into $\mathbb{E}^n_1$, $\tau(\phi) = -mH$ and $\tau_2(\phi) = m\Delta H$.

In this thesis we study on the *biconservative submanifolds* and *biconservative hypersurfaces* of the Lorentzian space forms and we also obtained some results related biharmonic ones. This work consists of seven sections and these sections were planned as follows:

In the first section, we give a brief history and philosophy of biharmonic and biconservative submanifolds and studies has been done so far.

In the second section, we give some basic notions of the submanifold theory on Lorentzian inner product space and biharmonic submanifolds.

In the third section, biconservative surfaces with constant mean curvature (CMC) in Minkowski 4-space $\mathbb{E}^4_1$ is studied. Firstly, we determine the canonical forms of the shape operator and then we give some examples of such submanifolds in $\mathbb{E}^4_1$. Later, we classify biconservative CMC submanifolds in $\mathbb{E}^4_1$. Then, we generalize all results to the CMC surfaces of $\mathbb{S}^4_1$ and $\mathbb{H}^4_1$.

In the fourth section, we examine the biconservative hypersurfaces in Minkowski 4-space $\mathbb{E}^4_1$. In particular, we study hypersurfaces with non-diagonalizable shape operator A satisfying

$$A(\nabla H) = -\frac{nH}{2}\nabla H,$$

where n and H are the dimension and the mean curvature of the hypersurface, respectively. We determine the canonical forms of the shape operator, the mean curvature, sectional curvature and Levi-Civita connection of this hypersurface. Afterwards we give the necessary and sufficient condition for this hypersurface to be biconservative. Later we classify the biconservative hypersurface in $\mathbb{E}^4_1$ and show the uniquniess of them.

In fifth section, we examine the biconservative hypersurfaces with certain shape operator in Minkowski 5 space $\mathbb{E}^5_1$. We give some non-existence theorems.

In the sixth section, we examine the biconservative submanifolds with mean curvature whose gradient is light-like in $\mathbb{E}^n_1$. We give some non-existence results.

In the last section, the obtained conclusions are shared and recommendations are made about the future of the problems.

LORENTZ UZAY FORMLARININ
BİHARMONİK VE BİKONSERVATİF ALTMANİFOLDLARI

ÖZET

1964'te Eels ve Sampson enerji fonksiyonelinin geometrik ve fiziksel özellikleri üzerine çalışırken harmonik tasvirlerin bir genelleştirilmesi olarak k-harmonik tasvir tanımını literatüre sundular. Sonrasında bir çok geometrici biharmonik tasvirler üzerine çalışmalara başladı.

İki yarı-Riemann manifold arasındaki bir $\phi : M \to N$ tasvirine,

$$\tau(\phi) = \operatorname{trace} \nabla d\phi$$

ile ϕ tasvirinin gerilme alanı gösterilmek üzere, eğer

$$E_2 = \frac{1}{2}\int_M \|\tau(\phi)\|^2 v_g$$

şeklinde tanımlanan bienerji fonksiyonelinin kritik noktası ise biharmoniktir denir. Özel olarak ϕ bir izometrik daldırma ise $\tau(\phi) = mH$ denklemi sağlanır ki burada m ve H, M manifoldunun, sırasıyla, boyutunu ve ortalama eğrilik vektörünü göstermektedir.

1980'li yılların ortalarında, Chen, Öklid uzayların sonlu tipten altmanifoldlarının yapısının anlaşılması üzerine çalışırken bu çalışmanın bir parçası olarak biharmonik altmalifoldları çalışmıştır. Öklid uzayların biharmonik altmanifoldlarının alternatif tanımını vermiştir. Bu tanımın, sonradan yarı-Öklid uzaylar için de doğru olduğu anlaşılmıştır. Bir $\mathbb{E}^n_s$ yarı-Öklid uzayının bir M altmanifolduna, eğer x yer vektörü, $\Delta^2 x = 0$ denklemini sağlıyorsa biharmoniktir denir ki burada Δ ile Laplace operatörü gösterilmiştir. Laplace-Beltrami formülünden dolayı bu koşul, M altmanifoldunun H ortalama eğrilik vektörünün

$$\Delta H = 0$$

denklemini sağlamasıdır.

Hemen hemen aynı zamanlarda Jiang bienerji fonksiyonelinin Euler-Lagrange fonksiyonu üzerine çalışırken bu fonksiyonun ϕ tasvirine karşılık gelen bigerilim alanı olduğunu göstermiş ve bunu $\tau_2(\phi)$ ile ifade etmiştir. Bu kavram aşağıdaki şekilde ifade edilir:

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi.$$

Burada R^N izometrik olarak daldırılan uzayın Riemann eğrilik tensörüdür. Ayrıca bir ϕ tasvirinin biharmonik olabilmesi için gerek ve yeter koşulun $\tau_2(\phi) = 0$ eşitliğinin sağlanması olduğunu göstermiştir. Bundan yaklaşık kırk yıl sonra Caddeo v.d. yaptıkları çalışmada izometrik daldırma $\phi : M \to \mathbb{E}^n_1$ için

$$\tau_2(\phi) = 0$$

eşitliği sağlanmasının gerek ve yeter koşulunun

$$\Delta H = 0$$

olduğu göstermişlerdir. Sonuç olarak, Chen ve Jiang tarafından verilen biharmoniklik tanımları Öklid ve yarı-Öklid uzayların altmanifoldları için birbirlerine denktirler.

Bikonservatif altmanifoldlar, biharmonik altmanifoldlardan ortaya çıkmıştır. Hilbert tarafından 1924'te tanımlanan stress-enerji tensörünün, $\operatorname{div} S_2 = -\langle \tau_2(\phi), d\phi \rangle$ olmak üzere, bienerji fonksiyonlarına genişletilmesi

$$\begin{aligned} S_2(X,Y) = & \frac{1}{2}\|\tau(\phi)\|^2 \langle X,Y \rangle + \langle d\phi, \nabla \tau(\phi) \rangle \langle X,Y \rangle \\ & - \langle d\phi(X), \nabla_Y \tau(\phi) \rangle - \langle d\phi(Y), \nabla_X \tau(\phi) \rangle \end{aligned}$$

şeklinde verilmiştir. Eğer $\operatorname{div} S_2 = 0$ ise ϕ tasvirine bikonservatif tasvir denir. Özel olarak, ϕ tasviri bir izometrik daldırma ise bu koşul $(\Delta^2 x)^T = 0$ denkleminin veya, Laplace-Beltrami formülünden dolayı, $(\Delta H)^T = 0$ denkleminin sağlanmasına denktir.

Bu tez çalışmasında Lorentz uzay formlarının bikonservatif altmanifoldları çalışılmıştır. Ayrıca yine Lorentz uzay formlarının biharmonik altmanifoldları ile ilgili bazı sonuçlar elde edilmiştir. Bu çalışma yedi bölümden oluşmaktadır ve aşağıdaki gibi planlanmıştır:

İlk bölümde biharmonik ve bikonservatif alt manifold fikrinin tarihsel gelişimi verilmiştir. Günümüze kadar olan çalışmalar özetlenmiştir.

İkinci bölümde bu tezde faydalanılacak temel tanım ve teoremlere yer verilmiştir. Biharmoniklik ve bikonservatiflik denklemleri verilmiştir.

Üçüncü bölümde, 4-boyutlu $\mathbb{E}^4_1$ Minkowski uzayında, karşıt boyutu 2 olan sabit ortalama eğrilikli (CMC) bikonservatif altmanifoldlar üzerine çalışılmıştır. Şekil operatörleri, bikonservatiflik denklemi için önemli olduğundan öncelikle altmanifoldun şekil operatörünün kanonik formları elde edilmiştir. 4-boyutlu $\mathbb{E}^4_1$ Minkowski uzayında , bikonservatif altmanifoldlar ile ilgili örnekler verilmiştir. Daha sonra $\mathbb{E}^4_1$ uzayının bu türden bikonservatif altmanifoldlarının sınıflandırılması yapılmış ve bu durum de Sitter $\mathbb{S}^4_1$ ve anti-de Sitter $\mathbb{H}^4_1$ uzayları için de incelenmiştir. Buna İlave olarak, de Sitter uzayındaki $\mathbb{S}^4_1$, bir bikonservatif yüzeyin biharmonik olabilmesinin gerek ve yeter koşulu gösterilirken anti de Sitter $\mathbb{H}^4_1$ uzayındaki biharmonik manifoldların varlığı incelenmiştir. Var olan biharmonik yüzeyler ile ilgili örnekler verilmiştir. Yine bu bölümde yarı-minimal alt manifoldların bikonservatiflik durumu incelenmiş ve bu özelliğe sahip bir yüzey parametrizasyonu verilmiştir.

Dördüncü bölümde, 4-boyutlu $\mathbb{E}^4_1$ Minkowski uzayında, köşegenleştirilemeyen şekil operatörüne sahip bikonservatif hiperyüzeyler incelenmiştir. Hiperyüzeyler için bikonservatiflik denklemi

$$A(\nabla H) = -\frac{nH}{2} \nabla H,$$

şeklindedir. Burada H ve n, sırasıyla, altmanifoldun ortalama eğriliği ve boyutudur. Bu bölümde öncelikle bir hiperyüzeyin şekil operatörünün kanonik formları belirlenmiş ve hemen ardından ortalama eğriliği, kesitsel eğriliği ve Levi-Civita konneksiyonları bulunmuştur. Daha sonra Frobenius Teoremi'nden faydalanılarak hiperyüzey üzerinde özel bir koordinat sistemi inşa edilmiş ve bir hiperyüzeyin bikonservatif olabilmesi için gerek ve yeter koşullar verilmiştir. Ayrıca bu tür hiperyüzeylerin teklik durumu incelenmiş ve üzerine bir teorem verilmiş ve daha sonra bikonservatif hiperyüzeylerin tam bir sınıflandırılması yapılmıştır.

Beşinci bölümde, 5-boyutlu $\mathbb{E}^5_1$ Minkowski uzayında iki asli eğrilikli, köşegenleştirilemeyen şekil operatörünün belirli bir kanonik formuna sahip hiperyüzeyler üzerinde çalışılmıştır. Bu bölümde bu türden bikonservatif hiperyüzeylerin varlık durumu incelenmiştir. Verilen teoremler ile bu duruma kesin bir sonuç getirilmiştir.

Son bölümde, keyfi boyutlu $\mathbb{E}^{n+1}_1$ Minkowski uzayında ortalama eğriliğin gradyenti ışıksal olan (MCGL) bikonservatif hiperyüzeyler üzerinde çalışılmıştır. Öncelikle bu hiperyüzeye ait şekil operatörü ve ikinci temel form katsayıları belirlenip ardından konneksiyon formları bulunmuştur. Bu tür bir hiperyüzeyin varlık durumu incelenmiş ve verilen bir teorem ile bu duruma kesin bir sonuç getirilmiştir.

Son bölümde elde edilen sonuçlar paylaşılmış ve problemlerin geleceği hakkında önerilerde bulunulmuştur.

1. INTRODUCTION

The study of biharmonic submanifolds began with the independent works of B.-Y. Chen and G.-Y. Jiang in the middle of 1980's after Eells and Lemaire introduced the notion of k-harmonic map to classify maps between two Riemannian manifolds [1]. For $k = 2$, a biharmonic map $\phi : M \to N$ is defined as a critical point of the bienergy functional $E_2(\phi) = \frac{1}{2}\int_M \|\tau(\phi)\|^2 v_g$, where $\tau(\phi)$ is the tension field associated to ϕ. By computing the first variational formula, a biharmonic map ϕ is characterized by the vanishing of the associated bitension field:4

$$\tau_2(\phi) := -\Delta\tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi = 0. \tag{1.1}$$

A biharmonic isometric immersion $\phi : M \to N$ or $(\phi(M))$ is called as biharmonic submanifold. The study of biharmonic submanifolds has attracted great attentions of geometers.

In Chen's program of studing the finite type submanifolds of Euclidean space, Chen gave an alternative definition of biharmonic submanifold of a Euclidean space to be a submanifold with harmonic mean curvature vector field, [2]. Let M be a submanifold of the Euclidean space $\mathbb{E}^n$ with the mean curvature vector H and position vector x. H is said to be harmonic if the equation $\Delta H = 0$ is satisfied. Note that this condition is equivalent to the equation $\Delta^2 x = 0$ because of the well-known Laplace-Beltrami Formula $\Delta x = mH$. Caddeo, Montaldo and Oniciuc seems to be the first persons to use the term of *biharmonic submanifolds of Riemannian manifolds* in [3].

In the mean time, G.-Y. Jiang, began to study biharmonic submanifolds of Riemannian manifolds as biharmonic isometric immersions: In [4], the first and second variational formulas of E_2 are defined. In particular, it is proved that the definition of Chen is equivalent to the definition of Eells and Lemaire.

One of the fundamental problems in the study of biharmonic submanifolds is to classify such submanifolds in a model space. So far, most of the work done has been focused on classification of biharmonic submanifolds of Riemannian (also semi Riemanninan) space forms.

It is easy to prove that a minimal isometric immersion is biharmonic. However, in [2] B.-Y. Chen claimed that the converse of this holds if the ambient space is Euclidean and the following conjecture proposed:

Chen's Biharmonic Conjecture: A biharmonic submanifold of Euclidean space is minimal.

Although many geometers has obtained affirmative partial solutions on this conjecture, the problem is still open, [5].

On the other hand; R Caddeo *et al.* showed that Chen's conjecture is not true on Euclidean sphere. One of the basic example is the canonical inclusion

$$i:\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}})\times\mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})\subset\mathbb{S}^{m_1+m_2},$$

which is a non-minimal biharmonic immersion, [3].

A submanifold is said to have parallel normalized mean curvature vector field (PNMCV) if the unit vector field in the direction of the mean curvature vector field is parallel in the normal bundle. Before proceed, we would like to add that a weaker version of Chen's biharmonic conjecture has been recently proposed in [6] as follows.

Chen's weaker Biharmonic Conjecture: There do not exist biharmonic submanifolds in Euclidean spaces with a parallel normalized mean curvature vector.

On the other hand, *biconservative submanifolds* arise as the vanishing of the stress-energy tensor of the bienergy. Biconservative hypersurfaces in $\mathbb{E}^3$ and $\mathbb{E}^4$ were studied for the first time in 1995 by Hasanis and Vlachos, [7]. In that paper, biconservative hypersurfaces was called as *H-hypersurfaces*.

Caddeo *et al.* introduced the concept of *biconservative immersions* from the principle of a stress-energy tensor of the bienergy. An isometric immersion $\phi: M\rightarrow N$ is said to be *biconservative* if its associated divergence of the stress-bienergy tensor is zero [8]. In [9], this tensor was expressed as

$$\begin{aligned}S_2(X,Y)=&\frac{1}{2}\|\tau(\phi)\|^2\langle X,Y\rangle+\langle d\phi,\nabla\tau(\phi)\rangle\langle X,Y\rangle\\&-\langle d\phi(X),\nabla_Y\tau(\phi)\rangle-\langle d\phi(Y),\nabla_X\tau(\phi)\rangle\end{aligned}\tag{1.2}$$

which yields $\operatorname{div} S_2 = -\langle \tau_2(\phi), d\phi \rangle$. Note that for an isometric immersion ϕ, $\operatorname{div} S_2 = (\tau(\phi))^T$. In general, a submanifold is called *biconservative* if $\operatorname{div} S_2 = 0$.

In the last decade the theory of biconservative submanifolds proved to be a very interesting research topic. Although a biharmonic submanifold is biconservative, the converse of this does not hold in general. This theory arose from the theory of biharmonic submanifolds, but the class of biconservative submanifolds is richer than the later one.

Results in [8] and [7] showed that biconservative hypersurfaces in Riemannian 3-space forms and Euclidean 4-space must be either a surface with constant mean curvature (CMC) or rotational surfaces.

In pseudo-Riemannian setting, *Chens's biharmonic conjecture* doesn't hold for submanifolds in a $\mathbb{E}^m_s$. For example in [10] and [11] Chen and Ishikawa gave some examples of non-minimal biharmonic surfaces in the semi-Euclidean space $\mathbb{E}^4_s$ for $s = 1, 2$. In particular in [10] an isometric immersion x into $\mathbb{E}^4_1$ defined by

$$x(u,v) = (\varphi(u,v), \varphi(u,v), u, v) \tag{1.3}$$

for a smooth function ϕ satisfying $\Delta\varphi \neq 0$ and $\Delta^2\varphi = 0$. This isometric immersion is biharmonic and its mean curvature vector is given by $H = -\frac{1}{2}(\Delta\varphi, \Delta\varphi, 0, 0)$. However, biharmonicity for hypersurfaces in pseudo-Euclidean space implies minimality in some special cases. For example, it was shown in [11] that any biharmonic surface in Minkowski 3-space is minimal, and in [12] that every biharmonic Lorentzian hypersurface in $\mathbb{E}^4_1$ is minimal. So, many geometers has been attrachted to study the biconservative hypersurfaces in pseudo-Riemannian space. However, in [13] the author showed that any non-CMC biconservative surface in Minkowski 3-space is locally either a revolution of surface or a null scroll. It is interesting that null scrolls, claimed to be biconservative, appear in the classification results, which have no counterparts in Euclidean case. However, by a direct computation it can be observed that a biconservative immersion is not biconservative. Moreover, it was shown that biconservative surfaces in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ are either CMC or rotational surfaces [13].

In Riemannian space forms, when then codimension is greater than or equal to 2, parallel mean curvature vector (PMCV) surfaces in space forms are automatically biconservative, while surfaces with constant mean curvature (CMC) are not. Montaldo *et al.* in [14] gave a complete classification of CMC biconservative surfaces. They proved that a CMC biconservative surface in 4-dimensional space form of non-zero constant sectional curvature with codimension 2 is PMC. In [15], Dorel *et al.* classified nonminimal biconservative surfaces with parallel mean curvature vector field in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

In [16] and [17] Şen and Turgay showed that PNMCV biconservative surface is congruent to certain rotational surface in $\mathbb{E}^4$. Moreover, in [16], it was shown that there is no biharmonic PNMCV surface in $\mathbb{E}^4$. In [18], PNMCV biconservative m dimensional submanifolds were studied in $\mathbb{E}^{m+2}$ and its canonical forms were obtained.

In this thesis biconservative and biharmonic submanifolds in Lorentzian space forms are studied. This thesis consists of seven sections and these are planned as follows.

In section 2, basic definitions and facts are summarized.

In section 3, the classifications of biconservative CMC surface in Lorentzian space forms, $\mathbb{E}^4_1$, $\mathbb{S}^4_1$ and $\mathbb{H}^4_1$ are obtained.

In section 4, the biconservative hypersurfaces with non-diagonalizable shape operator in $\mathbb{E}^4_1$ are completely classifed.

In section 5, nonexistence of the biconservative hypersurface with *certain* non-diagonalizable shape operator in $\mathbb{E}^5_1$ are proved.

In section 6, biconservative hypersurfaces in arbitrary dimension are studied.

The last section is devoted to the conclusions and recommendations about the problems discussed in this thesis.

2. PRELIMINARIES

In this chapter some useful definitions, theorems and lemmas for thesis are given by using [19] and [20].

2.1 Lorentzian inner product space

Definition 2.1.1. A symmetric bilinear form on a finite-dimensional real vector space V is an $\mathbb{R}$-bilinear function $B : V \times V \to \mathbb{R}$ such that $B(u,v) = B(v,u)$ for all $u, v \in V$.

A symmetric bilinear form B, for $v \in V$ and $v \neq 0$, is called

• positive definite if $B(v,v) > 0$,

• positive semi-definite if $B(v,v) \geq 0$,

• negative definite if $B(v,v) < 0$,

• negative semi-definite if $B(v,v) \leq 0$,

• nondegenerate if $B(u,v) = 0$ for all $u \in V$ implies $v = 0$.

Definition 2.1.2. The index of a symmetric bilinear form B on V is the dimension of a largest subspace $W \subset V$ on which $B|_W$ is negative definite.

Definition 2.1.3. An inner product g on a finite-dimensional real vector space V is a nondegenerate symmetric bilinear form. If a vector space V equipped with g then (V,g) means an inner product space.

Definition 2.1.4. Let (V,g) be an inner product space and $0 \neq v \in V$. Then, v is said to be

- space-like if $\langle v,v \rangle > 0$,
- time-like if $\langle v,v \rangle < 0$,
- null or light-like if $\langle v,v \rangle = 0$.

Definition 2.1.5. For an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of an inner product space V, we have

$$g(e_i, e_j) = \varepsilon_i \delta_{ij}, \qquad \varepsilon_i = g(e_i, e_i) = \pm 1, \tag{2.1}$$

where δ_{ij} is the *Kronecker delta*,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.2}$$

Every vector $v \in V$ can be expressed in a unique way as

$$v = \sum_{i=1}^{n} \varepsilon_i g(v, e_i) e_i \tag{2.3}$$

for an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of an inner product space V, the number of negative sings in the singature $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ is the index of V.

Lemma 2.1.6. *Let (V, g) be an inner product space with the index k. Then the count of time-like vector in an orthagonal (or orthonormal) base of V is equal to k.*

An inner product space $(V, \langle , \rangle)$ is said to be Lorentzian if its index is 1. In this case, a pseudo-orthonormal basis can be considered:

Definition 2.1.7. If V is a vector space with a Lorentzian inner product $\langle , \rangle$, $\{e_1, e_2; e_3, e_4 \cdots, e_n\}$ is called pseudo-orthonormal if

$$\langle e_1, e_1 \rangle = 0 = \langle e_2, e_2 \rangle = \langle e_1, e_i \rangle = \langle e_2, e_i \rangle, \langle e_1, e_2 \rangle = -1, \langle e_i, e_j \rangle = \delta_{ij} \tag{2.4}$$

are satisfied for $3 \leq i, j \leq n$.

2.2 Semi-Riemannian Manifold

Definition 2.2.1. A differentiable manifold of dimension n is a set M and family $\mathscr{B} = \{(U_\alpha, x_\alpha) : \alpha \in I\}$ of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that:

(1) $\bigcup_\alpha x_\alpha(U_\alpha) = M$.

(2) For any pair $\alpha, \beta \in I$ with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mapping $x_\beta^{-1} \circ x_\alpha$ is differentiable,

(3) The family $\mathscr{B}$ is maximal relative to the conditions (1) and (2).

Definition 2.2.2. Let M be a manifold and $p \in M$. A *tangent vector* at p is a real-valued function, i.e.;

(1)$X(af+bg)=aX(f)+bX(g)$,
(2)$X(fg)=X(f)g(p)+f(p)X(g)$,
where $a,b\in\mathbb{R}$ and f,g are differentiable functions defined on M.

Let T_pM be the set of all tangent vectors to M at p then the usual definitions of functional addition and scalar multiplications make T_pM a vector space over $\mathbb{R}$. So, T_pM is called the *tangent space* to M at p.

