ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

MANIFOLDS OF GENERALISED G-STRUCTURES IN STRING COMPACTIFICATIONS

Ph.D. THESIS

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Department of Mathematical Engineering

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SİCİM KOMPAKTİFİKASYONLARINDA GENELLEŞTİRİLMİŞ G-YAPISI OLAN MANİFOLDLAR

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FOREWORD

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March 2023 Emine DİRİÖZ
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TABLE OF CONTENTS

ABBREVIATIONS

RR	**:**	Ramond-Ramond fields
DFT	**:**	Double Field Theory
NATD	**:**	Non-Abelian T-duality

SYMBOLS

M	: Manifold
g	: Riemannian metric
J	: Almost complex structure
$\mathscr{J}$	: Generalised almost complex structure
G	: Lie group
TM	: Tangent bundle of M
T^*M	: Cotangent bundle of M
$TM \oplus T^*M$	: Generalised tangent bundle of M
∇	: connection on TM
B	: 2-form field
Φ	: Pure spinor
E	: Vector bundle
L(M)	: Linear frame bundle
D	: Connection on principal bundles
C^∞	: Smooth sections
$\mathscr{H}$	: Generalised metric
Cl(V,Q)	: Clifford algebra
$\bigwedge^\bullet V$	: Exterior algebra

MANIFOLDS OF GENERALISED G-STRUCTURES IN STRING COMPACTIFICATIONS

SUMMARY

A G-structure on a differentiable manifold M of dimension n can be described as a reduction of the linear frame bundle $L(M)$ of M to a Lie subgroup G of $GL(n,\mathbb{R})$. Such a reduction is equivalent to the existence of certain geometric structures on M, depending on what the subgroup G is. For example, an $O(n)$-structure corresponds to the existence of a Riemannian metric g. Similarly, by the existence of an almost complex structure J, the structure group reduces to $GL(n/2,\mathbb{C})$. If a Riemannian metric and an almost complex structure are compatible that is metric is hermitian then the structure group reduces to $SU(n/2)$.

In a similar fashion, a generalised G-structure can be described as a reduction of the structure group of the principal bundle associated to the generalised tangent bundle $TM \oplus T^*M$. The natural structure group of $TM \oplus T^*M$ is $O(n,n)$. The generalised G-structures also correspond to existence of certain geometrical objects. For example, the reduction of the structure group from $O(n,n)$ to $O(n) \times O(n)$ corresponds to the existence of a generalised metric. Similarly, on an even dimensional real manifold M a generalised almost complex structure is given by a reduction of the structure group from $O(n,n)$ to $U(n/2,n/2)$. A generalised almost complex structure is defined by the existence of a pure spinor which is a section of the exterior bundle $\bigwedge^\bullet T^*M$.

The main theme of this thesis is the study of manifolds of generalised G-structure relevant for string compactifications. Superstring theory is a quantum theory of gravity consistent in 10 dimensions. There are five consistent superstring theories and the low energy dynamics of massless space-time fields is governed by ten dimensional supergravity theories. The supergravity field equations are nonlinear partial differential equations which can be regarded as a generalisation of field equations of Einstein's theory of general relativity (GR). In a supersymmetric compactification of Type II string theory down to 4 dimensions, it is required that the structure group of the generalised tangent bundle $TM \oplus T^*M$ of the six dimensional internal manifold M is reduced from $SO(6,6)$ to $SU(3) \times SU(3)$. This is equivalent to the existence of two globally defined compatible pure spinors Φ_1 and Φ_2. Furthermore, these pure spinors should satisfy certain first order differential equations, namely supersymmetry equations. We show that these equations are covariant under certain $Pin(d,d)$ transformations. We also show that Non-Abelian T-duality (NATD) which is generated by a coordinate dependent $Pin(d,d)$ transformation is a particular solution generating transformation for these pure spinor equations. Our method is demonstrated by studying the NATD of a specific class of geometries with $SU(2)$ isometry and $SU(3)$-structure. Some of the manifolds belonging to this class are $AdS_5 \times T^{1,1}$, $AdS_5 \times Y^{p,q}$ and $AdS_5 \times S^5$. It is interesting to note that in each case, the internal

manifold is a Sasaki-Einstein manifold. We show that the transformed pure spinors are associated with an $SU(2)$-structure.