Definition 2.2.3. A metric tensor g on a differentiable manifold M is a symmetric non-degenerate $(0,2)$ tensor field on M of constant index, i.e., g assingns to each point $p\in M$ a scalar product g_p on the tangent space T_pM,

$$\begin{array}{rcl} g_p : T_pM\times T_pM & \to & \mathbb{R} \\ (X,Y) & \to & g(X,Y)_p=\langle X,Y\rangle_p \end{array}$$

and the index of g_p is the same for all $p\in M$. The pair (M,g) is called a *semi-Riemannian manifold* or a *pseudo-Riemannian manifold*. The index s $(0\le s\le \dim M)$ of the metric tensor g is called the index of semi-Riemannian manifold. If $s=0$ then M is called a *Riemanian manifold*, and each g_p is a positive definite inner product on T_pM. If $s=1$ then M is called a *Lorentz manifold* and corresponding metric is said to be *Lorentzian*.

Definition 2.2.4. Let (M,g) be a semi-Riemannian manifold. A vector field X on M is a map $X:M\to TM$ such that

$$X_p := X(p)\in T_pM$$

X is said to be space-like, time-like or null if for all $p\in M$ X_p is space-like, timelike or null.

Definition 2.2.5. Let M be a manifold with the local coordinates $x_1,x_2,\cdots,x_n$ and $\partial_{x_1},\partial_{x_2},\cdots,\partial_{x_n}$ be the usual coordinate vector fields on a coordinate neighborhood on M. Then the *tangent space* and p of M is

$$T_pM=\operatorname{span}\{\partial_{x_1},\partial_{x_2},\cdots,\partial_{x_n}\}.$$

Theorem 2.2.6. *If $x_1,x_2,\cdots,x_n$ is an coordinate system in M at p, then its coordinate vector $\partial_{x_1},\partial_{x_2},\cdots,\partial_{x_n}$ form a basis for the tangent space T_pM and for all $X\in T_pM$*

$$X=\sum_{i=1}^{n}X(x^i)\partial_{x_i}. \tag{2.5}$$

Definition 2.2.7. Let X and Y be differentiable vector fields on a differentiable manifold M. Then Lie bracket of X and Y is defined as

$$[X,Y] = XY - YX. \tag{2.6}$$

Let f be a dif and only ifrentiable function defined on M. Then

$$[X,Y](f) = X(Y(f)) - Y(X(f)). \tag{2.7}$$

Definition 2.2.8. A k-dimensional distribution D on M is a map which assigns to every point $p \in M$ a vector subspace D_p of T_pM.

Definition 2.2.9. A k-dimensional distribution D on M is said to be involutive if

$$X,Y \in D \Rightarrow [X,Y] \in D.$$

Corollary 2.2.10. *A 1-dimensional distribution is involutive.*

The following theorem is called Local Frobenius Theorem.

Theorem 2.2.11. *Let D be an involutive k-dimensional distribution on M. Then for every $p \in M$, there exists a coordinate patch $U, x_1, x_2, \cdots, x_n$ such that*

$$D = span(\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_k}). \tag{2.8}$$

We are going to give the following Lemma which can be prove by Local Frobenius Theorem.

Lemma 2.2.12. *If T' and T'' are two involutive distributions on manifold M which are complementary at every point of M, then for a point y of M, there exist a local coordinate system $x_1, x_2, \cdots, x_n$ starting from y such that $(\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_k})$ and $(\partial/\partial x_{k+1}, \cdots, \partial/\partial x_n)$ form local basis for T' and T'' respectively. In other words, for any set of constants $(c_1, \cdots, c_k, c_{k+1}, \cdots, c_n)$, the equations $x_i = c_i, 1 \leq i \leq k$ (resp., $x_j = c_j, k+1 \leq j \leq n$)) define an integral manifold of T'' (resp., T').*

We also need the following direct result of this Lemma.

Corollary 2.2.13. *If X,Y are two linearly independent vector fields on a 2-dimensional manifold and $[X,Y] = 0$, then there exists a coordinate system (s,t) such that $X = \partial_s$ and $Y = \partial_t$.*

Proposition 2.2.14. *If X, Y and Z are differentiable vector fields on manifold M and a, b are real numbers and f, g are differentiable functions, then*

(1) $[X,Y] = -[Y,X]$,
(2) $[aX+bY,Z] = a[X,Z]+b[Y,Z]$,
(3) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = 0$,
(4) $[fX,gY] = fg[X,Y]+fX(g)Y-gY(f)X$.
Moreover, if $x_1, x_2, \cdots, x_n$ is a local coordinate system, then $[\partial_{x_i}, \partial_{x_j}] = 0$ for all $1 \leq i, j \leq n$

Definition 2.2.15. Let $\chi(M)$ be the set of all vector fields on M. An affine connection ∇ on a differentiable manifold M is a mapping

$$\nabla : \chi(M) \times \chi(M) \to \chi(M), \qquad \nabla_X Y = \nabla(X,Y),$$

and which satisfies the following properties:

(1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
(2) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$,
(3) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,
for all $X, Y, Z \in \chi(M)$ and $f, g \in C^\infty(M)$.

Definition 2.2.16. Let M be an manifold and ∇ be a connection on M. Let U be an open set in M and $e_1, e_2, \cdots, e_n$ is a local frame field on U.

$$\nabla_X e_i = \sum_{j=1}^{n} \omega_{ij}(X) e_j \tag{2.9}$$

is called the *connection 1-forms* ω_{ij} on U.

Definition 2.2.17. There exists a uniqe connection ∇ on a pseudo-Riemannian manifold (M, g), satisfying

$$\nabla_X Y - \nabla_Y X = [X,Y] \text{ (torsion free)}, \tag{2.10}$$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \text{ (metric compatible)}. \tag{2.11}$$

This connection, called the Levi-Civita connection, is characterized by Kozsul formula, that is,

$$\begin{aligned} 2g(\nabla_X Y, Z) \;&=\; Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g([X,Y],Z)-g([Y,Z],X) \\ &\quad -g([Z,X],Y). \end{aligned} \tag{2.12}$$

Definition 2.2.18. Let M be a pseudo-Riemannian manifold and ∇ be a connection on M. Let U be an open neighbourhood in M and $x_1, x_2, \cdots, x_n$ are the local coordinates on M. Then

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \Gamma_{ij}^k \partial_{x_k}, \tag{2.13}$$

the functions Γ_{ij}^k are called Christoffel symbols of the connection ∇.

Definition 2.2.19. For a pseudo-Riemannian manifold (M, g) with Levi-Civita connection ∇, the function $R : \chi(M) \times \chi(M) \times \chi(M) \to \chi(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is a $(1,3)$ tensor field which is called the *Riemannian curvature tensor*. Moreover, we have $(0,4)$ tensor defined by $R(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle$.

Definition 2.2.20. At a point $p \in M$, a 2-dimensional linear subspace π of the tangent space T_pM is called a plane section. For a given basis $\{u, v\}$ of the plane section π, we define a real number by

$$Q(u,v) = \langle u,u \rangle \langle v,v \rangle - \langle u,v \rangle^2. \tag{2.14}$$

The plane section π is called nondegenerate if and only if $Q(u,v) \neq 0$.

Definition 2.2.21. For a nondegenerate plane section π at p, the number

$$K_p(u,v) = \frac{\langle R(u,v)v, u \rangle}{Q(u,v)} \tag{2.15}$$

is independent of the choice of the basis $\{u, v\}$ for π, which is called the *sectional curvature* $K(\pi)$ of π.

A pseudo-Riemannian manifold M is said to have constant curvature if its sectional curvature is constant. In this case its curvature tensor R satisfies

$$R(X,Y,Y,X) = cQ(X,Y), \tag{2.16}$$

where c is the sectional curvature of M.

Definition 2.2.22. The *Ricci tensor* of pseudo-Riemannian n manifold M, denoted by Ric, is a symmetric $(0,2)$ tensor defined by

$$\mathrm{Ric}\,(X,Y) = \mathrm{trace}\,\{Z \to R(Z,X)Y\}, \tag{2.17}$$

or equivalently

$$\operatorname{Ric}(X,Y) = \sum_{i=1}^{n} \varepsilon_i \langle R(e_i,X)Y, e_i \rangle, \tag{2.18}$$

where $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal frame. $\operatorname{Ric}(X,Y)$ is independent of the choice of orthonormal frame. *Ricci curvature* $\operatorname{Ric}(u)$ is defined by $\operatorname{Ric}(u) = \operatorname{Ric}(u,u)$.

Definition 2.2.23. The *scalar curvature* S of M is defined by

$$S = \sum_{i<j} K(e_i, e_j), \tag{2.19}$$

where $e_1, e_2, \cdots, e_n$ is an orthonormal frame of M. The scalar curvature S is indepenent of the choice of the orthonormal frame.

2.3 Submanifolds of Lorentzian Space Forms

Let $\mathbb{E}^n_s$ denote the semi-Euclidean n-space with index s whose metric tensor is given by

$$\tilde{g} = \langle \cdot, \cdot \rangle = -\sum_{i=1}^{s} dx_i \otimes dx_i + \sum_{j=s+1}^{n} dx_j \otimes dx_j, \tag{2.20}$$

where $x_1, x_2, \ldots, x_n$ is a Cartesian coordinate system in $\mathbb{R}^n$.

Let c be a nonzero real number and $x_0 \in \mathbb{E}^n_s$. We put

$$\mathbb{S}^{n-1}_s(x_0, c) = \{x \in \mathbb{E}^n_s : \langle x - x_0, x - x_0 \rangle = \frac{1}{c}\}, \tag{2.21}$$

$$\mathbb{H}^{n-1}_s(x_0, -c) = \{x \in \mathbb{E}^n_s : \langle x - x_0, x - x_0 \rangle = -\frac{1}{c}\}, \tag{2.22}$$

where $\langle , \rangle$ is the associated scalar product. (2.21) and (2.22) are known as a *pseudo sphere* and *pseudo-hyperbolic sphere*, respectively. The point x_0 is called the center of $\mathbb{S}^{n-1}_s(x_0, c)$ and $\mathbb{H}^{n-1}_s(x_0, -c)$. If x_0 is the origin of the pseudo-Euclidean spaces then, we denote $\mathbb{S}^{n-1}_s(0, c)$ and $\mathbb{H}^{n-1}_s(x, -c)$ by $\mathbb{S}^{n-1}_s(c)$ and $\mathbb{H}^{n-1}_s(-c)$, respectively. Throughout this thesis, we denote the m-dimensional *Lorentzian space form* with constant sectional curvature $\delta \in \{-1, 0, 1\}$ by $\mathbb{L}^m(\delta)$ by taking δ instead of c. In fact, we have

$$\mathbb{L}^m(\delta) = \begin{cases} \mathbb{S}^m_1 & \text{if } \delta = 1, \\ \mathbb{E}^m_1 & \text{if } \delta = 0, \\ \mathbb{H}^m_1 & \text{if } \delta = -1, \end{cases} \tag{2.23}$$

where $\mathbb{E}^m_1$, $\mathbb{S}^m_1$ and $\mathbb{H}^m_1$ stand for the m-dimensional *Minkowski*, *de Sitter* and *anti-de Sitter spaces*, respectively.

Definition 2.3.1. Let M,N be two differentiable manifolds of dimensions m,n, respectively. A differentiable mapping $x : M \to N$ is said to be an immersion if $dx_p : T_pM \to T_{x(p)}N$ is injective for all $p \in M$.

Definition 2.3.2. A manifold M is a *submanifold* of a manifold N provided that

1) M is a topological subspace of M.
2) The inclusion map $i : M \to N$ is smooth and its differential is one-to-one.

The dif and only iference $\dim N - \dim M$ is called the codimension of M in N. If N is a semi-Riemannian manifold and nondegenerate metric of submanifold M has a constant index at each point $p \in M$, then M is called a semi-Riemannian submanifold of semi-Riemannian N.

Let M be an m-dimensional semi-Riemannian submanifold of $\mathbb{L}^n(\delta)$. We denote by ∇ and $\tilde{\nabla}$ Levi-Civita connections of M and $\mathbb{L}^n(\delta)$, respectively. Then, Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.24}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi, \tag{2.25}$$

respectively, for any vector fields X,Y tangent to M and ξ normal to M, where h is the second fundamental form, A is the shape operator and $\nabla^\perp$ is the normal connection. Denote the curvature tensor of M and $\mathbb{L}^n(\delta)$ with R and $\tilde{R}$, respectively, and let $R^\perp$ stand for the normal curvature tensor of M (in $\mathbb{L}^n(\delta)$). Then, the integrability conditions, called Gauss, Ricci and Codazzi equations,

$$R(X,Y)Z = \delta(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.26}$$

$$R^\perp(X,Y)\xi = h(X,A_\xi Y) - h(A_\xi X,Y), \tag{2.27}$$

$$(\bar{\nabla}_Y h)(X,Z) = (\bar{\nabla}_X h)(Y,Z) \tag{2.28}$$

are satisfied, where the covariant derivative $\bar{\nabla}h$ of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{2.29}$$

Moreover, the second fundamental form and the shape operators are related by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X,Y\rangle. \tag{2.30}$$

On the other hand, the mean curvature vector H of M is defined by

$$H = \frac{1}{m} \text{trace}\, h \tag{2.31}$$

and its norm $\|H\| = |\langle H, H\rangle|^{1/2}$ is called the mean curvature of M.

Definition 2.3.3. A pseudo-Riemannian submanifold M is called CMC if its mean curvature is constant i.e., $\|H\|$ is constant.

Definition 2.3.4. A pseudo-Riemannian submanifold M is called *minimal* if the mean curvature vector H vanishes identically, i.e., $H = 0$. If $\|H\| = 0$ and $H \neq 0$ at each point of M, then M is said to be quasi-minimal.

Definition 2.3.5. The metric connection $\nabla^{\perp}$ defined by (2.24) is called the normal connection. A normal vector field ξ on M is said to be *parallel* if $\nabla^{\perp}\xi = 0$ holds identically. In particular, M is said to have *parallel mean curvature vector* if $\nabla^{\perp}H = 0$ holds identically.

Definition 2.3.6. Let M be a pseudo-Riemannian n-manifold and $f \in C^{\infty}(M)$. The *gradient* of f, denoted by ∇f, or by $\text{grad}\, f$, is a vector field dual to the differential df.

$$\langle \nabla f, X\rangle = df(X) = X(f), \qquad X \in \chi(M). \tag{2.32}$$

In terms of a coordinate system $\{x_1, x_2, \ldots, x_n\}$ of M, we have

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \tag{2.33}$$

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j} \partial_{x_j}. \tag{2.34}$$

Definition 2.3.7. If $X \in \chi(M)$ and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame, the *divergence* of X, denoted by $\text{div} X$, is defined by

$$\text{div} X = \sum_{i=1}^{n} \varepsilon_j \langle \nabla_{e_i} X, e_i\rangle \tag{2.35}$$

which is independent of the chosen frame.

Definition 2.3.8. The *Laplacian* of $f \in C^{\infty}(M)$, denoted by Δf, is defined by $\Delta f = -div(\nabla f)$. In terms of an orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$, we have

$$\Delta f = \sum_{i=1}^{n} \left(-\varepsilon_i e_i(e_i f) + (\nabla_{e_i} e_i) f\right), \tag{2.36}$$

where $\varepsilon_i = \langle e_i, e_i \rangle$. In terms of coordinate system $\{x_1, x_2, \ldots, x_n\}$, we have

$$\Delta f = \sum_{i,j=1}^{n} \left(-g^{ij} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^{k} \frac{\partial f}{\partial x_k} \right) \tag{2.37}$$

where g^{ij} is an inverse of metric g defined on M.

Definition 2.3.9. An immersion $x : M \to N$ between two manifolds M and N of dimension of m and n, respectively, is said to be *hypersurface* if codimension of the immersion is 1. If N is Lorentzian then M is said to be *Lorentzian hypersurface*.

Theorem 2.3.10. *[21] Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ and $\rho : (M, g) \to (\tilde{M}, \tilde{g})$ be connected hypersurfaces, where $(\tilde{M}, \tilde{g})$ is a Riemannian space form and let $\Phi : TM_f^{\perp} \to TM_{\rho}^{\perp}$ be a vector bundle isomorphism such that*

$$h_{\rho}(X, Y) = \Phi h_f(X, Y) \text{ or } h_{\rho}(X, Y) = -\Phi h_f(X, Y), \tag{2.38}$$

where h_{ρ}, h_f denote, respectively, the second fundamental forms of f and ρ. Then there exists an isometry $\tau : (\tilde{M}, \tilde{g}) \to (\tilde{M}, \tilde{g})$ such that $\rho = \tau \circ f$.

Definition 2.3.11. The trace of a bilinear mapping $\psi : V \times V \to \tilde{V}$ is

$$\text{trace} = \sum_{i=1}^{n} \varepsilon_i \psi(e_i, e_i), \tag{2.39}$$

where $\varepsilon_i = \langle e_i, e_i \rangle$ and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of V.

In terms of a pseudo-orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of V, we have

$$\text{trace}\, \psi = -\psi(e_1, e_2) - \psi(e_2, e_1) + \sum_{i=3}^{n} \varepsilon_i \psi(e_i, e_i.) \tag{2.40}$$

On the other hand, a linear endhomorphism $A : V \to V$ is said to be symmetric (or self-adjoint) if

$$\langle AX, Y \rangle = \langle X, AY \rangle \tag{2.41}$$

whenever $X, Y \in V$. Now, we will give the well-known following lemma.

Lemma 2.3.12. *[22] A symmetric endomorphism A of an inner product space $(V, \langle , \rangle)$ with a Lorentzian inner product can be put into one of four forms:*

1) When A is represented with respect to an orthonormal basis;

$$\textit{Case I. } A \sim \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}, \tag{2.42}$$

$$\textit{Case II. } A \sim \begin{bmatrix} a_0 & b_0 & & & \\ -b_0 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{bmatrix}. \tag{2.43}$$

2) When A is represented with respect to an pseudo-orthonormal basis;

$$\textit{Case III. } A \sim \begin{bmatrix} a_0 & 1 & & 0 & \\ 0 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ 0 & & & & a_{n-2} \end{bmatrix}, \tag{2.44}$$

$$\textit{Case IV. } A \sim \begin{bmatrix} a_0 & 0 & 0 & & & \\ 0 & a_0 & 1 & & & \\ -1 & 0 & a_0 & & & \\ & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_{n-3} \end{bmatrix}. \tag{2.45}$$

Here b_0 is assumed to be nonzero. In cases (2.42)*,* (2.43) *and* (2.44) *the eigenvalues are real, while $a_0 \pm ib_0$ are eigenvalues in case* (2.45)*.*

Remark 2.3.13. Assume that a symmetric endomorphism A has the matrix representation

$$A = \begin{bmatrix} a_0 & b & & & \\ & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{bmatrix} \tag{2.46}$$

with respect to a pseudo-orthonormal base $\{\tilde{e}_1, \tilde{e}_2; e_3, \ldots, e_n\}$. Then, by defining e_1 and e_2 by $e_1 = c\tilde{e}_1$ and $e_2 = \frac{1}{c}\tilde{e}_2$, one can get (2.44), where c is a suitable non-zero constant.

2.4 Biconservative Submanifolds

In this subsection, we give a summary of well-known facts about energy functionals and relations between biconservative submanifolds and semi-Riemannian space-forms.

Definition 2.4.1. Let $\phi : (M,g) \to (N,h)$ be a smooth map. Then the energy of ϕ is defined by

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 v_g, \tag{2.47}$$

where v_g is the volume element of g.

In terms of local coordinates $\{x^i\}$ on M and $\{u^\alpha\}$ on N then $d\phi = \phi_i^\alpha dx_i \partial u_\alpha$, where $\phi_i^\alpha = \frac{\partial}{\partial x_i}\phi^\alpha$ and

$$\|d\phi\|^2 = g^{ij}\phi_i^\alpha \phi_j^\beta h_{\alpha\beta}. \tag{2.48}$$

We shall derive the equation (2.48). First, for the unit tangent vector field X, we have

$$\begin{aligned} \|d\phi\|^2 &= h(d\phi(X), d\phi(X)), \qquad X = \lambda^i \partial_{x_i} \\ &= h\left(\phi_i^\alpha dx_i \partial u_\alpha(\lambda^i \partial_{x_i}), \phi_j^\beta dx_j \partial_{u_\beta}(\lambda^j \partial_{x_j})\right) \\ &= \phi_i^\alpha \phi_j^\beta \lambda^i \lambda^j h\left(\partial_{u_\alpha}, \partial_{u_\beta}\right). \end{aligned} \tag{2.49}$$

Note that $g(X,X) = 1$ gives $\lambda^i\lambda^j = g^{ij}$ since $g(X,X) = \lambda^i\lambda^j g(\partial_{x_i}, \partial_{x_j}) = 1$. So the above equation implies (2.48).

Definition 2.4.2. Let $\phi : M \to N$ be a smooth map, ϕ_t be a variation of ϕ such that $\phi_0 = \phi$ and $\frac{\partial \phi_t}{\partial_t}|_{t=0} = V$ and let V be a vector field on N. Then, $\tau(\phi) = tr_g \nabla d\phi$ is called tension field of ϕ, where $\nabla_V E(\phi_t) = \frac{\partial E(\phi_t)}{\partial_t}|_{t=0} = -\int \langle V, \tau\phi \rangle$.

In terms of local coordinates, we have

$$\tau(\phi) = (-\Delta\phi^\alpha + \Gamma_{\beta\gamma}^\alpha \phi_i^\beta \phi_i^\gamma g^{ij})\partial_{u_\alpha}. \tag{2.50}$$

Definition 2.4.3. A smooth map $\phi : (M,g) \to (N,h)$ is harmonic if it is a critical point of energy functional. This condition equivalent to $\tau(\phi) = 0$.

We want to notice that if $\phi = x$ is an isometric immersion, then $\tau(x) = mH$. So x is minimal if and only if it is harmonic.

Definition 2.4.4. A smooth map $\phi : (M,g) \to (N,h)$ is biharmonic if it is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M \|\tau(\phi)\|^2 v_g. \tag{2.51}$$

Moreover, the Euler–Lagrange equation associated with $E_2(\phi)$ is given by the vanishing of the bitension field as

$$\tau_2(\phi) = -\Delta^\phi \tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi = 0. \tag{2.52}$$

Definition 2.4.5. A map ϕ satisfying the condition

$$\langle \tau_2(\phi), d\phi \rangle = 0 \tag{2.53}$$

is said to be biconservative. Note that an isometric immersion $\phi = x$ is biconservative if and only if the tangential part of $\tau_2(x)$ vanishes identically, that is,

$$(\tau_2(x))^T = 0. \tag{2.54}$$

Let $x : (\Omega, g) \to (N, \tilde{g})$ be an isometric immersion between semi-Riemannian manifolds and let us put $M = x(\Omega)$. By splitting $\tau_2(x)$ into its tangential and normal components and considering (2.52), one can obtain the following proposition. (See, for example, [14]).

Proposition 2.4.6. *[14] x is biharmonic if and only if the equations*

$$m\operatorname{grad} \|H\|^2 + 4\operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + 4\operatorname{trace}(\tilde{R}(\cdot,H)\cdot)^T = 0 \tag{2.55}$$

and

$$-\Delta^\perp H + \operatorname{trace} h(A_H(\cdot),\cdot) + \operatorname{trace}(\tilde{R}(\cdot,H)\cdot)^\perp = 0 \tag{2.56}$$

are satisfied, where m is the dimension of M and $\Delta^\perp$ is the Laplacian associated with $\nabla^\perp$.

Note that (2.54) implies

Proposition 2.4.7. *[14] x is biconservative if and only if the equation* (2.55) *is satisfied.*

If N is a semi-Riemannian space form, we have

$$\operatorname{trace}(\tilde{R}(.,H).)^T = 0. \tag{2.57}$$

Therefore, we have the following result:

Proposition 2.4.8. *Let M be a hypersurface of the Minkowski space $\mathbb{E}_s^{n+1}$, $s = 0,1$ with the shape operator A and mean curvature H. M is said to be biconservative if the following equation*

$$A(\operatorname{grad} H) + \varepsilon \frac{nH}{2} \operatorname{grad} H = 0 \tag{2.58}$$

holds where $\varepsilon = \langle N, N \rangle$, i.e.,

$$\varepsilon = \begin{cases} -1, & \text{if } M \text{ is Riemannian} \\ 1, & \text{if } M \text{ is Lorentzian} \end{cases} \tag{2.59}$$

is satisfied.

Now, we consider the case $(N, \tilde{g}) = \mathbb{L}^n(\delta)$ and assume that M is a CMC surface. In this case, we have $\|H\| = \text{const.}$ and

$$\tilde{R}(X,H)Y = -\delta \langle X, Y \rangle H \tag{2.60}$$

whenever X, Y are tangent to M. Therefore, one can conclude that M is biconservative if and only if

$$\operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\cdot) = 0. \tag{2.61}$$

Moreover, the equation (2.56) turns into

$$-\Delta^\perp H + \operatorname{trace} h(A_H(\cdot), \cdot) - \delta m H = 0. \tag{2.62}$$

Remark 2.4.9. If M is a submanifold of $\mathbb{L}^n(\delta)$ with parallel mean curvature vector, then the equation (2.55) is trivially satisfied. Therefore, we are going to call a biconservative submanifold as '**proper**' if it has no open part with parallel mean curvature vector. We would like to note that surfaces in $\mathbb{L}^n(\delta)$ with parallel mean curvature vector are classified in [23] (See also [24]).

3. CMC SURFACE IN LORENTZIAN SPACE FORMS

In this section, we study biconservative CMC surfaces in $\mathbb{E}^4_1$. Note that the article [25] contains all results appearing in this section.

3.1 The Form of Shape Operators

When M is a Lorentzian surface there exists a semi-geodesic local frame field.

Proposition 3.1.1. *[26] Let M be a Lorentzian surface with the metric tensor g. Then, there exists a local coordinate system* (s,t) *such that*

$$g = -(dt \otimes ds + ds \otimes dt) + 2f ds \otimes ds. \tag{3.1}$$

Furthermore, the Levi-Civita connection of M satisfies

$$\nabla_{\partial_t}\partial_t = 0, \tag{3.2}$$

$$\nabla_{\partial_t}\partial_s = \nabla_{\partial_s}\partial_t = -f_t\partial_t, \tag{3.3}$$

$$\nabla_{\partial_s}\partial_s = f_t\partial_s + (2ff_t - f_s)\partial_t. \tag{3.4}$$

Remark 3.1.2. If M is a Lorentzian surface, then at each point p, there exist two linearly independent vectors v and w. Furthermore, any null vector is proportional to either v or w. The coordinate system (s,t) in the proceeding Lemma can be chosen so that ∂_t is proportional to either one of them.

Lemma 3.1.3. *Let M be a Lorentzian surface,* $p \in M$ *and A be a symmetric endomorphism of* T_pM*. Then, by choosing an appropriated base for* T_pM*, by Lemma 2.3.12, A can put into one of the following three canonical forms:*

Case (i). $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ *with respect to an orthonormal base* $\{e_1, e_2\}$,

Case (ii). $A = \begin{bmatrix} a_1 & 1 \\ 0 & a_1 \end{bmatrix}$ *with respect to pseudo-orthonormal base* $\{e_1, e_2\}$,

Case (iii). $A = \begin{bmatrix} a_1 & b \\ -b & a_1 \end{bmatrix}$ *with respect to orthonormal base* $\{e_1, e_2\}$,

where b is a non-zero constant.

On the other hand, if M is a surface in $\mathbb{L}^4(\delta)$, $\delta = \pm 1$, we are going to put $\hat{\nabla}$ for the Levi-Civita connection of $\mathbb{E}^5_\beta$, where $\beta = \frac{3-\delta}{2}$. Consider an isometric immersion $x:(\Omega, g) \hookrightarrow \mathbb{L}^4(\delta)$ with $x(\Omega) \subset M$. Let $i:\mathbb{L}^4(\delta) \subset \mathbb{R}^5$ be the inclusion and put $\hat{x} = i \circ x$. Then, we have

$$\hat{h}(X,Y) = i_*(h(X,Y)) - \delta g(X,Y)\hat{x}, \tag{3.5}$$

where $\hat{h}$ denotes the second fundamental form of M in $\mathbb{E}^5_\beta$.

Lemma 3.1.4. *Let M be a proper biconservative surface in $\mathbb{L}^4(\delta)$ with non-zero CMC and consider the orthonormal frame field $\{N_1, N_2\}$ of its normal bundle such that*

$$H = cN_1. \tag{3.6}$$

Then, we have two cases:

Case 1: The shape operator A_{N_2} has the matrix representation

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.7}$$

with respect to an approriately chosen pseudo-orthonormal frame field $\{e_1, e_2\}$ of the tangent bundle of M.

Case 2: A_{N_2} satisfies $A_{N_2}X = 0$ whenever X is tangent to M.

Proof. Let M be a proper biconservative surface with CMC. Define the 1-form ω_{34} by

$$\omega_{34}(X) = \langle \tilde{\nabla}_X N_1, N_2 \rangle. \tag{3.8}$$

Then, (2.61) takes the form

$$\operatorname{trace}\left(\langle \nabla^{\perp}_{\cdot} N_1, N_2 \rangle A_{N_2}(\cdot)\right) = 0. \tag{3.9}$$

Since A_{N_2} is symmetric, it can be put into one of three forms given in case (i), (ii) and (iii) of Lemma 3.1.3. We are going to consider these cases seperately. Note that in each of these cases we have

$$\operatorname{trace} A_{N_2} = 0 \tag{3.10}$$

since $H = cN_1$ and the equation (2.31).

Case (i). There is an orthonormal frame field $\{e_1, e_2\}$ such that

$$A_{N_2} = \begin{bmatrix} k_1 & 0 \\ 0 & -k_1 \end{bmatrix} \tag{3.11}$$

for a smooth function k_1. Thus, (3.9) turns into

$$\varepsilon\omega_{34}(e_1)k_1e_1 - \omega_{34}(e_2)k_1e_2 = 0, \tag{3.12}$$

where we put $\varepsilon = \langle e_1, e_1\rangle = \pm 1$. Therefore, since M is proper biconservative, we have $\omega_{34} \neq 0$. So, the equation above yields that the open subset $\mathscr{O} = \{p \in M | k_1(p) \neq 0\}$ must be empty. Consequently, we have *Case 2* of the Lemma.

Case (ii). There is a pseudo-orthonormal frame field such that

$$A_{N_2} = \begin{bmatrix} k_1 & 1 \\ 0 & k_1 \end{bmatrix} \tag{3.13}$$

for a smooth function k_1. By considering trace$A_{N_2} = 0$, we obtain *Case 1* of the lemma.

Case (iii). There is an orthonormal frame field $\{e_1, e_2\}$ so that

$$A_{N_2} = \begin{bmatrix} 0 & -\gamma \\ \gamma & 0 \end{bmatrix} \tag{3.14}$$

for a smooth non-vanishing function γ because $trA_{N_2} = 0$. In this case, (3.9) becomes

$$\varepsilon\omega_{34}(e_1)\gamma e_1 + \omega_{34}(e_2)\gamma e_2 = 0 \tag{3.15}$$

which gives $\omega_{34}(e_1) = \omega_{34}(e_2) = 0$ because $\gamma \neq 0$. However, this is not possible unless H is parallel. □

3.2 Examples of Biconservative Surfaces in $\mathbb{E}^4_1$

First, we obtain the following family of biconservative surfaces in $\mathbb{E}^4_1$ which has no counter part in the Euclidean 4-space.

Proposition 3.2.1. *Let $\alpha, \beta : I \to \mathbb{E}^4_1$ be smooth functions satisfying*

$$\langle \beta, \beta\rangle = 0, \quad \langle \beta', \beta'\rangle = c^2, \quad \langle \alpha', \beta\rangle = -1. \tag{3.16}$$

Then, the ruled surface given by

$$x(s,t) = \alpha(s) + t\beta(s) \tag{3.17}$$

has CMC and it is proper biconservative.

Proof. Let M be a surface given by (3.17) and assume that α, β satisfy (3.16). We define functions a_1, a_2, a_3, a_4 by

$$a_1 = \langle \alpha', \alpha'\rangle, \qquad a_2 = \langle \alpha', \beta'\rangle, \qquad a_3 = \langle \alpha', \beta''\rangle, \qquad a_4 = \langle \beta'', \beta''\rangle. \tag{3.18}$$

Then, $e_1 = \partial_t$ and $e_2 = \partial_s + f\partial_t$ form a pseudo-orthonormal frame field for the tangent bundle of M because (3.17) implies

$$\begin{aligned}
\langle e_1, e_1 \rangle &= \langle \partial_t, \partial_t \rangle = 0, && (3.19)\\
\langle \partial_s, \partial_s \rangle &= \langle \alpha' + t\beta', \alpha' + t\beta' \rangle \\
&= a_1 + 2ta_2 + t^2c^2, && (3.20)\\
\langle e_2, e_2 \rangle &= \langle \partial_s, \partial_s \rangle + 2f\langle \partial_s, \partial_t \rangle = 0, && (3.21)\\
\langle e_1, e_2 \rangle &= \langle \partial_s, \partial_t \rangle = \langle \alpha', \beta \rangle = -1, && (3.22)
\end{aligned}$$

where "$'$" denotes the ordinary derivative with respect to s and we put

$$f(s,t) = \frac{1}{2}\left(c^2t^2 + 2ta_2(s) + a_1(s)\right). \tag{3.23}$$

Also, we consider the orthonormal frame field $\{N_1, N_2\}$ of the normal bundle of M then one can choose N_1 as

$$N_1 = -\frac{1}{c}(\beta' + f_t\beta). \tag{3.24}$$

Since $\langle \beta', \beta \rangle = 0$,we have $\langle N_1, e_1 \rangle = 0$. Moreover, by a direct computation, we get

$$\begin{aligned}
\langle N_1, e_2 \rangle &= -\frac{1}{c}\langle \beta' + f_t\beta, \alpha' + t\beta' + f\beta \rangle, && (3.25)\\
f_t &= c^2t + a_2(s). && (3.26)
\end{aligned}$$

Replacing (3.26) into (3.25) and using (3.16) gives $\langle N_1, e_2 \rangle = 0$. So, N_1 is a unit normal vector.

We want to notice that

$$\tilde{\nabla}_{e_1} e_2 = \phi_1 e_2 + h(e_1, e_2) \tag{3.27}$$

and

$$\tilde{\nabla}_{e_1} e_1 = x_{tt} = 0. \tag{3.28}$$

So

$$\langle \tilde{\nabla}_{e_1} e_1, e_2 \rangle = -\langle e_1, \tilde{\nabla}_{e_1} e_2 \rangle = \phi_1 = 0. \tag{3.29}$$

It follows $\tilde{\nabla}_{e_1} e_2 = h(e_1, e_2)$. So, with a direct computation

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_2 &= \tilde{\nabla}_{\partial_t}(\alpha' + t\beta' + f\beta) && (3.30)\\
&= \beta' + f_t\beta && (3.31)\\
&= -cN_1, && (3.32)
\end{aligned}$$

and it follows that

$$H = -h(e_1, e_2) = cN_1. \tag{3.33}$$

Moreover, (3.28) also gives

$$h(e_1, e_1) = 0. \tag{3.34}$$

We want to notice that

$$A_{N_2}e_1 = -\langle A_{N_2}e_1, e_2\rangle e_1 - \langle A_{N_2}e_1, e_1\rangle e_2. \tag{3.35}$$

By (2.30), (3.35) becomes

$$A_{N_2}e_1 = -\langle h(e_1, e_2), N_2\rangle e_1 - \langle h(e_1, e_1), N_2\rangle e_2. \tag{3.36}$$

Therefore, (3.34) and (3.33) implies

$$A_{N_2}(e_1) = 0. \tag{3.37}$$

By a direct computation we get

$$\tilde{\nabla}_{e_1}N_1 = c\partial_t. \tag{3.38}$$

Note that the equation (3.33) yields that M has CMC. On the other hand, (3.37) and (3.38) imply

$$\operatorname{trace} A_{\nabla^{\perp}_{\cdot}H}(\cdot) = -A_{\nabla^{\perp}_{e_1}H}(e_2) - A_{\nabla^{\perp}_{e_2}H}(e_1) = 0 \tag{3.39}$$

which yields that M is also biconservative because (2.61) is satisfied. Now, we want to show that H is not parallel. To do this we must calculate $\nabla^{\perp}_{\partial_s}H$ due to $\nabla^{\perp}H = \nabla^{\perp}_{e_2}H = \nabla^{\perp}_{\partial_s}H$ because (3.38) gives $\nabla^{\perp}_{\partial_t}H = 0$ which means $\nabla^{\perp}_{e_1}H = 0$ So,

$$\tilde{\nabla}^{\perp}_{\partial_s}H = \tilde{\nabla}_{\partial_s}H + \tau_1 e_1 + \tau_2 e_2, \tag{3.40}$$

where

$$\tau_1 = \langle \tilde{\nabla}_{\partial_s}H, e_2\rangle, \tag{3.41}$$

$$\tau_2 = \langle \tilde{\nabla}_{\partial_s}H, e_1\rangle, \tag{3.42}$$

Note that a direct calculation by using (3.26) gives

$$\tilde{\nabla}_{\partial_s}H = -(\beta'' + a_2'\beta + (c^2t + a_2(s))\beta'). \tag{3.43}$$

Moreover, the first and the second equality of (3.16) gives

$$\langle \beta, \beta'' \rangle = -c^2. \tag{3.44}$$

So, by using (3.44) we get

$$\tau_2 = c^2 \tag{3.45}$$

and after direct calculcations and using (3.23), we have

$$\tau_1 = \left(\frac{2a_2^2 - a_1 + 2a_3 - 2a_2'}{2} + a_2 \left(2c^2 - 1\right) t - \frac{1}{2} c^2 \left(1 - 2c^2\right) t^2 \right). \tag{3.46}$$

Replacing (3.46) and (3.45) into (3.40) we get

$$\begin{aligned} \tilde{\nabla}_{\partial_s} H &= -\left(\frac{2a_2^2 - a_1 + 2a_3 - 2a_2'}{2} + a_2 \left(2c^2 - 1\right) t - \frac{1}{2} c^2 \left(1 - 2c^2\right) t^2 \right) e_1 - c^2 e_2 \\ &\quad + \nabla^{\perp}_{\partial_s} H. \end{aligned} \tag{3.47}$$

Now, we want to notice that $\nabla^{\perp}_{\partial_s} H = \xi N_2$ and it gives $\xi^2 = \langle \nabla^{\perp}_{\partial_s} H, \nabla^{\perp}_{\partial_s} H \rangle$. So, we can say, by using (3.40)

$$\xi = \sqrt{||\tilde{\nabla}_{\partial_s} H||^2 + 2\tau_1^2 + 2\tau_2^2 - 2\tau_1 \tau_2}. \tag{3.48}$$

After straightforward computations, (3.48) becomes

$$\xi = \sqrt{a_2^2 (c^2 - 2) + a_1 - 2a_3 + a_4 + 2a_2 (c^2 - 1)^2 t + (c^3 - c)^2 t^2} \tag{3.49}$$

Replacing (3.49) into (3.47) we get

$$\begin{aligned} \tilde{\nabla}_{\partial_s} H &= -\left(\tfrac{2a_2^2 - a_1 + 2a_3 - 2a_2'}{2} + a_2 \left(2c^2 - 1\right) t - \tfrac{1}{2} c^2 \left(1 - 2c^2\right) t^2 \right) e_1 - c^2 e_2 \\ &\quad + \xi N_2. \end{aligned} \tag{3.50}$$

Note that H is parallel if and only if ξ vanishes identical. Assume that H is parallel. If we take a derivative ξ with respect to t we get

$$2a_2 (c^2 - 1)^2 + 2t (c^3 - c)^2 = 0. \tag{3.51}$$

Rearranging the above equation we get

$$2(c^2 - 1)^2 \left(a_2(s) + c^2 t \right) = 0, \tag{3.52}$$

where c doesn't have to be 1 since we chose it arbitrary and nonzero. So, we have $a_2 = -c^2 t$. But it gives a contradiction because a_2 depends only on s. This yields that H is not parallel. □

Before we continue, we want to present an explicit example.

Example 3.2.2. The vector valued functions

$$\beta(s) = \frac{1}{\sqrt{2}}(\cosh(bs), \sinh(bs), \cos(as), \sin(as)) \tag{3.53}$$

$$\alpha(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{b}\sinh(bs), \frac{1}{b}\cosh(bs), -\frac{1}{a}\sin(as), \frac{1}{a}\cos(as)\right) \tag{3.54}$$

satisfy the conditions given in (3.16) for $c = \sqrt{(a^2+b^2)/2}$. Therefore, the CMC surface given by

$$x(s,t) = \frac{1}{\sqrt{2}} \begin{array}{c}\left(t\cosh(bs) + \frac{1}{b}\sinh(bs), t\sinh(bs) + \frac{1}{b}\cosh(bs),\right. \\ \left. t\cos(as) - \frac{1}{a}\sin(as), t\sin(as) + \frac{1}{a}\cos(as)\right)\end{array} \tag{3.55}$$

is biconservative because of Proposition 3.2.1.

In the next two propositions, we obtain two families of biconservative cylinders in $\mathbb{E}^4_1$. Note that there exists a similar family of CMC biconservative surface in the Euclidean 4-space (See [14, Proposition 5.2]).

Proposition 3.2.3. *Let M be the cylinder in $\mathbb{E}^4_1$ given by*

$$x(s,t) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), t)$$

for an arc-length parametrized curve $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{E}^3_1$ with a non-null normal vector field. Then M is proper biconservative and CMC if the curvature of α is constant and its torsion is non-vanishing.

Proof. By the hypothesis, the vector fields $N_1 = (n_1(s), n_2(s), n_3(s), 0)$ and $N_2 = (b_1(s), b_2(s), b_3(s), 0)$ form a local orthonormal frame field for the normal bundle of M, where $n = (n_1, n_2, n_3)$ and $b = (b_1, b_2, b_3)$ denote the unit normal and binormal vector fields of α in $\mathbb{E}^3_1$, respectively. Then one can find easily that

$$\tilde{\nabla}_{\partial_s}\partial_s = \kappa N_1(s), \qquad \tilde{\nabla}_{\partial_s}\partial_t = 0, \qquad \tilde{\nabla}_{\partial_t}\partial_t = 0 \tag{3.56}$$

which imply

$$h(\partial_s, \partial_s) = \kappa N_1(s), \qquad h(\partial_s, \partial_t) = 0, \qquad h(\partial_t, \partial_t) = 0. \tag{3.57}$$

By a direct computation using (3.57), we obtain

$$H = \varepsilon_1 \frac{\kappa}{2} N_1, \qquad A_{N_2} = 0, \qquad \nabla^{\perp}_{\partial_s} N_1 = \varepsilon_2 \tau N_2. \tag{3.58}$$

Then, for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ depending on the causality of n and b, respectively, κ and τ are the curvature and torsion of α, respectively. Now, if κ is constant, then M is CMC. In this case, $A_{N_2} = 0$ implies (2.61). $\square$

By a similar way, we have

Proposition 3.2.4. *Let M be the cylinder in $\mathbb{E}^4_1$ given by*

$$x(s,t) = (t, \alpha_1(s), \alpha_2(s), \alpha_3(s)) \tag{3.59}$$

for an arc-length parametrized curve $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{E}^3$. Then, M is proper biconservative and CMC if the curvature of α is constant and its torsion is non-vanishing.

We also want to give the following example of quasi-minimal biconservative surface in $\mathbb{E}^4_1$.

Example 3.2.5. [10] Consider the surface in $\mathbb{E}^4_1$ given by

$$x(u,v) = (\psi(u,v), u, v, \psi(u,v)) \tag{3.60}$$

for a smooth function ψ. A direct computation yields that its mean curvature vector is

$$H = \frac{\psi_{uu} + \psi_{vv}}{2}(-1,0,0,1) \tag{3.61}$$

and it satisfies $A_H = 0$ since linearity property of the shape operator

$$A_H = -\frac{\psi_{uu} + \psi_{vv}}{2}\nabla(-1,0,0,1) = 0. \tag{3.62}$$

A further computation shows that (2.61) is satisfied and H is not parallel if $\psi_{uu} + \psi_{vv}$ is not a constant.

3.3 Local Classification Theorem

In this subsection, we consider two cases given in Lemma 3.1.4 seperately in order to obtain the complete classification biconservative CMC surfaces in the Minkowski 4-space.