The plan of the thesis is as follows: in section 2, we study principal fiber bundles, vector bundles and linear frame bundles. Then, we study the concept of the reduction of the structure groups. We also give familiar examples of G-structures in detail. In section 3, we shortly review the relation between G-holonomy and torsion-free G-structures. In section 4, we study the basic concepts regarding the geometry on the generalised tangent bundle $TM \oplus T^{*}M$. This leads us to the definition of a generalised G-structure. Since our main interest is in $SU(3) \times SU(3)$-structures we give in a separate subsection the description of $SU(3) \times SU(3)$-structures and the associated pure spinors in detail. In section 5, we focus on the differential equations to be satisfied by the pure spinors for preservation of $\mathcal{N} = 1$ supersymmetry. We study the covariance of these equations under constant and non-constant $Pin(d,d)$ transformations. Then, we study Non-Abelian T-duality (NATD) transformations in detail, and we show the invariance of pure spinor equations under NATD. In section 6, we consider a specific class of geometries. We transform the pure spinors associated with the $SU(3)$-structure, and show that the resulting pure spinors determine an $SU(2)$ structure. We also study the NATD transformation of the metric, the B field and the Ramond-Ramond fields.

SİCİM KOMPAKTİFİKASYONLARINDA GENELLEŞTİRİLMİŞ G-YAPISI OLAN MANİFOLDLAR

ÖZET

Boyutu n olan türevlenebilir bir M manifoldu üzerinde, bir G-yapısı, lineer çerçeve demetinin temel alt demeti olarak tanımlanır. G grubu ise $GL(n,\mathbb{R})$ grubunun bir Lie alt grubudur. Doğrusal çerçeve demeti, TM teğet demetiyle ilişkili ana çerçeve demetidir. G-yapıları ile M üzerinde belirli geometrik nesnelerin varlığı arasında 1-1 karşılıklılık vardır. Örneğin, doğrusal çerçeve demetinin yapı grubunun $O(n)$'e indirgenmesi, bir Riemann metriği g'nin varlığına karşılık gelir. Benzer şekilde, hemen hemen kompleks bir J yapısının varlığıyla, yapı grubu $GL(n/2,\mathbb{C})$'ye indirgenir. Eğer Riemann metriği ve hemen hemen komplex yapı birbirleriyle uyumluysa yani bir Hermityen metriği h varsa, yapı grubu $SU(n/2)$'e düşer. Bu ise reel bir 2-form ω'nın varlığına ve belirli koşulları sağlayan kompleks ayrıştırılabilir bir $n/2$-form Ω'nın varlığına eşdeğerdir. Benzer şekilde, herhangi bir yedi boyutlu düzgün manifold M üzerinde G_2-yapısı, çerçeve demetinin yapı grubunun kompakt, istisnai Lie grubu G_2'ye indirgenmesi olarak tanımlanabilir. G_2, pozitif bir 3-form olan φ'yi koruyan $GL(7,\mathbb{R})$'nin bir alt grubudur.

G-yapıları ve G-holonomi kavramları yakından ilişkilidir. Bir konneksiyonun holonomi grubu, bir vektör demetinde veya bir temel lif demetinde global bir değişmez olarak tanımlanır. Holonomi grupları belirli bir teğet uzayının T_xM endomorfizmaları olarak tanımlanır. Bunlar belirli bir alt grup içinde değerler alan ve kapalı döngüler etrafında paralel taşıma vasıtasıyla oluşturulan endomorfizmalardır. Holonomi kavramı bizim için iki nedenden ötürü önemlidir. Birincisi, holonomi grubu kovaryant sabit tensörleri tanımlar. Örneğin, Riemann metriği g kovaryant olarak sabit olduğunda, holonomi grubu $O(n)$'in bir alt grubu olur. J'nin kovaryant olarak sabit olduğu durumda holonomi grubu $GL(n/2,\mathbb{C})$'nin bir alt grubudur. Eğer kapalı dejenere olmayan bir 2-form ω ve ayrışabilen kapalı bir $n/2$ form Ω'nın varlığını empoze edersek o zaman yapı grubu $SU(n/2)$'e düşer ve bu formların kovaryant türevleri sıfır ise o zaman holonomi grubu $SU(n/2)$'in bir alt grubu olacaktır. Eğer 3-form olan φ Levi-Civita konneksiyonuna göre paralelse, bu durumda holonomi grubu G_2'nin bir alt grubu olur. Holonomi grubunun bizim için önemli olmasının ikinci nedeni ise burulmasız olan konneksiyonlar için, G-holonomisi ile burulmasız G-yapılarının birbirlerine denk olmasıdır.