Proposition 3.3.1. *Let M be a proper biconservative surface in $\mathbb{E}^4_1$ satisfying the Case 1 of Lemma 3.1.4. Then, it is locally congruent to the surface described in Proposition 3.2.1.*

Proof. Assume that M satisfies the condition given in the Case 1 of Lemma 3.1.4 for a pseudo-orthanormal frame field $\{e_1, e_2, N_1, N_2\}$ and let ω_{34} be as defined in (3.8). We consider a local coordinate system (s,t) defined on the open set $\mathscr{O} \subset M$ satisfying the conditions given in Proposition 3.1.1 such that e_1 is propotional to ∂_t (See Remark 3.1.2). Let $x(s,t)$ be a local parametrization of $\mathscr{O} \subset M$. Then, we have

$$A_{N_2}(\tilde{e}_1) = 0 \text{ and } A_{N_2}(\tilde{e}_2) = \gamma \tilde{e}_1 \tag{3.63}$$

for a non-vanishing smooth function γ, where we define $\tilde{e}_1, \tilde{e}_2$ by

$$\tilde{e}_1 = \partial_t, \qquad \tilde{e}_2 = \partial_s + f\partial_t. \tag{3.64}$$

Note that (2.61) implies

$$\omega_{34}(e_1) = 0 \tag{3.65a}$$

and (2.30) gives

$$h(\tilde{e}_1, \tilde{e}_1) = h^3_{11} N_1, \tag{3.65b}$$

$$h(\tilde{e}_1, \tilde{e}_2) = cN_1, \tag{3.65c}$$

$$h(\tilde{e}_2, \tilde{e}_2) = h^3_{22} N_1 - \gamma N_2 \tag{3.65d}$$

for some functions h^3_{11} and h^3_{22}. We combine (3.65) with the Codazzi equation (2.28) for $X = e_2, Y = Z = e_1$ to get

$$e_2(h^3_{11}) = -2\langle \nabla_{e_2} e_1, e_2 \rangle h^3_{11}, \tag{3.66}$$

$$h^3_{11}\omega_{34}(e_2) = 0. \tag{3.67}$$

Since H is not parallel, (3.65a) and (3.67) imply $h^3_{11} = 0$. Consequently, (3.65b) and (3.2) give

$$\tilde{\nabla}_{\partial_t} \partial_t = 0 \tag{3.68}$$

which yields $x_{tt} = 0$. Therefore, we have (3.17) for some α, β. By considering (3.1), we get the first and the third equations in (3.16). On the other hand, $\nabla_{e_1} e_1 = 0$ means $\nabla_{e_1} e_2 = 0$ and so $\tilde{\nabla}_{e_1} e_2 = h(e_1, e_2) = -H$. So, by using (3.64), and a direct computation, we obtain

$$H = -(\beta' + f_t \beta) \tag{3.69}$$

which yields the second equation of (3.16) because H has CMC. Hence, $\mathscr{O}$ is congruent to the ruled surface given in Proposition 3.2.1. $\square$

Proposition 3.3.2. *Let M be a proper biconservative surface in $\mathbb{E}^4_1$ satisfying the Case 2 of Lemma 3.1.4. Then, it is locally congruent to one of two cylinders described in Proposition 3.2.3 and Proposition 3.2.4.*

Proof. Assume that M satisfies the condition given in the Case 2 of Lemma 3.1.4 for the frame field $\{N_1,N_2\}$ of the normal bundle of M, $p\in M$, and let ω_{34} be the 1-form by defined (3.8). Since M is proper biconservative, we have $\omega_{34}\neq 0$ outside of a subset of M with empty interior. Note that $H=2cN_1$ implies trace$A_{N_1}=2c$. We are going to consider three canonical forms of A_{N_1} given in Lemma 3.1.3 seperately.

Case (i). There is an orthonormal frame field $\{e_1,e_2\}$ such that

$$A_{N_1}=\begin{bmatrix} k_1 & 0 \\ 0 & 2c-k_1 \end{bmatrix} \tag{3.70}$$

for a smooth function k_1. We assume $\langle e_2,e_2\rangle=1$ and put $\varepsilon=\langle e_1,e_1\rangle\in\{-1,1\}$. In this case, by a direct computation using the Codazzi equation (2.28) for $X=e_1,Y=Z=e_2$ and $X=e_2,Y=Z=e_1$, we obtain

$$(2c-k_1)\omega_{34}(e_1)=0, \tag{3.71a}$$

$$\varepsilon k_1\omega_{34}(e_2)=0, \tag{3.71b}$$

$$e_1(k_1)=2c\phi_2, \tag{3.71c}$$

$$e_2(k_1)=-2c\phi_1, \tag{3.71d}$$

where we define ϕ_i by $\nabla_{e_i}e_1=\phi_i e_2$.

First we assume $\omega_{34}(e_1)=0$ on M. Then, $\omega_{34}(e_2)\neq 0$ and (3.71b) implies $k_1=0$. On the other hand, if $\omega_{34}(e_1)\neq 0$ on an open subset $\mathscr{O}$ of M, then (3.71a) and (3.71b) imply $k_1=2c$ and $\omega_{34}(e_2)=0$, seperately. In both cases, (3.71c) and (3.71d) yield that $\phi_1=\phi_2=0$ on M. Therefore, we have $\nabla_{e_i}e_j=0,\ i,j=1,2$ which implies the existence of a local coordinate system (s_1,s_2) such that $e_1=\partial_{s_1}$, $e_2=\partial_{s_2}$ defined in a neighborhood $\mathscr{N}_p$ of p. Let $x=x(s,t)$ be a local parametrization of $\mathscr{N}_p$. We put $s_1=s,s_2=t$ if $\omega_{34}(e_2)=0$ and $s_1=t,s_2=s$ if $\omega_{34}(e_1)=0$. In both cases, the Gauss-Weingarten formula turns into

$$\widetilde{\nabla}_{\partial_t}\partial_t=0,\qquad \widetilde{\nabla}_{\partial_t}\partial_s=0 \tag{3.72}$$

which gives $x_{tt}=x_{ts}=0$. Therefore, we have

$$x(s,t)=\alpha(s)+t\beta_0 \tag{3.73}$$

for an $\mathbb{R}^4$-valued function α and constant vector $\beta_0 \in \mathbb{E}^4_1$. By considering that $\{\partial_s, \partial_t\}$ an orthonormal frame field, we obtain that $\mathcal{N}_p$ is congruent to one of two cylinders given in Proposition 3.2.3 and Proposition 3.2.4.

Case (ii). Assume that there is a pseudo-orthonormal frame field $\{e_1, e_2\}$ such that

$$A_{N_1} = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}. \tag{3.74}$$

In this case, by combining $A_{N_2} = 0$ and (3.74) with (2.30), we get

$$h(e_1,e_1) = 0,\ h(e_1,e_2) = -cN_1, h(e_2,e_2) = -N_1. \tag{3.75}$$

By considering the Codazzi equation (2.28) two times by replacing,

• $X = e_1, Y = Z = e_2$ we get

$$\nabla^\perp_{e_1} h(e_2,e_2) - 2h(\nabla_{e_1}e_2,e_2) = \nabla^\perp_{e_2} h(e_1,e_2) - h(\nabla_{e_2}e_1,e_1) - h(e_1,\nabla_{e_2}e_2) \tag{3.76}$$

so,

$$-\omega_{34}(e_1)N_2 + 2\phi_1 N_1 = -c\omega_{34}(e_2)N_2. \tag{3.77}$$

Hence, we obtain

$$\phi_1 = 0 \tag{3.78}$$

$$\omega_{34}(e_1) = c\omega_{34}(e_2) \tag{3.79}$$

• $X = e_2, Y = Z = e_1$

$$\nabla^\perp_{e_2} h(e_1,e_1) - 2h(\nabla_{e_2}e_1,e_1) = \nabla^\perp_{e_1} h(e_2,e_1) - h(\nabla_{e_1}e_2,e_1) - h(e_2,\nabla_{e_1}e_1) \tag{3.80}$$

so,

$$0 = -c\omega_{34}(e_1)N_2, \tag{3.81}$$

which gives $\omega_{34}(e_1) = 0$. Therefore, H is parallel because of (3.79). Thus it remains to study the following case, since this kind of biconservative CMC surface has already been classsified , [23].

Case (iii). Assume that there is an orthonormal frame field $\{e_1, e_2\}$ such that

$$A_{N_1} = \begin{bmatrix} c & \gamma \\ -\gamma & c \end{bmatrix}$$

and $\langle e_1, e_1 \rangle = -1$, where γ is a smooth non-vanishing function. Note that we have

$$h(e_1,e_1) = -cN_1, \qquad h(e_1,e_2) = -\gamma N_2, \qquad h(e_2,e_2) = cN_1. \tag{3.82}$$

In this case, we use the Codazzi equation (2.28) to get

$$c\omega_{34}(e_1) + \gamma\omega_{34}(e_2) = \gamma\omega_{34}(e_1) - c\omega_{34}(e_2) = 0 \tag{3.83}$$

which implies

$$\gamma\omega_{34}(e_1) = c\omega_{34}(e_2) \tag{3.84}$$

$$-c\omega_{34}(e_1) = \gamma\omega_{34}(e_2) \tag{3.85}$$

Multiplying (3.84) by γ and (3.85) by $-c$ and adding these equtions, we get

$$(\gamma^2 + c^2)\omega_{34}(e_1) = 0. \tag{3.86}$$

Therefore, we have $\omega_{34}(e_1) = 0$. Furthermore, (3.84) gives $\omega_{34} = 0$ which yields a contradiction. □

By combining Proposition 3.3.1 and Proposition 3.3.2, we obtain the following classification theorem.

Theorem 3.3.3. *A surface M in $\mathbb{E}^4_1$ is non-zero CMC and biconservative if and only if it is locally congruent to one of the following four types of surfaces.*

(i). A surface with parallel mean curvature vector,

(ii). A ruled surface described in Proposition 3.2.1,

(iii). A cylinder described in Proposition 3.2.3,

(iv). A cylinder described in Proposition 3.2.4.

Remark 3.3.4. The surfaces given in the case (ii) and case (iv) of Theorem 3.3.3 are not proper biharmonic. On the other hand, if M is a cylinder given in the case (iii) of Theorem 3.3.3, then it is biharmonic if and only if its profile curve is appropriately chosen (See [10, Theorem 5.1] and [11, Theorem 5.1]).

Now, let M be a quasi-minimal surface in $\mathbb{E}^4_1$ and consider the pseudo-orthonormal frame field $\{N_1, N_2\}$ of its normal bundle such that

$$H = N_1 = -\frac{\operatorname{trace} A_{N_2}}{2} N_1 - \frac{\operatorname{trace} A_{N_1}}{2} N_2. \tag{3.87}$$

So, we have trace$A_{N_1} = 0$. Therefore, since M is Riemannian, we can choose orthonormal tangent vector fields e_1, e_2 so that

$$A_{N_1} = \begin{bmatrix} k_1 & 0 \\ 0 & -k_1 \end{bmatrix} \tag{3.88}$$

for some smooth functions k_1. Because of the definition of the quasi minimal surface, we can define ψ_1, ψ_2 by

$$\nabla^\perp_{e_i} N_1 = \psi_i N_1, \qquad \nabla^\perp_{e_i} N_2 = -\psi_i N_2. \tag{3.89}$$

Consequently, the biconservativity equation (2.61) implies

$$0 = \text{trace} A_{\nabla^\perp_{\cdot} H}(\cdot) = \psi_1 k_1 e_1 - \psi_2 k_1 e_2. \tag{3.90}$$

Thus we have :

$$\psi_1 k_1 = 0, \qquad \psi_2 k_1 = 0. \tag{3.91}$$

If $k_1 \neq 0$ then (3.91) implies $\psi_2 = \psi_1 = 0$ which yields that H is parallel. Thus we have $k_1 = 0$. Then $A_{N_1} = 0$. Therefore we have $A_H = 0$. By using the exactly same method in [10, Sect. 6], we observe that M is locally congruent to the surface given in Example 3.2.5. Therefore, we have

Proposition 3.3.5. *A quasi-minimal surface M in $\mathbb{E}^4_1$ is CMC and proper biconservative if and only if it is locally congruent to the surface given in Example 3.2.5 for a smooth function ψ such that $\psi_{uu} + \psi_{vv}$ is not a constant.*

3.4 CMC Surfaces in $\mathbb{S}^4_1$ and $\mathbb{H}^4_1$

In this section, we consider CMC surfaces in non-flat Lorentzian space forms. First, we obtain the following classification theorem.

Theorem 3.4.1. *Let M be a surface in $\mathbb{L}^4(\delta)$, $\delta = \pm 1$. Then, M has non-zero CMC and it is proper biconservative if and only if it is locally congruent to the ruled surface parametrized by* (3.17) *for some α, β satisfying*

$$\langle \alpha, \alpha \rangle = \delta, \qquad \langle \alpha, \beta \rangle = 0, \qquad \langle \beta, \beta \rangle = 0, \tag{3.92a}$$

$$\langle \beta, \beta \rangle = 0, \quad \langle \beta', \beta' \rangle = 1 + c^2, \qquad \langle \alpha', \beta \rangle = -1, \tag{3.92b}$$

where c is the mean curvature of M.

Proof. In order to prove the necessary condition, we assume that M is a proper biconservative CMC surface. First, we consider the subset

$$\mathscr{F} = \{p \in M | A_{N_2}(X) = 0 \text{ whenever } X \in T_pM\}$$

of M and assume that its interior $\tilde{\mathscr{O}}$ is not empty. In this case, similar to the proof of Proposition 3.3.2, we obtain that A_{N_1} has the matrix representation

$$A_{N_1} = \begin{bmatrix} 2c & 0 \\ 0 & 0 \end{bmatrix} \tag{3.93}$$

with respect to an orthonormal frame field $\{e_1, e_2\}$ on $\tilde{\mathscr{O}}$, where c is the mean curvature of M and recall that

$$\nabla_{e_i} e_1 = \phi_i e_2, \qquad \nabla_{e_i} e_2 = -\phi_i e_1, \qquad i = 1,2. \tag{3.94}$$

Let us consider the Codazzi equation (2.28) for $X = e_2, Y = Z = e_1$ we get $\phi_2 = 0$ and for $X = e_1, Y = Z = e_2$ we get $\phi_1 = 0$. So, we have $\nabla_{e_i} e_j = 0,\ i, j = 1,2$ which yields that $\tilde{\mathscr{O}}$ is flat. Then, we consider the Gauss equation (2.26) for $X = Z = e_1,\ Y = e_2$ to get $e_2 = 0$ on $\mathscr{O}$ which is not possible. Therefore, the interior of $\mathscr{F}$ is empty and Lemma 3.1.4 implies that A_{N_2} has the matrix representation given in (3.7) with respect to an appropriately chosen pseudo-orthonormal frame field $\{e_1, e_2\}$ of the tangent bundle of M.

We consider a local coordinate system (s,t) defined on the open set $\mathscr{O} \subset M$ satisfying the conditions given in Proposition 3.1.1 and define $\tilde{e}_1, \tilde{e}_2$ as given in (3.64). Consequently, by using the Codazzi equation, we get

$$h(\tilde{e}_1, \tilde{e}_1) = 0. \tag{3.95}$$

By using this equation, when $e_1 = \partial_t$, by (3.2) and (3.5) we obtain

$$\hat{\nabla}_{\partial_t} \partial_t = 0 \tag{3.96}$$

which gives $x_{tt} = 0$, where $x = x(s,t)$ is the local parametrization of $\mathscr{O}$. Therefore, $\mathscr{O}$ is congruent to a ruled surface (3.17) for some α, β. Note that M is a submanifold of $\mathbb{L}(\delta)$ then $\langle x, x \rangle = \delta$. So, the following calculation

$$\langle \alpha, \alpha \rangle + 2t \langle \alpha, \beta \rangle + t^2 \langle \beta, \beta \rangle = \delta \tag{3.97}$$

implies (3.92a), since $e_1 = \partial_t = \beta$ implies $\langle \beta, \beta \rangle = 0$ and the coefficient of $2t$ equal zero from being $\delta = \pm 1$ and also the metric (3.1) gives the first and the third equations in (3.92b), since

$$\langle \partial_s, \partial_t \rangle = -1, \tag{3.98}$$

$$\langle \alpha', \beta \rangle + t\langle \beta', \beta \rangle = -1, \tag{3.99}$$

where $\langle \beta', \beta \rangle = 0$ since β is a null vector.

Note that $\hat{\nabla}_{\partial_t} \partial_t = \hat{\nabla}_{e_1} e_1 = 0$ gives $\hat{\nabla}_{e_1} e_2 = 0$ and so $\hat{\nabla}_{e_1} e_2 = \hat{h}(\tilde{e}_1, \tilde{e}_2) = -\hat{H}$. By considering (3.5), (3.17) and (3.64) we obtain

$$-\hat{H} = -H + \delta x = \hat{\nabla}_{\tilde{e}_1} \tilde{e}_2 = \beta' + f_t \beta, \tag{3.100}$$

where $H = cN_1$, then

$$\langle -H + \delta x, -H + \delta x \rangle = \langle \beta' + f_t \beta, \beta' + f_t \beta \rangle \tag{3.101}$$

$$c^2 + \delta^3 = \langle \beta', \beta' \rangle \tag{3.102}$$

$$c^2 \pm 1 = \langle \beta', \beta' \rangle \tag{3.103}$$

from which we get the second equation in (3.92b). Hence, $\mathscr{O}$ is congruent to the ruled surface given in the theorem.

The proof of the sufficient condition follows from a direct computation similar to the proof of Proposition 3.2.1. □

Remark 3.4.2. In [14, Theorem 5.1], it was proved that there exists no CMC proper biconservative surface in the non-flat Riemannian space forms $\mathbb{S}^4$ and $\mathbb{H}^4$.

Next, we consider (2.62) for the surface given in Theorem 3.4.1 to obtain the classification of biharmonic CMC surfaces.

Let M be the proper biconservative CMC surface in $\mathbb{L}^4(\delta)$, $\delta = \pm 1$ parametrized by (3.17) for some vector valued functions α, β satisfying (3.92). We define $\tilde{e}_1, \tilde{e}_2$ as given in (3.64) to get (3.38), (3.33) and rearranging (3.100), we have

$$H = \delta \alpha + (\delta t - f_t)\beta - \beta'. \tag{3.104}$$

Note that (3.38) gives $A_{N_1} e_1 = c e_1$ from which it follows that

$$A_{N_1}(e_2) = -h_{22} e_1 - c e_2 \tag{3.105}$$

for a smooth function h_{22}. The Ricci equation (2.27) for $X = e_1, Y = e_2$ implies

$$R^{\perp}(e_1, e_2)H = h(e_1, A_H e_2) - h(A_H e_1, e_2). \tag{3.106}$$

We know that $\nabla_{e_1} e_1 = 0$ and $\nabla_{e_2} e_1$ is proportional to e_1. So, by combining all of them and (3.33) with (3.106), we get

$$R^{\perp}(e_1, e_2)H = 0. \tag{3.107}$$

By considering the left hand side of (3.107), we get

$$-\Delta^{\perp} H = 0. \tag{3.108}$$

On the other hand, by using (2.30) and (3.104), we get

$$\operatorname{trace} h(A_H(\cdot), \cdot) = 2c^2 H. \tag{3.109}$$

By considering (3.108) and (3.109), we conclude that (2.62) is equivalent to

$$2(c^2 - \delta)H = 0. \tag{3.110}$$

Hence, we have the following results.

Theorem 3.4.3. *Let M be a proper biconservative surface in the de Sitter space $\mathbb{S}^4_1$ with the constant mean curvature $c \neq 0$. Then, M is biharmonic if and only if $c = 1$.*

Theorem 3.4.4. *There exists no proper biharmonic surface in the anti-de Sitter space $\mathbb{H}^4_1$ with non-zero constant mean curvature.*

Next, we want to present an explicit example:

Example 3.4.5. The vector valued functions

$$\beta(s) = \frac{1}{\sqrt{2}}(\cosh(bs), \sinh(bs), \cos(as), \sin(as), 0)$$

$$\alpha(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{b}\sinh(bs), \frac{1}{b}\cosh(bs), -\frac{1}{a}\sin(as), \frac{1}{a}\cos(as), 2 - \frac{1}{a^2} - \frac{1}{b^2}\right)$$

satisfy the conditions given in (3.92) for $\delta = 1$ and c satisfying $a^2 + b^2 = 2(1 + c^2)$. Therefore, the ruled surface

$$\begin{aligned} x(s,t) = \tfrac{1}{\sqrt{2}}\big(&\tfrac{1}{b}\sinh(bs) + t\cosh(bs), \tfrac{1}{b}\cosh(bs) + \sinh(bs), \\ &-\tfrac{1}{a}\sin(as) + \cos(as), \tfrac{1}{a}\cos(as) + \sin(as), 2 - \tfrac{1}{a^2} - \tfrac{1}{b^2}\big) \end{aligned}$$

is a proper biconservative surface in $\mathbb{S}^4_1$ with the constant curvature c because of Proposition 3.2.1. Furthermore, this surface is biharmonic in $\mathbb{S}^4_1$ if $a^2 + b^2 = 4$.

4. HYPERSURFACE IN MINKOWSKI 4-SPACE

In this section, we consider biconservative hypersurfaces with non-diagonalizable shape operator in $\mathbb{E}^4_1$. Before we proceed, we would like to refer to [27] for the complete classification of biconservative hypersurfaces with diagonalizable shape operator.

Remark 4.0.1. Before we proceed we want to notice that H means the mean curvature for hypersurfaces in Lorentzian space forms while it means the mean curvature vector for submanifolds of codimension 2.

Note that the results appearing in this section are contained in [28].

4.1 Biconservative Hypersurfaces in $\mathbb{E}^4_1$

Let M be a proper biconservative Lorentzian hypersurface with a non-diagonalizable shape operator A in the Minkowski space $\mathbb{E}^4_1$. Then, from the equation (2.58) we have

$$A(\operatorname{grad} H) = \frac{-3H}{2}\operatorname{grad} H, \tag{4.1}$$

where H is the mean curvature. Therefore $\operatorname{grad} H$ is a principal direction of M with the corresponding principle curvature $\frac{-3H}{2}$.

Remark 4.1.1. If M has constant mean curvature, i.e., $\operatorname{grad} H = 0$, then the equation (4.1) is satisfied trivially. Therefore, we are going to call a biconservative hypersurface as '*proper*' if $\operatorname{grad} H$ does not vanish at any point.

Now, assume that A is non-diagonalizable and we consider the canonical forms of A given in Lemma 2.3.12. Consider the Case IV of Lemma 2.3.12

$$A \sim \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_1 & 1 \\ -1 & 0 & k_1 \end{bmatrix} \tag{4.2}$$

from (4.1) we have $k_1 = -\frac{3H}{2}$, and so $\operatorname{trace} A = -\frac{9H}{2}$ but we have $\operatorname{trace} A = 3H$ from (2.31). So it gives $H = 0$ which is a contradiction. So, Case IV of Lemma 2.3.12 is not possible. Moreover, A can not have the form given in the Case II of Lemma 2.3.12 due to [29, Theorem 1.1]. Hence, we have the following two cases subject to the casuality of $\operatorname{grad} H$.

Case (i). There exists a pseudo-orthonormal base field $\{e_1, e_2, e_3\}$ such that

$$A \sim \begin{bmatrix} -3H/2 & 1 & 0 \\ 0 & -3H/2 & 0 \\ 0 & 0 & 6H \end{bmatrix}, \tag{4.3}$$

where the null vector e_1 is proportional to $\operatorname{grad} H$.

Case (ii). There exists a pseudo-orthonormal base field $\{e_1, e_2, e_3\}$ such that

$$A \sim \begin{bmatrix} 9H/4 & 1 & 0 \\ 0 & 9H/4 & 0 \\ 0 & 0 & -3H/2 \end{bmatrix}, \tag{4.4}$$

where the space-like vector e_3 is proportional to $\operatorname{grad} H$.

Note that the Levi-Civita connection of M has the form

$$\nabla_{e_i} e_1 = \phi_i e_1 + \omega_{13}(e_i) e_3, \tag{4.5a}$$

$$\nabla_{e_i} e_2 = -\phi_i e_2 + \omega_{23}(e_i) e_3, \tag{4.5b}$$

$$\nabla_{e_i} e_3 = \omega_{23}(e_i) e_1 + \omega_{13}(e_i) e_2 \tag{4.5c}$$

for $i = 1, 2, 3$. Here we define the connection forms ω_{jk} with $\omega_{jk}(e_i) = \langle \nabla_{e_i} e_j, e_k \rangle$ and, for simplicity, we put $\phi_i = -\omega_{12}(e_i)$. In the following lemma, we prove that the Case (i) above is not possible.

Lemma 4.1.2. *Let M be a proper biconservative hypersurface of $\mathbb{E}^4_1$ with non-diagonalizable shape operator. Then, its shape operator A has the matrix represantation given in* (4.4) *for a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ such that $e_3 = \frac{\operatorname{grad} H}{\|\operatorname{grad} H\|}$.*

Proof. To obtain a contradiction we assume that A has the matrix represantation given in (4.3) and e_1 is proportional to $\operatorname{grad} H$. Recall that gradient of H has the form

$$\operatorname{grad} H = -e_2(H) e_1 - e_1(H) e_2 + e_3(H) e_3. \tag{4.6}$$

So, we have

$$e_1(H) = e_3(H) = 0, \qquad e_2(H) \neq 0. \tag{4.7}$$

Note that the second fundemantal form of M satisfies

$$h(e_1,e_1)=0, \qquad h(e_1,e_2)=\frac{3H}{2}, \qquad h(e_2,e_2)=-1, \qquad h(e_3,e_3)=6H. \quad (4.8)$$

First, we use the Codazzi equation (2.28) for $X=e_i$, $Y=e_j$ and $Z=e_k$ for each triple in (i,j,k).

- The triple $(2,3,1)$ implies

$$\omega_{23}(e_2)=-\omega_{13}(e_2) \quad (4.9)$$

- The triple $(1,3,2)$ gives

$$\omega_{13}(e_1)=\frac{15H}{2}\omega_{23}(e_1) \quad (4.10)$$

- The triple $(3,1,1)$ gives

$$\omega_{13}(e_1)=0. \quad (4.11)$$

By replacing (4.11) into (4.10) we have $\omega_{23}(e_1)=0$.

- The triple $(1,3,3)$ gives

$$\omega_{13}(e_3)=0. \quad (4.12)$$

- The triple $(3,2,2)$ gives

$$0=-\omega_{23}(e_2)h(e_1,e_2)-\omega_{13}(e_2)h(e_2,e_2)-\omega_{23}(e_2)h(e_3,e_3) \quad (4.13)$$

by using (4.9) we have

$$\omega_{23}(e_2)=0 \quad (4.14)$$

- The triple $(2,3,3)$ gives

$$\omega_{23}(e_3)=-\frac{8}{5}\frac{e_2(H)}{H}. \quad (4.15)$$

Note that by considering (4.7), from (4.15) we get

$$e_1\left(\omega_{23}(e_3)\right)=\frac{8\phi_1}{5}\frac{e_2(H)}{H}. \quad (4.16)$$

By a further computation we get

$$R(e_1,e_3,e_3,e_2)=-e_1\left(\omega_{23}(e_3)\right)-\phi_1\omega_{23}(e_3)=0, \quad (4.17)$$

where we use (4.15) and (4.16) in the last equality. However, by combining (4.17) with the Gauss equation (2.26) for $X=e_1$ and $Y=Z=e_3$, we get $H=0$ which yields a contradiction. □

4.2 Uniqueness of Biconservative Hypersurfaces

Let (Ω, g) be a 3-dimensional Lorentzian manifold which admits a proper biconservative hypersurface with non-diagonalizable shape operator into $\mathbb{E}^4_1$. First, we prove that H, e_1, e_2 and e_3 in Lemma 4.1.2 are intrinsic, i.e., they can be uniquely determined by considering the metric g of Ω.

Lemma 4.2.1. *Let (Ω, g) be a 3-dimensional Lorentzian manifold. Assume that (Ω, g) admits a proper biconservative isometric immersion x with a non-diagonalizable shape operator into $\mathbb{E}^4_1$ with the mean curvature H. Then, the function H^2 and the vector fields E_1, E_2, E_3 can be determined intrinsically, where E_1, E_2, E_3 are defined by*

$$e_1 = x_* E_1, \quad e_2 = x_* E_2, \quad e_3 = x_* E_3.$$

Proof. By considering (4.4) and the Gauss equation, after direct calculations, we get

$$K(E_1, E_2) = K(e_1, e_2) = -\langle R(E_1, E_2)E_2, E_1\rangle = -k_1^2, \tag{4.18}$$

and

$$\begin{aligned} K(E_1, E_3) + K(E_2, E_3) &= -(\langle R(E_1, E_3)E_3, E_2\rangle + \langle R(E_2, E_3)E_3, E_1\rangle) && (4.19)\\ &= -2k_1k_3. && (4.20)\end{aligned}$$

So the scalar curvature S of (Ω, g) is

$$S = K(E_1, E_2) + K(E_1, E_3) + K(E_2, E_3) = -k_1^2 - 2k_1k_3. \tag{4.21}$$

Since we have $k_1 = \frac{9}{4}H$ and $k_3 = \frac{-3}{2}H$, the equation above gives

$$S = \frac{27}{16}H^2 \tag{4.22}$$

which shows that H^2 is intrinsic. Also (4.22) implies

$$E_3 = \frac{\operatorname{grad} S}{\| \operatorname{grad} S \|}. \tag{4.23}$$

Therefore, E_3 can be determined intrinsically.

Now, we are going to show that E_1 and E_2 can be determined by considering the curvature tensor R of Ω. Note that the Gauss equation and (4.4) yields

$$R(E_1,E_3)E_1 = 0, \tag{4.24}$$
$$R(E_3,E_2)E_3 = -\frac{3H}{2}E_1, \tag{4.25}$$
$$R(E_2,E_3)E_2 = \frac{3H}{2}E_3. \tag{4.26}$$

Assume that $\tilde{E}_1$, $\tilde{E}_2$ are another couple of null vectors satisfying

$$g(\tilde{E}_1,E_3) = g(\tilde{E}_2,E_3) = 0, \tag{4.27}$$
$$R(\tilde{E}_1,E_3)\tilde{E}_1 = 0, \tag{4.28}$$
$$R(E_3,\tilde{E}_2)E_3 = -\frac{3H}{2}\tilde{E}_1. \tag{4.29}$$

Then, because of (4.27), we have either $\tilde{E}_1 = \mu E_1,\ \tilde{E}_2 = \frac{1}{\mu}E_2$ or $\tilde{E}_1 = \mu E_2,\ \tilde{E}_2 = \frac{1}{\mu}E_1$ for a non-vanishing function μ. Note that if $\tilde{E}_1 = \mu E_2,\ \tilde{E}_2 = \frac{1}{\mu}E_1$ then (4.28) implies

$$R(E_2,E_3)E_2 = 0 \tag{4.30}$$

which is a contradiction because of (4.26). Therefore, we have $\tilde{E}_1 = \mu E_1,\ \tilde{E}_2 = \frac{1}{\mu}E_2$ from (4.29) we get

$$R(E_3,\frac{1}{\mu}E_2)E_3 = -\frac{3H}{2}\mu E_1.$$

However, this equation and (4.25) yield $\mu^2 = 1$. Therefore, we have $\tilde{E}_1 = \pm E_1$, $\tilde{E}_2 = \pm E_2$. □

Now, we are ready to prove the following uniqueness theorem:

Theorem 4.2.2. *Let (Ω,g) be a 3-dimensional, connected Lorentzian manifold. If (Ω,g) admits two proper biconservative isometric immersions $x,\tilde{x} : (\Omega,g) \hookrightarrow \mathbb{E}^4_1$ with the shape operator given in Lemma 4.1.2, then there exists an isometry $\tau : \mathbb{E}^4_1 \to \mathbb{E}^4_1$ such that $\tilde{x} = \tau \circ x$.*

Proof. Let the unit normal vectors of x and $\tilde{x}$ be N and $\tilde{N}$, respectively. Note that because of Lemma 4.1.2 and Lemma 4.2.1, the shape operators of x and $\tilde{x}$ have the forms

$$A^x = \begin{bmatrix} 9H/4 & 1 & 0 \\ 0 & 9H/4 & 0 \\ 0 & 0 & -3H/2 \end{bmatrix}, \qquad A^{\tilde{x}} = \begin{bmatrix} 9\varepsilon H/4 & 1 & 0 \\ 0 & 9\varepsilon H/4 & 0 \\ 0 & 0 & -3\varepsilon H/2 \end{bmatrix} \tag{4.31}$$

for a function $H : \Omega \to \mathbb{R}$, where $\varepsilon = \pm 1$. Now, we define the vector bundle isomorphism $\Phi : T\Omega_x^{\perp} \to T\Omega_{\tilde{x}}^{\perp}$ by

$$\Phi(N) = \varepsilon \tilde{N} \tag{4.32}$$

which satisfies the condition of Theorem 2.3.10. □

4.3 Local Classification of Biconservative Hypersurfaces

In this subsection, we obtain the local classification of proper biconservative hypersurfaces with non-diagonalizable shape operator.

Assume that M is a proper biconservative hypersurface with a non-diagonalizable shape operator in $\mathbb{E}^4_1$. Then, by Lemma 4.1.2, the shape operator A of M has the form (4.4), where we have $e_3 = \frac{\operatorname{grad} H}{\|\operatorname{grad} H\|}$. We put $k_1 = \frac{9H}{4}$ and $k_3 = -\frac{3H}{2}$.

In the next lemma, we get the connection forms of M.

Lemma 4.3.1. *Let M be a proper biconservative hypersurface in $\mathbb{E}^4_1$ with the shape operator A given in Lemma 4.1.2. Then, the connection forms of M defined in* (4.5) *satisfy*

$$\begin{aligned} \phi_1 = \omega_{13}(e_3) &= \omega_{13}(e_1) = \omega_{23}(e_3) = 0, && (4.33)\\ \omega_{13}(e_1) &= \omega_{23}(e_2), && (4.34)\\ -e_3(k_1) &= \omega_{13}(e_2)(k_1 - k_3), \qquad \omega_{13}(e_2) = \omega_{23}(e_1). && (4.35) \end{aligned}$$

Proof. Assume that the shape operator A of M has the form given in Lemma 4.1.2. Note that the equation (4.4) is equivalent to

$$Ae_1 = k_1 e_1,\ Ae_2 = e_1 + k_1 e_2,\ Ae_3 = k_3 e_3 \tag{4.36}$$

from which we have

$$\begin{aligned} h(e_1,e_1) = h(e_1,e_3) = h(e_2,e_3) = 0, \quad h(e_1,e_2) = -k_1 N,\\ h(e_2,e_2) = -N, \quad h(e_3,e_3) = k_3 N. \end{aligned} \tag{4.37}$$

Since $\operatorname{grad} H$ is proportional to e_3, we have

$$e_3(k_1) \neq 0, \qquad e_1(k_1) = e_2(k_1) = 0, \tag{4.38}$$

$$e_3(k_3) \neq 0, \qquad e_1(k_3) = e_2(k_3) = 0. \tag{4.39}$$

By taking into account the equations (4.37), (4.38) and (4.39), we consider the Codazzi equation (2.28) for $X = e_i$, $Y = e_j$ and $Z = e_k$ for some triple (i, j, k). From the triple $(1,2,2)$, $(1,3,3)$, $(2,3,3)$ and $(1,3,1)$, we get (4.33), and the triple $(2,3,1)$ implies (4.35). On the other hand, by combining the equation $[e_1, e_2](k_1) = 0$ with (4.5) and (4.38), we obtain (4.34). □

As a consequence of Lemma 4.3.1, the equations in (4.5) imply

$$[e_1,e_2] = -\phi_2 e_1, \quad (4.40a)$$

$$[e_1,e_3] = -(\omega_{32}(e_1)+\phi_3)e_1, \quad (4.40b)$$

$$[e_2,e_3] = \omega_{23}(e_2)e_1-(\omega_{31}(e_2)-\phi_3)e_2. \quad (4.40c)$$

Now, let $p \in M$. Because of (4.40a), the distrubition $D = \mathrm{span}\{e_1,e_2\}$ is involutive. Obviously, $D^\perp = \mathrm{span}\{e_3\}$ is also involutive. Thus, there exists a local coordinate system (s,t,u) in a neighborhood of p such that

$$D = \mathrm{span}\{\partial_s,\partial_t\} \text{ and } D^\perp = \mathrm{span}\{\partial_u\}. \quad (4.41)$$

Therefore, we have

$$e_1 = b_{11}\partial_s + b_{12}\partial_t, \quad (4.42a)$$

$$e_2 = b_{21}\partial_s + b_{22}\partial_t, \quad (4.42b)$$

$$e_3 = b_{33}\partial_u \quad (4.42c)$$

for some smooth functions b_{ij}.

Lemma 4.3.2. *By redefining (s,t,u) properly, we can assume $b_{12}=0, b_{33}=1$ and*

$$(b_{22})_s = 0. \quad (4.43)$$

Proof. Put $\tilde{e}_1 = fe_1$. Then, the equations (4.40a)-(4.42a) imply $[\tilde{e}_1,e_2] = \xi_1\tilde{e}_1$, $[e_1,e_3] = \xi_2\tilde{e}_1$, $[e_2,e_3] = \xi_3\tilde{e}_1+\xi_4 e_2$ and $\tilde{e}_1 = \tilde{b}_{11}\partial_s+\tilde{b}_{12}\partial_t$ for some smooth functions ξ_i and $\tilde{b}_{1j}$. If we choose f as a solution of

$$e_3(f) = f\xi_2, \quad (4.44)$$

then we get $[\tilde{e}_1,e_3]=0$. So, we have

$$[\tilde{e_1},e_3] = (\tilde{b}_{11}(b_{33})_s+\tilde{b}_{12}(b_{33})_t)\partial_u - b_{33}((b_{11})_u\partial_s+(b_{12})_u\partial_t) = 0 \quad (4.45)$$

from which we get

$$(\tilde{b}_{11})_u = 0, (\tilde{b}_{12})_u = 0 \quad (4.46)$$

and

$$\tilde{b}_{11}(b_{33})_s+\tilde{b}_{12}(b_{33})_t = 0. \quad (4.47)$$

Further, the equation $[e_2, e_3] = \xi_3\tilde{e}_1 + \xi_4 e_2$ implies

$$[e_2, e_3] = (b_{21}(b_{33})_s + b_{22}(b_{33})_t)\partial_u - b_{33}((b_{21})_u\partial_s + (b_{22})_u)\partial_t) \quad (4.48)$$

$$= A_1\partial_s + A_2\partial_t \quad (4.49)$$

from which we obtain

$$b_{21}(b_{33})_s + b_{22}(b_{33})_t = 0. \quad (4.50)$$

By combining (4.47) and (4.50), we obtain

$$\begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} (b_{33})_s \\ (b_{33})_t \end{bmatrix} = 0. \quad (4.51)$$

Therefore, we have

$$(b_{33})_s = (b_{33})_t = 0 \quad (4.52)$$

since $\tilde{e}_1$, e_2 are linearly independent.

Now, consider a coordinate change $(s,t,u) \to (S,T,U)$ with the form

$$S = \varphi(s,t), \quad (4.53)$$

$$T = \psi(s,t), \quad (4.54)$$

$$U = \tau(u), \quad (4.55)$$

where τ, φ and ψ are the solutions of

$$\tilde{b}_{11}\varphi_s + \tilde{b}_{12}\varphi_t = 1, \quad (4.56)$$

$$\tilde{b}_{11}\psi_s + \tilde{b}_{12}\psi_t = 0, \quad (4.57)$$

$$b_{33}\tau'(u) = 1. \quad (4.58)$$

A direct computation yields $\tilde{e}_1 = \partial_S, e_3 = \partial_U$. Therefore, we proved that we can redefine (s,t,u) so that (4.42a)-(4.42c) are satisfied for $b_{11} = \frac{1}{f}$, $b_{12} = 0$ and $b_{33} = 1$. Consequently, (4.40a) implies

$$[(1/f)\partial_s, b_{21}\partial_s + b_{22}\partial_t] = -\frac{\phi_2}{f}\partial_s, \quad (4.59)$$

from which we get (4.43). □

As a result of Lemma 4.3.2, we have

$$e_1 = \frac{1}{f(s,t,u)}\partial_s, \quad (4.60a)$$

$$e_2 = b_{21}(s,t,u)\partial_s + b_{22}(t,u)\partial_t, \quad (4.60b)$$

$$e_3 = \partial_u \quad (4.60c)$$

for some smooth functions f, b_{21}, b_{22}. Consequently, we have

$$k_1 = k_1(u), \qquad k_3 = k_3(u). \tag{4.61}$$

Since $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal frame field, the metric tensor g has the form

$$g = g_{12}(ds \otimes dt) + g_{22}dt^2 + du^2, \tag{4.62a}$$

where we put

$$g_{12} = \langle \partial_s, \partial_t \rangle = \frac{-f}{b_{22}}, \tag{4.62b}$$

$$g_{22} = \langle \partial_t, \partial_t \rangle = \frac{2fb_{21}}{b_{22}^2}. \tag{4.62c}$$

By a direct computation considering (4.62a), we get

$$\begin{array}{rcl} \nabla_{\partial_s}\partial_s & = & \Gamma_{ss}^s \partial_s, \\ \nabla_{\partial_s}\partial_t & = & \Gamma_{st}^s \partial_s + \Gamma_{st}^u \partial_u, \\ \nabla_{\partial_s}\partial_u & = & \Gamma_{su}^s \partial_s, \\ \nabla_{\partial_t}\partial_t & = & \Gamma_{tt}^s \partial_s + \Gamma_{tt}^t \partial_t + \Gamma_{tt}^u \partial_u, \\ \nabla_{\partial_t}\partial_u & = & \Gamma_{tu}^s \partial_s + \Gamma_{tu}^t \partial_t, \\ \nabla_{\partial_u}\partial_u & = & 0 \end{array} \tag{4.63a}$$

for some smooth functions Γ_{ij}^k, $i,j,k \in \{s,t,u\}$ such that

$$\Gamma_{ss}^s = \frac{f_s}{f}, \tag{4.63b}$$

$$\Gamma_{st}^s = -\frac{(b_{21})_s}{fb_{22}}, \tag{4.63c}$$

$$\Gamma_{st}^u = -\frac{(g_{12})_u}{2}, \tag{4.63d}$$

$$\Gamma_{su}^s = \Gamma_{tu}^t = \omega_{23}(e_1) = \frac{1}{2}\left(\frac{f_u}{f} - \frac{(b_{22})_u}{b_{22}}\right), \tag{4.63e}$$

$$\Gamma_{ut}^s = -\left(\frac{b_{21}}{b_{22}}\right)_u. \tag{4.63f}$$

Now, in order to get a local parametrization of M in a neighborhood of p, we consider an isometric immersion $x : (\Omega, g) \hookrightarrow \mathbb{E}_1^4$ with $x(\Omega) = M$ for some $\Omega \subset \mathbb{R}^3$. Because of $h(e_1, e_1) = 0$ and (4.60a), we have $h(\partial_s, \partial_s) = 0$. Therefore, (4.63a) and (4.63b) give

$$x_{ss} = \frac{f_s}{f} x_s. \tag{4.64}$$

On the other hand, by combining the equations (4.63e) and (4.35), we get

$$-\frac{3k_1'}{5k_1} = \frac{1}{2}\left(\frac{f_u}{f} - \frac{b_{22,u}}{b_{22}}\right) \tag{4.65}$$

from which we obtain

$$f(s,t,u) = k_1(u)^{\frac{-6}{5}} b_{22}(t,u)\gamma(s,t) \tag{4.66}$$

for a smooth function γ, where "$'$" denotes the ordinary derivative with respect to u. Note that (4.66) and (4.64) imply

$$x_{ss} = \frac{\gamma_s}{\gamma} x_s \tag{4.67}$$

which yields

$$x_s = \gamma c_1(t,u) \tag{4.68}$$

for a $\mathbb{E}_1^4$-valued function c_1.

Next, by combining (4.63a) and (4.63e) with $h(e_1,e_3) = 0$, we get

$$x_{su} = \frac{1}{2}\left(\frac{f_u}{f} - \frac{(b_{22})_u}{b_{22}}\right) x_s. \tag{4.69}$$

By taking into account (4.68), we consider (4.69) to obtain $c_1(t,u) = k_1^{\frac{-3}{5}} c_{11}(t)$. Consequently, (4.68) turns into

$$x_s = \gamma k_1^{\frac{-3}{5}} c_{11}(t). \tag{4.70}$$

By solving this equation, we obtain

$$x(s,t,u) = \tilde{\gamma} k_1(u)^{-\frac{3}{5}} c_{11}(t) + c_2(t,u) \tag{4.71}$$

for a smooth $\mathbb{R}^4$-valued function c_2, where we put $\tilde{\gamma} = \int \gamma ds$. Next, we define new parameters (s_1,s_2,s_3) with

$$s_1 = \tilde{\gamma}(s,t), \qquad s_2 = t, \qquad s_3 = c(u), \tag{4.72}$$

where we put $c(u) = k_1^{\frac{-3}{5}}$. By combining (4.71) with (4.72), we obtain the following result. Note that (4.72) implies

$$\begin{aligned} \partial_s &= \gamma \partial_{s_1}, \\ \partial_t &= \tilde{\gamma}_t \partial_{s_1} + \partial_{s_2}, \\ \partial_u &= c' \partial_{s_3}, \end{aligned} \tag{4.73}$$

and (4.71) turns into

$$x(s_1,s_2,s_3) = s_1 s_3 \alpha(s_2) + \beta(s_2,s_3) \tag{4.74}$$

for some smooth $\mathbb{E}_1^4$-valued functions α, β.

On the other hand, (4.62b), (4.66) and (4.72) imply

$$\langle \partial_s, \partial_t \rangle = -s_3^2 \gamma. \tag{4.75}$$

Moreover, from (4.73) the left hand side of (4.75) takes the form

$$\langle \partial_s, \partial_t \rangle = \gamma s_3 \langle \alpha, \beta_{s_2} \rangle. \tag{4.76}$$

Consequently, (4.75) implies

$$\langle \alpha, \beta_{s_2} \rangle = -s_3. \tag{4.77a}$$

By a further computation, considering (4.62a), (4.73) and (4.77a), we get

$$\begin{aligned} \langle \alpha, \alpha \rangle &= 0, && (4.77b)\\ \langle \alpha, \beta_{s_3} \rangle &= 0, && (4.77c)\\ \langle \alpha', \beta_{s_3} \rangle &= 1, && (4.77d)\\ \langle \beta_{s_2}, \beta_{s_3} \rangle &= 0, && (4.77e)\\ \langle \beta_{s_3}, \beta_{s_3} \rangle &= b(s_3)^2, && (4.77f) \end{aligned}$$

where we put $b(s_3) = \frac{1}{c'(u)}$ which is a non-vanishing smooth function.

By summing up all of the results, we obtain the following theorem:

Theorem 4.3.3. *Let M be a proper biconservative hypersurface in $\mathbb{E}^4_1$ with non-diagonalizable shape operator. Then, M has the local parametrization given by* (4.74) *for some smooth $\mathbb{E}^4_1$-valued functions α, β satisfying* (4.77).

Now, we want to get necessary and sufficient conditions that α, β satisfy in order the hypersurface constructed in Theorem 4.3.3 to be biconservative.