Bu tezde amacımız $TM \oplus T^*M$ üzerinde tanımlanan genelleştirilmiş G-yapılarını çalışmaktır. $TM \oplus T^*M$ genelleştirilmiş teğet demeti olarak adlandırılır ve TM'in kesitlerinde tanımlı olan Lie parantezi, $TM \oplus T^*M$'in kesitlerinde tanımlı Courant parantezi ile yer değiştirilir. Genelleştirilmiş G-yapısını genelleştirilmiş teğet demetiyle ilişkili olan genelleştirilmiş çerçeve demetinin yapı grubunun bir indirgenmesi olarak tanımlayacağız. $TM \oplus T^*M$'in doğal yapı grubu $O(n,n)$'dir. Yukarıda

tartıştığımız geleneksel G-yapılarında olduğu gibi, genelleştirilmiş G-yapıları da belirli geometrik nesnelerin varlığına karşılık gelir. Örneğin, yapı grubunun $O(n,n)$'den $O(n) \times O(n)$'e indirgenmesi, genelleştirilmiş bir metriğin varlığına karşılık gelir. Genelleştirilmiş metrik, Riemannian metriği g ve bir 2-form alanı B'den oluşan $TM \oplus T^*M$ üzerinde tanımlı bir metriktir. Benzer şekilde, çift boyutlu bir reel manifold M üzerinde, genelleştirilmiş hemen hemen kompleks yapının varlığı, yapı grubunun $O(n,n)$'den $U(n/2,n/2)$'e indirgenmesine karşılık gelir. Ayrıca, $O(n) \times O(n)$ ve $U(n/2,n/2)$'nin kesişimi $U(n/2) \times U(n/2)$-yapısına bir indirgenme sağlar. Bu ise iki uyumlu genelleştirilmiş hemen hemen kompleks yapının varlığına denktir. Dahası, yapı grubu, normu sıfırdan farklı saf bir spinörünün varlığıyla $SU(n/2,n/2)$'e indirgenir. Saf spinörler $\bigwedge^\bullet T^*M$ kesitleridir yani homojen olmayan diferansiyel formlar olarak tanımlanabilir. Saf bir spinörün maksimum izotropik bir sıfır uzayı vardır. Yapı grubu, iki uyumlu saf spinörün varlığıyla $SU(n/2) \times SU(n/2)$'e indirgenebilir. Uyumluluk koşulu ise spinörlerin ilişkili oldukları genelleştirilmiş hemen hemen kompleks yapıların uyumluluğuyla ilgilidir. Öncelikli ilgi alanımız $SU(3) \times SU(3)$- yapıları ve bu yapıyla ilişkili olan homojen olmayan diferansiyel formlar yani saf spinörlerdir.

Bu tezdeki temel amacımız, sicim kompaktifikasyonlarıyla ilgili olan genelleştirilmiş G-yapısı olan manifoldları çalışmaktır. Süpersicim teorisi, 10 boyutta tutarlı bir kuantum kütleçekimi teorisidir. 5 tane tutarlı süpersicim teorisi bulunur ve kütlesiz uzay-zaman alanlarının düşük enerji dinamikleri on boyutlu süper kütleçekim teorileriyle yönetilir. Süper kütleçekim alan denklemleri, Einstein'ın genel görelilik teorisinin (GR) alan denklemlerinin genelleştirilmesi olarak düşünülebilecek doğrusal olmayan kısmi diferansiyel denklemlerdir. GR'de ana dinamik alan Riemann metriğidir. Sicim kuramında, Riemann metriğine ek olarak 2-form olan bir B alanı, dilaton adı verilen bir skaler alan ve Ramond-Ramond (RR) alanları adı verilen bir grup p-form alanları da mevcuttur. Süper kütleçekimindeki standart formülasyonda p farklı değerler alır. Tip (m) IIA durumunda RR alan şiddeti $p = 0,2,4$ dereceli formlar iken Tip IIB durumunda $p = 1,3,5$ dereceli formlardır. Bu formların Hodge düallerinin de dikkate alınmasıyla elde edilen demokratik formülasyondaysa Tip (m) IIA durumunda RR alan şiddeti $p = 0,2,4,6,8,10$ dereceli formlar iken Tip IIB durumunda $p = 1,3,5,7,9$ dereceli formlardır. Sicim teorisini 4 boyutlu fiziksel teorilerle ilişkilendirmek için, 4 boyutlu $M_{1,3}$ Minkowski uzay-zamanına kompaktifikasyona karşılık gelen $M_{1,3} \times Y_6$ biçimindeki çözümleri dikkate almak gerekir. Y_6 iç manifold adı verilen kompakt bir manifolddur. Ortaya çıkan 4 boyutlu teorinin gerçekçi olabilmesi için iç manifoldun belirli özellikleri sağlaması gerekmektedir. Örneğin 4 boyutlu teoride, süpersimetri adı verilen belirli bir simetrinin olması istenir. Örneğin, RR p-form alanının sıfırlandığı tip II sicim teorisinin kompaktifikasyonunda $\mathcal{N} = 2$ süpersimetri için, iç manifoldun kompleks boyutu 3 olan bir Calabi-Yau manifoldu olması gerekir. Calabi-Yau manifoldu, birinci Chern sınıfının sıfırlandığı kompleks bir Kähler manifoldudur. Bu durumda holonomisi $SU(3)$ olan Ricci-flat metrik kabul ettiği gosterilmiştir.