Let M be a hypersurface parametrized by (4.74) and assume that α, β satisfy the conditions given in (4.77) for a non-vanishing smooth function b. Then, by using (4.77), one can observe that the vector fields e_1, e_2, e_3 given by

$$e_1 = \partial_{s_1}, \qquad e_2 = b_{21}\partial_{s_1} + \frac{1}{s_3^2}\partial_{s_2}, \qquad e_3 = \frac{1}{b}\partial_{s_3} \tag{4.78}$$

form a pseudo-orthonormal base for the tangent bundle of M and the unit normal vector field of M is

$$N = \frac{1}{\sqrt{a^2 - b'^2}} \left(-\frac{b'}{b} (s_1 \alpha + \beta_{s_3}) + \beta_{s_3 s_3} \right), \tag{4.79}$$

where b_{21} is a smooth function and

$$a = \langle \beta_{s_3s_3}, \beta_{s_3s_3} \rangle. \tag{4.80}$$

A direct computation yields $h(e_1, e_1) = h(e_1, e_3) = 0$ and

$$\tilde{\nabla}_{e_3} e_3 = \frac{\sqrt{a^2 - b'^2}}{b^2} N. \tag{4.81}$$

Moreover, the equation $h(e_2, e_3) = 0$ is satisfied if and only if $(s_1\alpha' + \beta_{23})^\perp = 0$ which is equivalent to

$$\langle \beta_{s_3s_3}, \beta_{s_2s_3} \rangle = 0. \tag{4.82}$$

By a further computation considering (4.78) and (4.79), we get

$$Ae_1 = -\tilde{\nabla}_{e_1} N = \frac{b'}{bs_3\sqrt{a^2 - b'^2}} e_1. \tag{4.83}$$

Now, (4.81) and (4.83) yield that under the condition (4.82), M has non-diagolizable shape operator and its principle directions are e_1 and e_3 with the corresponding principle curvatures

$$k_1 = \frac{b'}{bs_3\sqrt{a^2 - b'^2}}, \qquad k_3 = \frac{\sqrt{a^2 - b'^2}}{b^2}, \tag{4.84}$$

respectively. Hence, M is biconservative if and only if $2k_1 + 3k_3 = 0$ and $k_3 = k_3(s_3)$ which is equivalent to

$$\langle \beta_{s_3s_3}, \beta_{s_3s_3} \rangle = b'^2 + \frac{2b'b}{3s_3} \tag{4.85}$$

because of (4.80) and (4.84).

By summing up these results, we get the following:

Proposition 4.3.4. *M is a proper biconservative with non-diagonalizable shape operator if and only if it can be locally parametrized by* (4.74) *for some smooth $\mathbb{E}^4_1$-valued functions α, β satisfying* (4.77)*,* (4.82) *and* (4.85).

Next, we would like to construct an example of biconservative hypersurfaces with a non-diagonalizable shape operator.

Proposition 4.3.5. *Let $\zeta : I \to \mathbb{E}^4_1$ be a null curve which admits a pseudo-orthonormal base field $\{T, U; \alpha_1, \alpha_2\}$ of vector fields defined on ζ such that*

$$\begin{aligned} \langle U,U\rangle = \langle T,T\rangle &= 0, \quad \langle T,U\rangle = -1, && (4.86a)\\ \zeta' &= T, && (4.86b)\\ \alpha_1' &= A_3 T, && (4.86c)\\ \alpha_2' &= A_2 U + A_4 T, && (4.86d) \end{aligned}$$

where I is an open interval and $A_2, A_3, A_4 : I \to \mathbb{R}$ are some smooth functions. Then, the hypersurface M in $\mathbb{E}^4_1$ parametrized by

$$x(s,u,w) = \zeta(s) + uT(s) + w\alpha_1(s) + f(w)\alpha_2(s), \qquad w \in I,\ (u,s) \in \Omega \tag{4.87}$$

has a non-diagonalizable shape operator and it is proper biconservative if and only if f is a smooth function satisfying

$$-3f(w)f''(w) + 2f'(w)^2 + 2 = 0, \tag{4.88}$$

where Ω is an open subset in $\mathbb{R}^2$.

Proof. By using (4.86), we observe that the equations

$$\begin{aligned} U' &= A_3\alpha_1 + A_4\alpha_2 - A_1 U, && (4.89a)\\ T' &= A_2\alpha_2 + A_1 T && (4.89b) \end{aligned}$$

are satisfied for a smooth function A_1. By a direct computation we obtain that the vector fields e_1, e_2, e_3 defined by

$$e_1 = \partial_u, \tag{4.90}$$

$$e_2 = -\frac{2uA_1 f + 2wA_3 f + 2A_4 f^2 + u^2(-A_2) + 2f}{2A_2 f^2}\partial_u + \frac{1}{A_2 f}\partial_s, \tag{4.91}$$

$$e_3 = -\frac{uf'}{f\sqrt{f'^2+1}}\partial_u - \frac{1}{\sqrt{f'^2+1}}\partial_w \tag{4.92}$$

form a pseudo-orthonormal base field of the tangent bundle of M, and the unit normal vector field of M is

$$N = -\frac{u}{f\sqrt{f'^2+1}}T + \frac{f'}{\sqrt{f'^2+1}}\alpha_1 - \frac{1}{\sqrt{f'^2+1}}\alpha_2. \tag{4.93}$$

Moreover, the matrix representation of the shape operator A of M with respect to the base $\{e_1,e_2,e_3\}$ is

$$\begin{bmatrix} \frac{1}{f(w)\sqrt{f'(w)^2+1}} & -\frac{A_3(s)(w+f(w)f'(w))+1}{f(w)^2A_2(s)\sqrt{f'(w)^2+1}} & 0 \\ 0 & \frac{1}{f(w)\sqrt{f'(w)^2+1}} & 0 \\ 0 & 0 & -\frac{f''(w)}{\left(f'(w)^2+1\right)^{3/2}} \end{bmatrix}. \tag{4.94}$$

Therefore, M has a non-diagonalizable shape operator with the principle curvatures

$$k_1 = \frac{1}{f(w)\sqrt{f'(w)^2+1}}, \tag{4.95}$$

$$k_3 = -\frac{f''(w)}{(f'(w)^2+1)^{3/2}} \tag{4.96}$$

and the vector field e_3 is a principle direction of M. Because $e_1(k_1)=e_1(k_3)=e_2(k_1)=e_2(k_3)=0$, e_3 is proportional to grad$H$. Therefore, M is biconservative if and only if $2k_1+3k_3=0$ which is equivalent to (4.88). □

Remark 4.3.6. By redefine the parameter w properly, one can assume $A_1=1$ in (4.89). Moreover, given $s_0\in I$, smooth functions $A_i: I\to\mathbb{R}$, $i=1,2,3,4$, a point $p\in\mathbb{R}^4$ and a pseudo-orthonormal base $\{T^0,U^0,{\alpha_1}^0,{\alpha_2}^0\}$ of $\mathbb{E}^4_1$, there exists a unique null curve ζ passing through p at s_0 satisfying the condition described by (4.86), and

$$T(s_0)=T^0,\quad U(s_0)=U^0,\quad \alpha_1(s_0)={\alpha_1}^0,\quad \alpha_2(s_0)={\alpha_2}^0. \tag{4.97}$$

Remark 4.3.7. A member M of the family of hypersurfaces given by (4.86) and (4.87) might be considered as a generalization of null-scrolls, Weingarten surfaces or rotational surfaces in the Minkowski 4-space. Note that if f is a non-zero constant, then M becomes an isoparametric hypersurface expressed in [22, Example 3.2].

Now, we are ready to prove the main result.

Theorem 4.3.8. *M is poper biconservative hypersurface in $\mathbb{E}^4_1$ with a non-diagonalizable shape operator if and only if it is locally congruent to the hypersurface described in Proposition 4.3.5.*

Proof. The sufficient condition has already proved and we are going to prove the necessary condition. Assume that M is biconservative and its shape operator is non-diagonalizable. In this case, Proposition 4.3.4 implies that M can be parametrized by (4.74) for some α, β and we have (4.77), (4.82) and (4.85). Moreover, the principle curvatures of M are given by (4.84).

First, we claim that the vectors $\beta_{s_3}(s_2,s_3), \beta_{s_3s_3}(s_2,s_3), \beta_{s_3s_3s_3}(s_2,s_3)$ are linearly dependent for any s_2, s_3. Note that the vectors $\alpha(s_2)$ and $\beta_{s_2}(s_2,s_3)$ are linearly independent by (4.77a) and (4.77b). So, the subspace $V_{s_2,s_3} = (\operatorname{span}\{\alpha(s_2), \beta_{s_2}(s_2,s_3)\})^{\perp}$ of $\mathbb{E}^4_1$ has dimension 2. By considering (4.77), (4.82) and (4.85) we observe that

$$\beta_{s_3}(s_2,s_3), \beta_{s_3s_3}(s_2,s_3), \beta_{s_3s_3s_3}(s_2,s_3) \in V_{s_2,s_3}.$$

This proves our first claim. Therefore, we have

$$A\beta_{s_3}(s_2,s_3) + B\beta_{s_3s_3}(s_2,s_3) + C\beta_{s_3s_3s_3}(s_2,s_3) = 0 \tag{4.98}$$

for some $A, B, C \in \mathbb{R}$. However, getting inner product of both sides of this equation with $\alpha'(s_2)$ and using (4.77d) give $A = 0$. Therefore, from (4.82) and (4.85) we have

$$\beta_{s_3s_3s_3} = \tilde{f}(s_3)\beta_{s_3s_3} \tag{4.99}$$

for a smooth function $\tilde{f}$. By solving this equation, we obtain

$$\beta(s_2,s_3) = \zeta(s_2) + f(s_3)\alpha_1(s_2) + s_3\alpha_2(s_2) \tag{4.100}$$

for some $\mathbb{E}^4_1$-valued smooth functions $\zeta, \alpha_1, \alpha_2$ and a function f. Note that because of (4.77f), (4.84) and (4.85), $\alpha_1(s_2)$ and $\alpha_2(s_2)$ are non-zero vectors for any s_2.

By combining (4.100) with (4.77), (4.82) and (4.85) we obtain

$$\begin{aligned}
\langle \alpha_i, \alpha_j \rangle &= c_{ij}, \qquad i,j = 1,2, && \text{(4.101a)}\\
\langle \alpha_1, \alpha \rangle = \langle \alpha_2, \alpha \rangle &= 0, && \text{(4.101b)}\\
\langle \alpha_1', \alpha_2 \rangle = \langle \alpha_1, \alpha_2' \rangle &= 0, && \text{(4.101c)}\\
\langle \alpha_1', \alpha \rangle &= 0, && \text{(4.101d)}\\
\langle \alpha_2', \alpha \rangle &= -1, && \text{(4.101e)}\\
\langle \zeta', \alpha \rangle = \langle \zeta', \alpha_1 \rangle = \langle \zeta', \alpha_2 \rangle &= 0 && \text{(4.101f)}
\end{aligned}$$

for some constants c_{ij}.

Now, we claim that

$$\zeta'(s_2) = \xi_1(s_2)\alpha(s_2) + \xi_2(s_2)\alpha_1(s_2) + \xi_3(s_2)\alpha_2(s_2) \tag{4.102}$$

for some smooth functions ξ_1, ξ_2, ξ_3. Note that because of (4.101a), (4.101c)-(4.101e), getting inner product of the equation

$$c_1\alpha(s_2) + c_2\alpha_1(s_2) + c_3\alpha_2(s_2) = 0 \tag{4.103}$$

with α_2 and α' yields $c_2 = 0$ and $c_3 = 0$ and with α_2' yields $c_1 = 0$, respectively. Therefore, the vectors $\alpha(s_2), \alpha_1(s_2), \alpha_2(s_2)$ are linearly independent for any s_2. On the other hand, by (4.101b),(4.101f) and (4.77b), we have

$$\{\alpha, \alpha_1, \alpha_2, \tau'\} \in (\mathrm{span}\{\alpha\})^{\perp}. \tag{4.104}$$

Note that $(\mathrm{span}\{\alpha\})^{\perp}$ has dimension 3, because of (4.77a). Therefore, (4.104) yields that $\alpha, \alpha_1, \alpha_2, \tau'$ are linearly dependent.

Therefore, we prove (4.102). Consequently, ζ' is a null-vector by (4.101f) and (4.102), i.e., ζ is a null curve and

$$\alpha(s_2) = \xi(s_2)\zeta'(s_2) \tag{4.105}$$

for a function ξ because of (4.101f). By combining (4.100) and (4.105) with (4.74), we get

$$x(s_1, s_2, s_3) = \zeta(s_2) + s_1 s_3 \xi(s_2)\zeta'(s_2) + f(s_3)\alpha_1(s_2) + s_3\alpha_2(s_2).$$

By defining new coordinates s, u, w by $u = s_1 s_3 \xi(s_2)$, $s = s_2$ and $s_3 = w$ and letting $\zeta' = T$, we obtain that M can be parametrized as given in (4.87). On the other hand, because of (4.101a), one may redefine the coordinate s and the vector fields α_1 and α_2 to be orthonormal vector fields in $T^{\perp}\zeta$ such that (4.101) is still satisfied. Next, let U be the null vector field on ζ defined by

$$\langle T, U \rangle = -1, \qquad \langle \alpha_1, U \rangle = \langle \alpha_2, U \rangle = 0. \tag{4.106}$$

By considering (4.101) again, we obtain that the pseudo-orthonormal base field $\{T, U, \alpha_1, \alpha_2\}$ satisfies the conditions given in (4.86). $\square$

5. HYPERSURFACES IN MINKOWSKI 5-SPACE

In this section, we consider biconservative hypersurfaces with non-diagonalizable shape operator and at most two distinct priciple curvatures in the Minkowski 5-space. We want to note that all results obtained in this section are published in [30].

5.1 The Form of Shape operators

Let M be a proper biconservative Lorentzian hypersurface in the Minkowski space $\mathbb{E}^5_1$. Then, the biconservativity equation turns into

$$A(\operatorname{grad} H) = -2H \operatorname{grad} H, \tag{5.1}$$

where H is the mean curvature function. Therefore, $\operatorname{grad} H$ is a principal direction of M with the corresponding principle curvature $-2H$.

Now, assume that A is non-diagonalizable and it has at most two distinct principle curvatures. By considering the canonical forms of A given in Lemma 2.3.12 and (5.1), we observe that there are two cases subject to the casuality of $\operatorname{grad} H$:

Case (1). $\operatorname{grad} H$ is a null vector. In this case, we have two subcases: If A has the form given in Case (ii) of Lemma 2.3.12, then

$$A = \begin{bmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & -2H & 0 \\ 0 & 0 & 0 & 10H \end{bmatrix}, \tag{5.2}$$

where $\operatorname{grad} H$ is proportional to e_1. On the other hand, if A has the form given in Case (iii) of Lemma 2.3.12, then $\operatorname{grad} H$ is proportional to e_2 and we have

$$A = \begin{bmatrix} -2H & 0 & 0 & 0 \\ 0 & -2H & 1 & 0 \\ -1 & 0 & -2H & 0 \\ 0 & 0 & 0 & 10H \end{bmatrix}. \tag{5.3}$$

Case (2). $\operatorname{grad} H$ is a space-like vector. In this case, e_4 can be chosen to be proportional to $\operatorname{grad} H$ and, similar to the previous case, we have the subcases

$$A = \begin{bmatrix} 2H & 0 & 0 & 0 \\ 0 & 2H & 1 & 0 \\ -1 & 0 & 2H & 0 \\ 0 & 0 & 0 & -2H \end{bmatrix} \tag{5.4}$$

and

$$A \sim \begin{bmatrix} 2H & 1 & 0 & 0 \\ 0 & 2H & 0 & 0 \\ 0 & 0 & 2H & 0 \\ 0 & 0 & 0 & -2H \end{bmatrix}. \tag{5.5}$$

Note that in all cases above vector fields e_1, e_2, e_3, e_4 form a pseudo-orthonormal base for TM. Consequently, the Levi-Civita connection of M has the form

$$\nabla_{e_i} e_1 = \phi_i e_1 + \omega_{13}(e_i)e_3 + \omega_{14}(e_i)e_4 \tag{5.6a}$$

$$\nabla_{e_i} e_2 = -\phi_i e_2 + \omega_{23}(e_i)e_3 + \omega_{24}(e_i)e_4 \tag{5.6b}$$

$$\nabla_{e_i} e_3 = \omega_{23}(e_i)e_1 + \omega_{13}(e_i)e_2 + \omega_{34}(e_i)e_4 \tag{5.6c}$$

$$\nabla_{e_i} e_4 = \omega_{24}(e_i)e_1 + \omega_{14}(e_i)e_2 - \omega_{34}(e_i)e_3 \tag{5.6d}$$

for $i = 1,2,3,4$, where we define the connection forms ω_{jk} with $\omega_{jk}(e_i) = \langle \nabla_{e_i} e_j, e_k \rangle$ and, for simplicity, we put $\phi_i = -\omega_{12}(e_i)$.

5.2 $\operatorname{grad} H$ is lightlike

In the following theorems, we prove that the Case (1) described above is not possible.

Theorem 5.2.1. *There is a no proper biconservative hypersurface in $\mathbb{E}^5_1$ with the shape operator given by* (5.3).

Proof. First, towards contradiction we assume that A has the matrix representation given in (5.3) and e_2 is proportional to $\operatorname{grad} H$. In this case, we have

$$e_2(H) = e_3(H) = e_4(H) = 0, \qquad e_1(H) \neq 0 \tag{5.7}$$

and note that the second fundamental form of the hypersurface M satisfies

$$h(e_1,e_3) = -N, \quad h(e_1,e_2) = 2HN, \quad h(e_3,e_3) = -2HN, \quad h(e_4,e_4) = 10HN \tag{5.8}$$

and $h(e_i, e_j) = 0$ for the remaining couple of (i, j).

We use the Codazzi equation (2.28) for $X = e_i$, $Y = e_j$ and $Z = e_k$ for each triple (i, j, k). The triple $(2,4,4),(3,4,4),(2,1,3),(1,2,2),(4,2,2),(3,4,2)$ and $(1,4,2)$ imply

$$\omega_{24}(e_4) = \omega_{34}(e_4) = \phi_2 = \omega_{23}(e_2) = \omega_{24}(e_2) = \omega_{24}(e_1) = 0 \tag{5.9}$$

from which we obtain the Levi-Civita connection of M as

$$\nabla_{e_4} e_4 = \omega_{14}(e_4)e_2, \tag{5.10a}$$

$$\nabla_{e_1} e_4 = \omega_{14}(e_1)e_2 - \omega_{34}(e_1)e_3, \tag{5.10b}$$

$$\nabla_{e_2} e_2 = 0, \tag{5.10c}$$

$$\nabla_{e_1} e_2 = -\phi_1 e_2 + \omega_{23}(e_1)e_3. \tag{5.10d}$$

So it follows from the Gauss equation

$$R(e_1, e_4, e_4, e_2) = 20H^2. \tag{5.11}$$

But, by replacing (5.10) into (5.11) we get $H = 0$ which is a contradiction. Therefore the proof is completed. □

Theorem 5.2.2. *There is a no proper biconservative hypersurface in* $\mathbb{E}^5_1$ *with the shape operator given by* (5.2).

Proof. Assume that A has the matrix representation given in (5.2), where e_1 is proportional to $\operatorname{grad} H$. In this case, we have

$$e_1(H) = e_3(H) = e_4(H) = 0, \qquad e_2(H) \neq 0. \tag{5.12}$$

and note that the second fundamental form of the hypersurface M satisfies

$$h(e_1, e_2) = 2HN = -h(e_3, e_3), \quad h(e_2, e_2) = -N, \quad h(e_4, e_4) = 10HN \tag{5.13}$$

and $h(e_i, e_j) = 0$ for the remaining couple of (i, j).

First, we use the Codazzi equation (2.28) for $X = e_i$, $Y = e_j$ and $Z = e_k$ for each triple in (i, j, k). The triple $(1,2,2),(2,4,4),(4,1,1),(1,4,2),(4,1,3)$ imply

$$e_2(H) = -\phi_1 \tag{5.14}$$

$$e_2(H) = \frac{-6}{5} H \omega_{24}(e_4) \tag{5.15}$$

$$\omega_{14}(e_1) = \omega_{24}(e_1) = \omega_{34}(e_1) = \omega_{14}(e_4) = 0 \tag{5.16}$$

where $\phi_1 = -\omega_{12}(e_i)$. By the (5.12) one can obtain $[e_1, e_2](H) = e_1 e_2(H)$. So, we have

$$e_1 e_2(H) = [e_1, e_2](H) = (\nabla_{e_1} e_2 - \nabla_{e_2} e_1)(H) = -\phi_1 e_2(H) \tag{5.17}$$

because of (5.12), (5.14) and (5.16).

By taking derivative of (5.15) along e_1 and replacing (5.17), we get

$$e_1(\omega_{24}(e_4)) = -\phi_1 \omega_{24}(e_4). \tag{5.18}$$

Note that the Gauss Equation (2.26) for $X = e_4, Y = e_1, Z = e_2, W = e_4$ gives

$$R(e_4, e_1, e_2, e_4) = -20H^2. \tag{5.19}$$

Substituting (5.16) and (5.18) into (5.19) gives

$$0 = 20H^2. \tag{5.20}$$

However, the equation (5.20) yields a contradiction. Hence, the proof is completed. □

5.3 grad H is spacelike

In this subsection, we consider biconservative hypersurfaces of $\mathbb{E}^5_1$ whose shape operator A has the form given in (5.4). In this case, the shape operator has the form

$$A(e_1) = 2He_1 - e_3,\ A(e_2) = 2He_2,\ A(e_3) = e_2 + 2He_3,\ A(e_4) = -2He_4. \tag{5.21}$$

Moreover, we have

$$e_4 = \frac{\operatorname{grad} H}{\| \operatorname{grad} H \|} \tag{5.22}$$

because the vector field e_4 is proportional to grad H.

On the other hand, the second fundamental form of M is

$$h(e_1, e_1) = h(e_2, e_2) = h(e_1, e_4) = h(e_3, e_4) = h(e_2, e_3) = h(e_2, e_4) = 0 \tag{5.23}$$

$$h(e_1, e_2) = -2HN = -h(e_3, e_3),\ h(e_1, e_3) = -N,\ h(e_4, e_4) = -2HN. \tag{5.24}$$

Thus, we obtain that

$$h(e_1, e_2) = h(e_4, e_4) = -h(e_3, e_3).$$

First, we apply the Codazzi equation (2.28) for $X = e_i$, $Y = e_j$, $Z = e_k$ for each triple (i, j, k).

• The triple $(4,3,3)$ implies

$$e_4(2H) + 2\omega_{23}(e_4) = \omega_{24}(e_3) + 2H\omega_{34}(e_3). \tag{5.25}$$

• The triple $(2,4,4)$ gives

$$\omega_{24}(e_4) = 0. \tag{5.26}$$

• The triple $(3,4,4)$ implies

$$\omega_{34}(e_4) = 0. \tag{5.27}$$

• From the triple $(1,4,4)$, we get

$$4H\omega_{14}(e_4) = \omega_{34}(e_4). \tag{5.28}$$

So, (5.27) implies

$$\omega_{14}(e_4) = 0. \tag{5.29}$$

• The triple $(1,2,3)$ gives

$$\phi_2 = 0. \tag{5.30}$$

• The triple $(4,2,3)$ gives

$$\omega_{34}(e_2) = 0. \tag{5.31}$$

• The triple $(3,4,2)$ gives

$$\omega_{34}(e_3) = 0. \tag{5.32}$$

• The triple $(2,3,4)$ gives

$$\omega_{34}(e_2) = \omega_{24}(e_3). \tag{5.33}$$

So, (5.32) implies

$$\omega_{24}(e_3) = 0. \tag{5.34}$$

• The triple $(1,2,4)$ implies

$$-4H(\omega_{14}(e_2) - \omega_{24}(e_1)) = \omega_{34}(e_2). \tag{5.35}$$

So from (5.31) we have

$$\omega_{14}(e_2) = \omega_{24}(e_1). \tag{5.36}$$

• The triple $(4,1,2)$ gives

$$-e_4(2H)+\omega_{23}(e_4)=4H\omega_{24}(e_1). \tag{5.37}$$

so by replacing (5.25),(5.32) and (5.34) we have

$$\omega_{23}(e_4)=\frac{4H}{3}\omega_{24}(e_1). \tag{5.38}$$

• The triple $(2,4,1)$ gives

$$-e_4(2H)+\omega_{23}(e_4)=4H\omega_{14}(e_2)-\omega_{34}(e_2). \tag{5.39}$$

So, by combining (5.31) and (5.38) with (5.39) we obtain

$$e_4(H)=-\omega_{23}(e_4). \tag{5.40}$$

• The triple $(4,2,4)$ with (5.26)

$$\omega_{24}(e_2)=0. \tag{5.41}$$

By summing up all results that we have found so far, we obtain the following lemma:

Lemma 5.3.1. *Let M be a proper biconservative hypersurface in $\mathbb{E}^5_1$ whose shape operator A has the form given in* (5.4)*. Then, the Levi-Civita connection of M satisfies*

$$\begin{array}{rclrcl}
\nabla_{e_1}e_4 &=& \omega_{24}(e_1)e_1+\omega_{14}(e_1)e_2-\omega_{34}(e_1)e_3, & \nabla_{e_2}e_4 &=& \omega_{24}(e_1)e_2,\\
\nabla_{e_3}e_4 &=& \omega_{14}(e_3)e_2, & \nabla_{e_4}e_2 &=& \omega_{23}(e_4)e_3,\\
\nabla_{e_4}e_3 &=& \omega_{23}(e_4)e_1+\omega_{13}(e_4)e_2, & \nabla_{e_4}e_4 &=& 0.
\end{array}$$

Next, by considering Lemma 5.3.1, we obtain

$$\begin{aligned}
R(e_3,e_4)e_4 &= \nabla_{e_3}\nabla_{e_4}e_4-\nabla_{e_4}\nabla_{e_3}e_4-\nabla_{[e_3,e_4]}e_4\\
&= -e_4(\omega_{14}(e_3))e_2-\omega_{14}(e_3)\nabla_{e_4}e_2+\omega_{23}(e_4)\nabla_{e_1}e_4\\
&\quad -(-\omega_{13}(e_4)+\omega_{14}(e_3))\nabla_{e_2}e_4.
\end{aligned} \tag{5.42}$$

On the other hand, the Gauss Equation (2.26) for $X=e_3, Y=Z=e_4, W=e_2$ implies

$$\langle R(e_3,e_4)e_4,e_2\rangle=-h(e_3,e_4)h(e_4,e_2)+h(e_3,e_2)h(e_4,e_4)=0. \tag{5.43}$$

Therefore, by getting inner product of both sides of (5.42) with e_2 and by using (5.26) and (5.41) we have

$$\omega_{23}(e_4)\omega_{24}(e_1)=0. \tag{5.44}$$

Substituting (5.38) into (5.44), we obtain $\omega_{23}(e_4) = 0$. Finally, by (5.40) we have H is a constant. However, this yields a contradiction because we assume that H is not constant. So, we have the following non-existence result.

Theorem 5.3.2. *There is a no proper biconservative hypersurface in* $\mathbb{E}^5_1$ *with the shape operator given by* (5.4).

6. SUBMANIFOLDS IN MINKOWSKI SPACE IN ARBITRARY DIMENSION

In this section, we mainly focus on submanifolds with mean curvature whose gradient is light-like. In section 6.1, we study hypersurfaces of $\mathbb{E}^{n+1}_1$.

Definition 6.0.1. Let M be a submanifold of the Minkowski space $\mathbb{E}^{n+1}_1$. M is said to be MCGL if grad $\|H\|$ is light-like.

Before we proceed, we would like to refer to [31] for a study of biharmonic MCGL hypersurfaces.

6.1 Biconservative Hypersurfaces with Mean Curvature Function Whose Gradient is Light-like in $\mathbb{E}^{n+1}_1$

Let M be a biconservative hypersurface in $\mathbb{E}^{n+1}_1$ with a non-diagonlaziable hypersurface. Then, by Lemma 2.3.12, there exists a pseudo-orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of the tangent bundle of M such that the shape operator A has the matrix representation of either

$$\text{Case 1: } A \sim \begin{bmatrix} k_1 & 1 & & & & \\ 0 & k_1 & & & & \\ & & k_3 & & & \\ & & & k_4 & & \\ & & & & \ddots & \\ & & & & & k_n \end{bmatrix}, \tag{6.1}$$

or

$$\text{Case 2: } A \sim \begin{bmatrix} k_1 & 0 & 0 & & & \\ 0 & k_1 & 1 & & & \\ -1 & 0 & k_1 & & & \\ & & & k_4 & & \\ & & & & \ddots & \\ & & & & & k_n \end{bmatrix}. \tag{6.2}$$

Note that the Levi-Civita connection of ∇ of M satisfies

$$\nabla_{e_i} e_1 = \phi_i e_1 + \sum_{\beta=3}^{n} \omega_{1\beta}(e_i) e_\beta, \tag{6.3}$$

$$\nabla_{e_i} e_2 = -\phi_i e_2 + \sum_{\beta=3}^{n} \omega_{2\beta}(e_i) e_\beta, \tag{6.4}$$

$$\nabla_{e_i} e_\alpha = \omega_{2\alpha}(e_i) e_1 + \omega_{1\alpha}(e_i) e_2 + \sum_{\beta=3}^{n} \omega_{\alpha\beta}(e_i) e_\beta, \tag{6.5}$$

where $\phi_i = \langle \nabla_{e_i} e_2, e_1 \rangle = -\omega_{12}(e_i)$ and $k_1, k_2, \ldots, k_n$ are smooth functions and $\alpha, \beta = 3, 4, \ldots, n$.

Remark 6.1.1. Note that if the matrix representation of A has the form given (6.1), then the principal curvatures of M satisfies

$$2k_1 + k_3 + \cdots + k_n = nH. \tag{6.6}$$

In the other case, the equation

$$3k_1 + k_4 + \cdots + k_n = nH \tag{6.7}$$

holds. We are going to study these two cases, seperately.

Now we study hypersurfaces with the shape operator given by (6.1). In this case, the second fundamental form of M satisfies

$$h(e_1, e_2) = -k_1 N, \qquad h(e_2, e_2) = -N, \qquad h(e_\alpha, e_\alpha) = k_\alpha N \tag{6.8}$$

and

$$h(e_1, e_1) = h(e_1, e_\alpha) = h(e_\alpha, e_\beta) = 0 \tag{6.9}$$

for $\alpha \neq \beta$. Now, M is biconservative, the biconservativity equation (2.58) gives

$$A(\nabla H) = -\frac{nH}{2} \nabla H. \tag{6.10}$$

Thus one can choose ∇H to the propotional to light-like vector e_1 and $-2k_1 = nH$ where H denotes the mean curvature of M. Recall that the definition of gradient of H, i.e., ∇H, with respect to pseudo-orthonormal base $\{e_1, e_2, \ldots, e_n\}$ such that

$$\nabla H = -e_2(H) e_1 - e_1(H) e_2 + \sum_{\alpha=3}^{n} e_\alpha(H) e_\alpha. \tag{6.11}$$

So, we have

$$e_2(k_1) \neq 0 \qquad e_1(k_1) = e_\alpha(k_1) = 0, \qquad \alpha = 3, 4, \ldots, n. \tag{6.12}$$

Now, we use the Codazzi equation (2.28).

• The triple $(e_1, e_\alpha, e_\alpha)$ implies

$$e_1(k_\alpha) = \psi_\alpha(k_1 - k_\alpha), \tag{6.13}$$

where we put $\psi_\alpha = \omega_{1\alpha}(e_\alpha)$.

• The triple $(e_2, e_\alpha, e_\alpha)$ gives

$$e_2(k_\alpha) = \Phi_\alpha(k_1 - k_\alpha) + \psi_\alpha, \tag{6.14}$$

where we put $\Phi_\alpha = \omega_{2\alpha}(e_\alpha)$.

• The triple $(e_\alpha, e_1, e_1), (e_\alpha, e_1, e_2)$ and (e_α, e_2, e_1) imply

$$\omega_{1\alpha}(e_1) = \omega_{2\alpha}(e_1) = \omega_{1\alpha}(e_2) = 0. \tag{6.15}$$

• The triple (e_1, e_α, e_β) and (e_α, e_β, e_1) imply

$$\omega_{\alpha\beta}(e_1)(k_\alpha - k_\beta) = \omega_{1\beta}(e_\alpha)(k_1 - k_\beta) = \omega_{1\alpha}(e_\beta)(k_1 - k_\alpha), \tag{6.16}$$

and $[e_\alpha, e_\beta](k_1) = 0$ from which we have

$$\omega_{1\alpha}(e_\beta) = \omega_{1\beta}(e_\alpha). \tag{6.17}$$

Because $k_1 \neq k_\alpha$, we have

$$\omega_{\alpha\beta}(e_1) = \omega_{1\beta}(e_\alpha) = \omega_{1\alpha}(e_\beta) = 0. \tag{6.18}$$

• The triple (e_1, e_2, e_2) gives

$$e_2(k_1) = 2\phi_1. \tag{6.19}$$

Morever; we have

$$[e_1, e_2](k_1) = e_1 e_2(k_1) \Rightarrow e_1(\phi_1) = -\phi_1^2. \tag{6.20}$$

Now we use the Gauss Equations

$$R(e_\alpha, e_1, e_2, e_\alpha) = h(a_\alpha, e_\alpha)h(e_1, e_2) \tag{6.21}$$

to obtain

$$\langle \nabla_{e_\alpha}\nabla_{e_1}e_2 - \nabla_{e_1}\nabla_{e_\alpha}e_2 - \nabla_{[e_\alpha,e_1]}e_2, e_\alpha\rangle = k_1 k_\alpha. \tag{6.22}$$

Next, we compute the left hand side of the equation (6.22) by using the equalities (6.15) and (6.18). Note that we have

$$\langle \nabla_{e_\alpha}\nabla_{e_1}e_2, e_\alpha\rangle = \langle \nabla_{e_\alpha}(-\phi_1 e_2), e_\alpha\rangle = -\phi_1 \Phi_\alpha, \tag{6.23}$$

$$\langle \nabla_{e_1}\nabla_{e_\alpha}e_2, e_\alpha\rangle = e_1(\Phi_\alpha). \tag{6.24}$$

Furthermore, the equation $[e_\alpha, e_1] = \nabla_{e_\alpha}e_1$ gives

$$\langle \nabla_{[e_\alpha,e_1]}e_2, e_\alpha\rangle = \langle \nabla_{(\phi_\alpha e_1 + \psi_\alpha e_\alpha)}e_2, e_\alpha\rangle = \psi_\alpha \Phi_\alpha. \tag{6.25}$$

By putting (6.23), (6.24), (6.25) into (6.22) we get

$$e_1(\Phi_\alpha) + \Phi_\alpha(\phi_1 + \psi_\alpha) = k_1 k_\alpha. \tag{6.26}$$

Now, the Gauss equation implies

$$R(e_\alpha, e_1, e_1, e_\alpha) = 0 \tag{6.27}$$

due to (6.1). Therefore, we have

$$\langle \nabla_{e_\alpha}\nabla_{e_1}e_1 - \nabla_{e_1}\nabla_{e_\alpha}e_1 - \nabla_{[e_\alpha,e_1]}e_1, e_\alpha\rangle = 0. \tag{6.28}$$

Direct computations for the left hand side of the equation (6.28) by using equalities (6.15) and (6.18) gives us

$$e_1(\psi_\alpha) = \psi_\alpha(\phi_1 - \psi_\alpha). \tag{6.29}$$

We are going to use the following lemmas:

Lemma 6.1.2. *Assume that $k_1 \neq k_\alpha$ for some $\alpha = 3,4,\ldots,n$. Then $e_1(k_\alpha) = 0$ implies $k_\alpha = 0$.*

Proof. Let $e_1(k_\alpha) = 0$ then $\psi_\alpha = 0$ from (6.13). So (6.14) becomes

$$e_2(k_\alpha) = \Phi_\alpha(k_1 - k_\alpha) \tag{6.30}$$

we have $\nabla_{e_2}e_1(k_\alpha) = 0$. Since $\omega_{2\alpha}(e_1) = 0$,from (6.15) we have

$$\begin{aligned} [e_1, e_2](k_\alpha) &= \nabla_{e_1}e_2(k_\alpha) \\ &= -\phi_1 e_2(k_\alpha) \end{aligned} \tag{6.31}$$

Moreover, by taking the derivative of both side of (6.30) along e_1, we get

$$-\phi_1 e_2(k_\alpha) = e_1(\Phi_\alpha)(k_1 - k_\alpha). \tag{6.32}$$

Subsituiting (6.30) into the equation above and by simplifying with $k_1 - k_\alpha$, because $k_1 \neq k_\alpha$, we get

$$-\phi_1 \Phi_\alpha = e_1(\Phi_\alpha). \tag{6.33}$$

So, by (6.26) we have $k_1 k_\alpha = 0$ which yields $k_\alpha = 0$ since $\nabla H = e_2(k_1)e_1$ is light-like so it cannot be vanish. Thus k_1 cannot be zero. □

Remark 6.1.3. We want to notice that if every $\psi_\alpha = 0$ for all $\alpha = 3,4,\dots,n$, then (6.6) gives $k_1 = 0$ which is a contradiction.

Lemma 6.1.4. *We have* grad $\left(\frac{k_\alpha}{k_1}\right) \neq 0$ *for all* $\alpha = 3,4,\dots,n$.

Proof. Assume that $k_1 = \lambda k_\alpha$ for some $\lambda \in \mathbb{R} - \{1\}$. Then, we have

$$0 = \lambda e_1(k_\alpha). \tag{6.34}$$

So, $e_1(k_\alpha) = 0$ and this means $k_\alpha = 0$ by Lemma 6.1.2. So, we have $k_1 = 0$ which is a contradiction. □

We want to notice that this lemma says distinct principle curvatures are not multiple of each other. Consider distinct principal curvatures $K_1, K_2, \dots, K_p$ with the corresponding multiplicities $v_1, v_2, \dots, v_n$, respectively as in [31]. Then (6.6) becomes

$$v_2 K_2 + v_3 K_3 + \cdots + v_p K_p = -(2 + v_1) K_1, \tag{6.35}$$

where we put $K_1 = k_1$ and $p \leq n$. So, we give the following lemmas.

Lemma 6.1.5. *By the definition above, we have*

$$v_2(K_1 - K_2) + v_3(K_1 - K_3) + \cdots + v_p(K_1 - K_p) \neq 0. \tag{6.36}$$

Proof. Assume that

$$v_2(K_1 - K_2) + v_3(K_1 - K_3) + \cdots + v_p(K_1 - K_p) = 0. \tag{6.37}$$

Then, we have

$$(v_2 + v_3 + \cdots + v_p) K_1 = v_2 K_2 + v_3 K_3 + \cdots + v_p K_p. \tag{6.38}$$

By replacing (6.35) in the right hand side of (6.38), we get

$$(n-v_1)K_1 = -(2+v_1)K_1$$

which is not possible. □

Lemma 6.1.6. *By the definition above, we have*

$$v_3(K_1-K_3)+v_4(K_1-K_4)+\cdots+v_p(K_1-K_p) \neq 0. \tag{6.39}$$

Proof. Assume that

$$v_3(K_1-K_3)+v_4(K_1-K_4)+\cdots+v_p(K_1-K_p) = 0. \tag{6.40}$$

Then by using (6.35), we get

$$(2+v_1+v_3+\cdots+v_p)K_1 = -v_2K_2. \tag{6.41}$$

However, this yields a contradiction by Lemma 6.1.4. □

Lemma 6.1.7. *Let* $S=\{2,3,\ldots,n\}$*,* $R\subset S$*,* $R^c = S-R = \{j_1, j_2, \cdots, j_r\}$*.*
If $\sum_{a\in R} v_a(K_1-K_a)=0$ *then*

$$\begin{bmatrix} \psi_{j_1} & \psi_{j_2} & \cdots & \psi_{j_r} \\ \psi^2_{j_1} & \psi^2_{j_2} & \cdots & \psi^2_{j_r} \\ \vdots & \vdots & \ddots & \vdots \\ \psi^r_{j_1} & \psi^r_{j_2} & \cdots & \psi^r_{j_r} \end{bmatrix} \begin{bmatrix} v_{j_1}(K_1-K_{j_1}) \\ v_{j_2}(K_1-K_{j_2}) \\ \vdots \\ v_{j_r}(K_1-K_{j_r}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6.42}$$

Proof. From (6.6) we have

$$\sum_{a\in R} v_aK_a + \sum_{b\in R^c} v_bK_b = -(2+v_1)K_1. \tag{6.43}$$

Assume that $\sum_{a\in R} v_a(K_1-K_a)=0$. Then,

$$\sum_{a\in R} v_aK_1 = \sum_{a\in R} v_aK_a. \tag{6.44}$$

By replacing (6.43) we get

$$\sum_{a\in R} v_aK_1 = -(2+v_1)K_1 - \sum_{b\in R^c} v_bK_b. \tag{6.45}$$

Then, for an available positive constant λ, we have

$$\lambda K_1 = \sum_{b\in R^c} v_bK_b. \tag{6.46}$$

If we take the derivative of (6.46) along e_1 by r times and use (6.13) and (6.26) we get matrix equation (6.42) that we want to show. □

Now, we inquiry the case of distinct principal curvatures till at most 5 for the (6.1).

Proposition 6.1.8. *Let M be an MCGL hypersurface in* $\mathbb{E}_1^{n+1}$ *with the shape operator given by* (6.1). *If M has 2 distinct curvatures, then it cannot be proper biconservative.*

Proof. Consider (6.35) for $p = 2$. Then,

$$v_1 K_1 + v_2 K_2 = nH. \tag{6.47}$$

So,

$$v_2 K_2 = -(2 + v_1) K_1. \tag{6.48}$$

It contradicts with Lemma 6.1.4. ☐

Proposition 6.1.9. *Let M be an MCGL hypersurface in* $\mathbb{E}_1^{n+1}$ *with the shape operator given by* (6.1). *If M has 3 distinct curvatures, then it cannot be proper biconservative.*

Proof. Consider (6.35) for $p = 3$. Then,

$$v_2 K_2 + v_3 K_3 = -(2 + v_1) K_1. \tag{6.49}$$

Taking the derivative of (6.49) along e_1 two times by using (6.13) and (6.29), we come across the matrix in [31, Lemma 3.1] So,

$$\begin{bmatrix} \psi_2 & \psi_3 \\ \psi_2^2 & \psi_3^2 \end{bmatrix} \begin{bmatrix} v_2(K_1 - K_2) \\ v_3(K_1 - K_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6.50}$$

For simplicity we put $\psi_2 = a, \psi_3 = b$. Note that if $\psi_2 = a = 0$, then $\psi_3 = 0$ and this gives a contradiction by Remark 6.1.3. So we can apply row echelon process to coefficents matrix and we get

$$\begin{bmatrix} a & b \\ 0 & b(b-a) \end{bmatrix} \begin{bmatrix} v_2(K_1 - K_2) \\ v_3(K_1 - K_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad a \neq 0. \tag{6.51}$$

So, the case is reduced for solving the following equation

$$b(b-a) v_3 (K_1 - K_3) = 0. \tag{6.52}$$

Now, we examine the solution of the equation just above under the following cases:

Case 1: Putting $b = a$ into (6.51) gives

$$a(v_2(K_1 - K_2) + v_3(K_1 - K_3)) = 0. \tag{6.53}$$

However, this is impossible by hypothesis $a \neq 0$ and by Lemma 6.1.5.

Case 2: If $b = 0$ then $K_3 = 0$ and $v_3 = 0$ by Lemma 6.1.2. Then (6.49) becomes $v_2 K_2 = -(2 + v_1) K_1$. This gives $K_1 = 0$ which is a contradiction by Lemma 6.1.4.

□

Therefore there is no biconservative hypersurface with at most 3 distinct principal curvatures with the shape operator given by (6.1).

Proposition 6.1.10. *Let M be MCGL hypersurface in* $\mathbb{E}_1^{n+1}$ *with the shape operator given by* (6.1). *If M has 4 distinct curvatures, then it cannot be proper biconservative.*

Proof. Consider (6.35) for $p = 4$. By following similar procedures as in the Proposition 6.1.9 we get the following matrix equation

$$\begin{bmatrix} \psi_2 & \psi_3 & \psi_4 \\ \psi_2^2 & \psi_3^2 & \psi_4^2 \\ \psi_2^3 & \psi_3^3 & \psi_4^3 \end{bmatrix} \begin{bmatrix} v_2(K_1 - K_2) \\ v_3(K_1 - K_3) \\ v_4(K_1 - K_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.54}$$

Note that if $\psi_2 = 0$ then (6.54) implies

$$\psi_4(\psi_4 - \psi_3) v_4 (K_1 - K_4) = 0. \tag{6.55}$$

So, putting $\psi_4 = 0$ into (6.54) implies $\psi_3 = 0$ since $\psi_2 = 0$. This contradicts Remark 6.1.3. In other case, if $\psi_4 = \psi_3$ and putting this into (6.54) gives

$$\psi_3(v_3(K_1 - K_3) + v_4(K_1 - K_4)) = 0.$$

By Lemma 6.1.6 $\psi_3 = 0$. It follows $\psi_2 = \psi_3 = \psi_4 = 0$ which is a contradiction by Remark 6.1.3.

So, $\psi_2 \neq 0$ and we can apply the row echelon process to the (6.54). Hence we have

$$\begin{bmatrix} a & b & c \\ 0 & b(b-a) & c(c-a) \\ 0 & 0 & c(c-a)(c-b) \end{bmatrix} \begin{bmatrix} v_2(K_1 - K_2) \\ v_3(K_1 - K_3) \\ v_4(K_1 - K_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a \neq 0, b \neq -a \tag{6.56}$$

where $a = \psi_2, b = \psi_3, c = \psi_4$ for simplicity. So, the case is reduced to the solution of the following equation

$$c(c-a)(c-b) v_4 (K_1 - K_4) = 0. \tag{6.57}$$

The realization of (6.57) depends on following Cases;

Case 1: $c = 0$ then $K_4 = 0$ and $v_4 = 0$ by Lemma 6.1.2. So, by replacing $c = 0$ into

(6.56), we get (6.51) of Proposition 6.1.9 and it has been shown already.