Sıfırlanmayan p-form alanlarının varlığında 4 boyutta Tip II sicim teorisinin süpersimetrik kompaktifikasyonunda, altı boyutlu iç manifold M'nin genelleştirilmiş teğet demeti $TM \oplus T^*M$'nin yapı grubunun $SO(6,6)$'dan $SU(3) \times SU(3)$'a düşürülmesi gerekmektedir. Bu, iki uyumlu saf spinörün Φ_1 ve Φ_2 varlığına denktir. Saf spinörler tarafından sağlanan denklemler, arka plan geometrisinin süpersimetrik olmasını sağlar. Bu spinörler aşağıdaki birinci dereceden diferansiyel denklemleri, yani süpersimetri denklemlerini sağlamalıdır.

$$d(e^{2A-\phi}e^B \wedge \Phi_1) = 0, \tag{1}$$
$$d(e^{2A-\phi}e^B \wedge \Phi_2) = e^{2A-\phi}dA \wedge e^B \wedge \bar{\Phi}_2 + \frac{i}{8}e^{3A}e^B \wedge \lambda(*_6 F). \tag{2}$$

Burada, A skaler bir fonksiyon, B bir 2-form alanı, ϕ skaler bir alan, $*_6$ altı boyutlu iç manifoldda Hodge yıldız operatörü, F homojen olmayan bir diferansiyel form ve Φ_1, Φ_2 saf spinörlerdir. Tüm bu terimler bölüm 5'te detaylıca incelenecektir. $\mathcal{N} = 1$ süpersimetrinin korunması ile ilişkili saf spinör denklemlerinin, $Pin(d,d)$ dönüşümleri altında kovaryant olduğunu kanıtlamayı amaçlıyoruz.

Bu tezde, koordinatlara bağlı bir $O(d,d)$ dönüşümü olan Abelian Olmayan T-düalitesi (NATD) ile saf spinörlerin nasıl dönüştürüleceğini göstereceğiz. Bu dönüşüm Riemann metriği, B alanı ve dilaton üzerinde etkilidir. Öte yandan spinor alanları, $Pin(d,d)$ ve $O(d,d)$ arasındaki çift kaplama homomorfizması dikkate alınarak elde edilen bir $Pin(d,d)$ dönüşümü ile dönecektir. Bu tezde NATD'nin saf spinor denklemleri için çözüm üreten bir dönüşüm olduğunu göstereceğiz. Bunu yapmak için, saf spinör denklemlerini $Pin(d,d)$ kovaryant olarak yeniden yazmamız gerekmektedir. Ancak NATD koordinat bağımlı bir $Pin(d,d)$ dönüşümü olduğu için bu analizi sonuçlandırmak yeterli olmayacaktır. Bölüm 5.2.2'de tartışılacak olan genişletilmiş bir analize ihtiyacımız olacak. Bu analiz sayesinde saf spinörlerin NAT düallerinin literatürdeki diferansiyel denklemleri sağlamaya devam ettiği ve bu nedenle düal arka planın en az $\mathcal{N} = 1$ süpersimetriyi koruduğu gösterilecektir.

Metodumuzu göstermek için, Tip IIB süper kütleçekiminin çözümleri olduğu bilinen belirli bir geometri tipine odaklanacağız. Çalışacağımız geometri, topolojik olarak $M_{1,3} \times \mathcal{M}_3 \times S^3$ şeklinde olacaktır, böylece bir $SU(2)$ izometrisi vardır. Bu tezin 6. bölümünde göreceğimiz gibi, metrik ve 5-formlar için aşağıdaki yapıyı inceleyeceğiz.