Case 2: $c-a=0$ then putting $c=a$ into (6.56) gives

$$b(b-a)v_3(K_1-K_3)=0. \tag{6.58}$$

We shall examine the solution of (6.58) in the following cases:

Case 2.a: $b=0$ then $K_3=0=v_3$ by Lemma 6.1.2. It means by (6.56)

$$a(v_2(K_1-K_2)+v_4(K_1-K_4))=0, \tag{6.59}$$

but it is not possible by Lemma 6.1.5 and our hypothesis $a\neq 0$.

Case 2.b: $b=a$ then $b=c=a$ and by (6.56) gives

$$a(v_2(K_1-K_2)+v_3(K_1-K_3)+v_4(K_1-K_4))=0, \tag{6.60}$$

but it is not possible by Lemma 6.1.5 and our hypothesis $a\neq 0$ again.

Therefore we can say $c\neq a$.

Case 3: Putting $c=b$ into (6.56) gives

$$b(b-a)(v_3(K_1-K_3)+v_4(K_1-K_4))=0. \tag{6.61}$$

Firstly, we want to notice that $v_3(K_1-K_3)+v_4(K_1-K_4)\neq 0$ by Lemma 6.1.6. Now we shall examine the solution of (6.61) in following Cases:

Case 3.a: $b=0$ then $c=0$ and it is equivalent to the Case 1.

Case 3.b: $b=a$ then $b=c=a$ and it is equivalent to the Case 2.b.

So we can say $c\neq b$. It follows that (6.57) is possible if and only if $c=0$. □

Therefore, there is no biconservative hypersurface with four distinct principal curvatures with the shape operator given by Case 1.

Proposition 6.1.11. *Let M be an MCGL hypersurface in* $\mathbb{E}^{n+1}_1$ *with the shape operator given by* (6.1). *If M has 5 distinct curvatures, then it cannot be proper biconservative.*

Proof. Consider (6.35) for $p=5$. By following similar process in the Proposition 6.1.10 we have the following matrix equation

$$\begin{bmatrix} a & b & c & d \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & 0 & c(c-a)(c-b) & d(d-a)(d-b) \\ 0 & 0 & 0 & d(d-a)(d-b)(d-c) \end{bmatrix} \begin{bmatrix} v_2\Delta_{12} \\ v_3\Delta_{13} \\ v_4\Delta_{14} \\ v_5\Delta_{15} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6.62}$$

where $a \neq 0, b \neq -a$ and $\Delta_{ij} = K_i - K_j$. So problem is reduced to the solution of the following equation

$$d(d-a)(d-b)(d-c)v_5(K_1-K_5) = 0. \tag{6.63}$$

The realization of (6.63) depends on the following cases;

Case 1: $d = 0$ then $K_5 = 0$ and $v_5 = 0$ by Lemma 6.1.2. So, by replacing $d = 0$ into (6.62), it is reduced to (6.56) and it has been shown already.

Case 2: Putting $d = c$ into (6.62) gives

$$c(c-a)(c-b)(v_4(K_1-K_4)+v_5(K_1-K_5)) = 0. \tag{6.64}$$

We shall examine the solution of (6.64) in the following cases:

Case 2.a: $c = 0$ then $d = 0$. So, this condition is equivalent to the Case 1.

Case 2.b: $c = b$ then $c = b = d$. So, putting it into (6.62) gives

$$b(b-a)(v_3(K_1-K_3)+v_4(K_1-K_4)+v_5(K_1-K_5)) = 0. \tag{6.65}$$

Firstly, we want to notice that $v_3(K_1-K_3)+v_4(K_1-K_4)+v_5(K_1-K_5) \neq 0$ by Lemma 6.1.6. Now we shall examine the solution of (6.65) such that if $b = 0$ then $d = 0$ and this gives Case 1 else if $b = a$ then $d = c = b = a$ and putting it into (6.62) gives

$$a(v_2(K_1-K_2)+v_3(K_1-K_3)+v_4(K_1-K_4)+v_5(K_1-K_5)) = 0, \tag{6.66}$$

but (6.66) is impossible by Lemma 6.1.5 and our hypothesis $a \neq 0$. Therefore $c = b$ if and only if $c = 0$.

Case 2.c: $c = a$ then $d = c = a$. So, putting it into (6.62) gives

$$b(b-a)v_3(K_1-K_3) = 0. \tag{6.67}$$

Now we shall examine the solution of (6.67) such that if $b = a$ then $d = c = b = a$ and putting it into (6.62) gives (6.66) else if $b = 0$ then putting it into (6.62) gives

$$a(v_2(K_1-K_2)+v_4(K_1+K_4)+v_5(K_1-K_5)) = 0, \tag{6.68}$$

but (6.68) is impossible by Lemma 6.1.6. Therefore $c \neq a$.

Case 2.d: $v_4(K_1-K_4)+v_5(K_1-K_5) = 0$ then Lemma 6.1.7 gives the matrix (6.51). So, it is obvious that this Case is equivalent to the Case in 6.1.9.

Therefore we can say $d = c$ if and only if $d = 0$.

Case 3: Putting $d = b$ into the matrix (6.62) then

$$c(c-a)(c-b)v_4(K_1-K_4) = 0. \tag{6.69}$$

We shall examine the solution of (6.69) in the following cases:

Case 3.a: Putting $c = 0$ into the matrix (6.62) with Case 3 gives

$$b(b-a)(v_3(K_1-K_3)+v_5(K_1-K_5)) = 0. \tag{6.70}$$

We shall examine the solution (6.70).

Case 3.a.i: $b = 0$ gives $d = 0$ by Case 3. So, this case is equivalent to the Case 1.

Case 3.a.ii: $b = a$ gives $b = a = d$ by Case 3. So, by replacing them into the matrix (6.62) gives

$$a(v_2(K_1-K_2)+v_3(K_1-K_3)+v_5(K_1-K_5), \tag{6.71}$$

but this gives a contradiction by Lemma 6.1.5.

Case 3.a.iii: $v_3(K_1-K_3)+v_5(K_1-K_5) = 0$ then by Lemma 6.1.7 and Case 3.a , we have $av_2(K_1-K_2) = 0$ which is impossible by hypothesis $a \neq 0$ and from being K_1 and K_2 are distinct.

Case 3.b: $c = b$ then $c = b = d$. So, it is obvious that this Case is equaivalent to the Case 2.b.

Case 3.c: Putting $c = a$ into the matrix (6.62) gives

$$b(b-a)(v_3(K_1-K_3)+v_5(K_1-K_5)) = 0. \tag{6.72}$$

Now we shall examine the solution of (6.72).

Case 3.c.i: $b = 0$ then $d = 0$ by Case 3 and it means $K_5 = 0$ by Lemma 6.1.2. So, this condition is equivalent to the Case 1.

Case 3.c.ii: $b = a$ then $d = c = b = a$ by Case 3 and Case 3.c. So, putting it into the matrix (6.62) gives the equation (6.66) but it was impossible by Lemma 6.1.5 and our hypothesis $a \neq 0$.

Case 3.c.iii: $v_3(K_1-K_4)+v_5(K_1-K_5) = 0$ then Lemma 6.1.7 gives

$$\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_2(K_1-K_2) \\ v_4(K_1-K_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6.73}$$

So, we have

$$a(v_2(K_1-K_2)+v_4(K_1-K_4)) = 0. \tag{6.74}$$

We want to notice that $v_2(K_1 - K_2) + v_4(K_1 - K_4) \neq 0$ by Lemma 6.1.5 and $a \neq 0$ already. So (6.74) is impossible.

Therefore we can say $d = b$ if and only if $d = 0$.

Case 4: Putting $d = a$ into the matrix (6.62) gives

$$c(c-a)(c-b)v_4(K_1 - K_4) = 0. \tag{6.75}$$

We shall examine the solution of (6.75) in the following cases:

Case 4.a: Putting $c = b$ into the matrix (6.62) gives

$$b(b-a)(v_3(K_1 - K_3) + v_4(K_1 - K_4)) = 0. \tag{6.76}$$

Now we shall examine the solution of (6.76).

Case 4.a.i: $b = 0$ gives $c = 0$ by Case 4. By replacing them into the matrix (6.62) gives

$$a(v_2(K_1 - K_2) + v_5(K_1 - K_5)) = 0, \tag{6.77}$$

but this gives a contradiction by Lemma 6.1.5 since $K_3 = K_4 = 0$.

Case 4.a.ii: $b = a$ gives $b = a = c = d$ by Case 4 and Case 4.a. So, by replacing them into the matrix (6.62) gives

$$a(v_2(K_1 - K_2) + v_3(K_1 - K_3) + v_4(K_1 - K_4) + v_5(K_1 - K_5)) = 0. \tag{6.78}$$

This gives a contradiction by Lemma 6.1.5.

Case 4.a.iii: $v_3(K_1 - K_3) + v_4(K_1 - K_4) = 0$ gives

$$a(v_2(K_1 - K_2) + v_5(K_1 - K_5)) = 0 \tag{6.79}$$

by Lemma 6.1.7 and from being $d = a$, but this gives a contradiction by Lemma 6.1.5.

Therefore, when we back to the Case 4.a, one can say $c \neq b$.

Case 4.b: Putting $c = a$ into the matrix (6.62) gives

$$b(b-a)(v_3(K_1 - K_3)) = 0. \tag{6.80}$$

Now we shall examine the solution of (6.80).

Case 4.b.i: $b = 0$ gives $K_3 = 0$ by Lemma 6.1.2 and note that $c = a = d$ by Case 4 and Case 4.b. So, by replacing all of them into the matrix (6.62), we get

$$a(v_2(K_1 - K_2) + v_4(K_1 - K_4) + v_5(K_1 - K_5)) = 0, \tag{6.81}$$

but this gives a contradiction by Lemma 6.1.5.

Case 4.b.ii: $b = a$ then $b = a = c = d$ by Case 4 and Case 4.b. So by replacing all of them into the matrix (6.62), we get

$$a(v_2(K_1 - K_2) + v_3(K_1 - K_3) + v_4(K_1 - K_4) + v_5(K_1 - K_5)) = 0, \tag{6.82}$$

but this gives a contradiction by Lemma 6.1.5.

Therefore, when we back to the Case 4.a, one can say $c \neq a$.

Case 4.c: Putting $c = 0$ into the matrix (6.62) gives

$$b(b - a)(v_3(K_1 - K_3)) = 0 \tag{6.83}$$

and note that $K_4 = v_4 = 0$. Now we shall examine the solution of (6.83).

Case 4.c.i: $b = 0$ gives $K_3 = v_3 = 0$. So, by replacing all of them into the matrix (6.62) gives

$$a(v_2(K_1 - K_2) + v_5(K_1 - K_5)) = 0, \tag{6.84}$$

but this gives a contradiction by Lemma 6.1.5.

Case 4.c.ii: $b = a$ gives $b = d = a$ by Case 4. So, by replacing them into the matrix (6.62) gives

$$a(v_2(K_1 - K_2) + v_3(K_1 - K_3) + v_5(K_1 - K_5)) = 0, \tag{6.85}$$

but this gives a contradiction by Lemma 6.1.5. So $b \neq a$, when we back to the Case 4.c. So, we have $c \neq 0$.

Therefore, (6.63) is satisfied if and only if $d = 0$ and it means that five distinct case is reduced to four distinct cases and they have been shown already. □

Now, we study on hypersurfaces with the shape operator given by (6.2). In this case, the second fundamental form of M satisfies

$$h(e_1, e_2) = -k_1 N = -h(e_3, e_3), \qquad h(e_1, e_3) = -N, \qquad h(e_\alpha, e_\alpha) = k_\alpha N, \tag{6.86}$$

$$h(e_1,e_1)=h(e_1,e_\alpha)=h(e_2,e_2)=0=h(e_2,e_3)=h(e_2,e_\alpha)=h(e_\alpha,e_\beta)=0, \quad (6.87)$$

for $\alpha \neq \beta$ and $\alpha,\beta = 3,4,\dots,n$. Now, assume that if the surface is biconservative hypersurface, then we have (6.10). Thus one can choose ∇H is proportional to light-like vector e_2 and say $-2k_1 = nH$. So

$$e_2(k_1)=e_\alpha(k_1)=0, \qquad e_1(k_1)\neq 0, \qquad \alpha=3,4,\dots,n. \quad (6.88)$$

Consider distinct principal curvatures $K_1,K_2,\dots,K_p$ with its multiplicities $v_1,v_2,\dots,v_n$ as in [31]. Then (6.7) becomes

$$v_2K_2+\cdots+v_pK_p=-(2+v_1)K_1. \quad (6.89)$$

Now we use the Codazzi equation (2.28).

• The triple (e_2,e_α,e_α) implies

$$e_2(k_\alpha)=\Phi_\alpha(k_1-k_\alpha). \quad (6.90)$$

•The triple (e_2,e_α,e_β) gives

$$\omega_{\alpha\beta}(e_2)(k_\alpha-k_\beta)=\omega_{2\beta}(e_\alpha)(k_1-k_\beta). \quad (6.91)$$

• The triple (e_β,e_2,e_α) gives

$$\omega_{2\alpha}(e_\beta)(k_1-k_\alpha)=\omega_{\alpha\beta}(e_2)(k_\alpha-k_\beta). \quad (6.92)$$

Note that $[e_\alpha,e_\beta](k_1)=0$. So, it implies

$$\omega_{2\beta}(e_\alpha)=\omega_{2\alpha}(e_\beta). \quad (6.93)$$

By combining (6.93), (6.91) and (6.92), we get

$$\omega_{2\beta}(e_\alpha)=\omega_{2\alpha}(e_\beta)=\omega_{\alpha\beta}(e_2)=0. \quad (6.94)$$

• The triple (e_1,e_2,e_3) implies

$$\phi_2=0. \quad (6.95)$$

Moreover, $[e_2,e_\alpha](k_1)=0$ gives

$$\omega_{2\alpha}(e_2)=0. \quad (6.96)$$

Now we use Gauss equations (2.26) to set $R(e_\alpha, e_2, e_2, e_\alpha) = 0$. Then (6.94), (6.95) and (6.96) give

$$e_2(\Phi_\alpha) = -\Phi_\alpha^2 \tag{6.97}$$

and $R(e_\alpha, e_1, e_2, e_\alpha) = k_1 k_\alpha$. So, we have

$$e_1(\Phi_\alpha) = -\Phi_\alpha^2(\phi_1 - \psi_\alpha) + k_1 k_\alpha. \tag{6.98}$$

It is obvious that if each $\Phi_\alpha = 0$ then it gives $k_1 = 0$ contradiction as in Remark 6.1.3. Now, we get the derivative of (6.89) along e_2 p times by using (6.90) and (6.98), to obtain

$$\begin{bmatrix} \Phi_2 & \Phi_3 & \cdots & \Phi_p \\ \Phi_2^2 & \Phi_3^2 & \cdots & \Phi_p^2 \\ \vdots & \ddots & & \vdots \\ \Phi_2^p & \Phi_3^p & \cdots & \Phi_p^p \end{bmatrix} \begin{bmatrix} v_2(K_1 - K_2) \\ v_3(K_1 - K_3) \\ \vdots \\ v_p(K_1 - K_P) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6.99}$$

We showed that the hypersurface with the matrix given in (6.99) cannot be biconservative. We want to notice that we obtain the same of Lemma 6.1.5 and Lemma 6.1.6 by (6.89). So, by using the same techniques, we obtain

Theorem 6.1.12. *There exists no MCGL biconservative hypersurface with at most 5 distinct principal curvatures and the shape operator given by* (6.1) *or* (6.2) *in the Minkowski space* $\mathbb{E}_1^{n+1}$.

7. CONCLUSIONS

In this thesis, biconservative submanifolds and biconservative hypersurfaces in Lorentzian space forms were studied. In section 3, some facts about biconservative CMC submanifolds were obtained. In Theorem 3.3.3, four types of surfaces were obtained to which every biconservative CMC surfaces in $\mathbb{E}^4_1$ are locally congruent. Moreover, in $\mathbb{S}^4_1$ it was shown that biconservative CMC surface are ruled surfaces satisfying some special cases. Also, necessary and sufficient condition for the such surface to be biharmonic was given. Finally, it was shown that there is no proper biharmonic surface in $\mathbb{H}^4_1$.

In section 4, proper biconservative hypersurfaces with a non-diagonalizable shape operator in $\mathbb{E}^4_1$ were studied. In Theorem 4.3.8, it was shown that such hypersurfaces are locally congruent to a rotational surface satisfying some special cases.

In section 5, we studied on the biconservative hypersurfaces with non-diagonalizable shaper operator and at most two principle curvatures in Minkowski 5-space $\mathbb{E}^5_1$. It was shown that there is no proper biconservative hypersurface in $\mathbb{E}^5_1$ with the shape operator given in (5.2), (5.3) and (5.4).

In section 6, biconservative hypersurfaces in Minkowski spaces $\mathbb{E}^{n+1}_1$ were studied. We mainly focus on hypersurfaces with mean curvature whose gradient is light-like. We determined its shape operator to be at most 5 distinct principal curvatures. It was shown that there is no biconservative MCGL hypersurface with at most 5 distinct principal curvatures in the Minkowski space $\mathbb{E}^{n+1}_1$.

In the future, what has been done in Section 4 in $\mathbb{E}^4_1$ can be considered in arbitrary dimension $\mathbb{E}^n_1$. So, the generalization of the surface given in Section 3 can be obtained by a generalization of the method used.

REFERENCES

[1] **Eells, J. and Lemaire, L.** (1983). Selected topics in harmonic maps, *: CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence,Rhode Island.

[2] **Chen, B.Y.** (1991). Some open problems and conjectures on submanifold of finite type, *Soochow J. Math.*, *17*, 169–188.

[3] **Caddeo, R.**, **Montaldo, S. and Oniciuc, C.** (2001). Biharmonic submanifolds of $\mathbb{S}^3$, *Internat. J. Math.*, *8*(12).

[4] **Jiang, G.Y.** (1986). 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. Ser.*, *4*, 130–144.

[5] **Chen, B.Y.** (2013). Recent developments of biharmonic conjecture and modified biharmonic conjectures, *arXiv:1307.0245*, *7*(8).

[6] **Chen, B.Y.** (2019). Chen's Biharmonic conjecture and submanifolds with parallel normalized mean curvature vector, *Mathematics*, *7*(8).

[7] **Hasanis, T. and Vlachos, I.** (1995). Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field, *Math. Nachr.*, *172*, 145–169.

[8] **Caddeo, R.**, **Montaldo, S.**, **Oniciuc, C. and Piu, P.** (2014). Surfaces in the three-dimensional space forms with divergence-free stress-bienergy tensor, *Ann. Mat. Pura Appl.*, *193*, 529–550.

[9] **Jiang, G.Y.** (1987). The conservation law for 2-harmonic maps between Riemannian manifolds, *Acta Math. Sinica*, *30*, 220–225.

[10] **Chen, B. Y.and Ishikawa, S.** (1991). Biharmonic surfaces in pseudo-Euclidean spaces, *Kyushu J. Math.*, *42*, 323–347.

[11] **Chen, B. Y.and Ishikawa, S.** (1998). Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces, *Kyushu J. Math.*, *52*(1), 167–185.

[12] **Arvanitoyeorgosa, A.**, **Defever, F.**, **Kaimakamis, G. and Papantoniou, V.** (2007). Biharmonic Lorentzian hypersurfaces in $\mathbb{E}^4_1$, *Pacific J. Math.*, *229*(2), 293–305.

[13] **Fu, Y.** (2013). On bi-conservative surfaces in Minkowski 3-space, *J. Geometry Phys.*, *66*, 71–79.

[14] **Montaldo, S.**, **Oniciuc, C. and Ratto, A.** (2016). Biconservative surfaces, *J. Geom. Anal.*, *26*, 313–329.

[15] **Fetcu, D.**, **Oniciuc, C. and Pinhero, A.L.** (2015). CMC biconservative surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, *Journal of Mathematical Analysis and Applications*, *425*(1), 588–609.

[16] **Yeğin Şen, R.** (2018). On biconservative surfaces in 4-dimensional Euclidean space, *J. Math. Anal. Appl.*, *460*, 565–581.

[17] **Yeğin Şen, R. and Turgay, N.C.** (2017). On biconservative surfaces in Euclidean space, *arXiv preprint arXiv:1704.05270*.

[18] **Yeğin Şen, R.** (2022). Biconservative submanifolds with parallel normalized mean curvature vector field in Euclidean spaces, *Bulletin of the Iranian Mathematical Society*.

[19] **Chen, B.Y.**, (2011). Pseudo-Riemannian geometry, δ-invariants and applications, World Scientific Publishing Co. Pte. Ltd, USA.

[20] **O'Neill, B.**, (1983). Semi–Riemannian geometry with applications to relativity, Academic Press, London.

[21] **Dajczer, M.**, **Antonucci, M.**, **Oliveira, G.**, **Lima-Filho, P. and Tojeiro, R.**, (1990). Submanifolds and Isometric Immersions, Publish or Perish, Houston.

[22] **Magid, M.A.** (1985). Lorentzian Isoparametric Hypersurfaces, *Pasific J. Math.*, *118*(1), 165–197.

[23] **Fu, Y. and Hou, Z.H.** (2010). Classification of Lorentzian surfaces with parallel mean curvature vector in pseudo-Euclidean spaces, *J. Math. Anal. App.*, *371*, 25–40.

[24] **Chen, B.Y. and Van der Veken, J.** (2010). Classification of marginally trapped surfaces with parallel mean curvature vector in Lorentzian space forms, *Houston J. Math*, *36*, 421–449.

[25] **Kayhan, A. and Turgay, N.C.** (2022). Biconservative surfaces with constant mean curvature in Lorentzian space forms, *flomath.*, *(submitted)*.

[26] **Dursun, U. and Turgay, N.C.** (2016). Classification of minimal Lorentzian surfaces in $\mathbb{S}^4_1$ with constant Gaussian and normal curvatures, *Taiwanese J. Math.*, *20*, 1295–1311.

[27] **Fu, Y. and Turgay, N.C.** (2016). Complete classification of biconservative hypersurfaces with diagonalizable shape operator in the Minkowski 4-space, *Int. J. Math.*, *27*(5), 1650041.

[28] **Kayhan, A. and Turgay, N.C.** (2022). Biconservative hypersurfaces in $\mathbb{E}^4_1$ with non-diagonalizable shape operator, *Mediterr J. Math.*, *(accepted)*.

[29] **Deepika, K.** (2017). On biconservative Lorentz hypersurface with non-diagonalizable shape operator, *Mediterr. J. Math.*, *127*(14).

[30] **Kayhan, A. and Turgay, N.C.** (2022). Biconservative hypersurfaces in Minkowski 5-space, *J. Geom.*, *113*(43).

[31] **Turgay, N.C.** (2016). A classiffication of biharmonic hypersurfaces in the Minkowski spaces of arbitrary dimension, *Hacettepe Journal of Mathematics and Statistics.*, *45*(4), 1125–1134.

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