$$ds^2 = e^{2A}dx_{1,3}^2 + ds^2(\mathcal{M}_3) + \sum_{i=1}^{3}\left(e^i\right)^2, \tag{3}$$
$$\mathcal{F}_5 = \mathcal{F}_2 \wedge e^1 \wedge e^2 \wedge e^3$$
$$F_5 = (1+*)\mathcal{F}_5 = \mathcal{F}_2 \wedge e^1 \wedge e^2 \wedge e^3 - e^{4A} *_3 \mathcal{F}_2 \wedge Vol_4$$

Ayrıca geometrinin bir $SU(3)$-yapısını kabul ettiğini varsayacağız. $SU(3)$ yapısı, $SU(3) \times SU(3)$ yapısının özel bir durumudur. $SU(3)$ yapısıyla ilişkili saf spinörler aşağıdaki gibidir.

$$\Phi_1 = -\frac{i}{8}e^{i\theta_-}e^A\Omega\,, \quad \Phi_2 = \frac{1}{8}e^{i\theta_+}e^A e^{-iJ} \tag{4}$$

Burada A bir fonksiyondur, J reel bir 2-form ve Ω belirli koşulları sağlayan bir 3-formdur. $SU(3)$ saf spinörlerini, B-alanını, metrik ve dilatonu NATD dönüşümü altında dönüştüreceğiz. Dönüşümden sonra elde ettiğimiz yeni geometri Tip II süper kütleçekimi denklemlerini otomatik olarak çözecektir. $SU(3)$ saf spinörlerinin NATD dönüşümünden sonra elde ettiğimiz yeni saf spinörlerin $SU(2)$ saf spinörleri olduğunu gösterebileceğiz. Yeni saf spinörler (5.1), (5.2)'de verilen süpersimetri denklemlerini çözecektir.

1. INTRODUCTION

A G-structure on a differentiable manifold M of dimension n can be described as a reduction of the linear frame bundle $L(M)$ of M to a Lie subgroup G of $GL(n,\mathbb{R})$, [1,2]. The linear frame bundle is the principal frame bundle associated to the tangent bundle TM. Such a reduction is equivalent to the existence of certain geometric structures on M, depending on what the subgroup G is. For example, the reduction of the structure group of the linear frame bundle to $O(n)$ corresponds to the existence of a Riemannian metric g, [2,3]. Similarly, by the existence of an almost complex structure J, the structure group reduces to $GL(n/2,\mathbb{C})$, [1]–[3]. If a Riemannian metric and an almost complex structure are compatible that is metric is hermitian then the structure group reduces to $SU(n/2)$, [1]–[3]. It is equivalent to the existence of a real 2-form ω and a complex decomposable $n/2$ form Ω satisfying certain conditions. Likewise, a G_2-structure can be defined on any seven dimensional smooth manifold M as a reduction of the structure group of the frame bundle to the compact, exceptional Lie group G_2. It is defined as the subgroup of the general linear group $GL(7,\mathbb{R})$ which preserves a positive 3-form φ, [1]–[3].

G-structures and G-holonomy are closely related. Holonomy group of a connection is described on a vector or a principal bundle as a global invariant, [1,2,4]. Typically, holonomy groups are defined to be endomorphisms of a fixed tangent space T_xM to the underlying manifold that take values within a particular subgroup arising from parallel transport around closed loops, [1,2,4]. There are two important aspects that are crucial for us to be aware of: firstly, the holonomy group is responsible for defining the covariantly constant tensors on M. For example, when Riemannian metric g is covariantly constant, then the holonomy group become a subgroup of $O(n)$. In the case when J is covariantly constant, the holonomy group is a subgroup of $GL(n/2,\mathbb{C})$. If we impose the existence of a closed non-degenerate 2-form ω arising from h, J and a closed complex decomposable $n/2$ form Ω ; $d\omega = 0$, $d\Omega = 0$, then the holonomy group will become a subgroup of $SU(n/2)$. Existence of G_2 holonomy imposes that φ has to be parallel with respect to the Levi-Civita connection. Secondly, when we restrict ourselves to torsion-free connections, G-holonomy groups will become equivalent to torsion-free G-structures, [1,2,4].

In a similar fashion, a generalised G-structure can be described as a reduction of the structure group of the principal bundle associated to the generalised tangent bundle $TM \oplus T^*M$, [5,6]. As we will discuss in Section 4 the natural structure group of $TM \oplus T^*M$ is $O(n,n)$. As in the reduction of the conventional G-structures we discussed above, the generalised G-structures are equivalent to the existence of certain geometric structures on M. For example, the reduction of the structure group from $O(n,n)$ to $O(n) \times O(n)$ corresponds to the existence of a generalised metric, [5,6]. The generalised metric is defined on $TM \oplus T^*M$ which consists of Riemannian metric g and a B-field B, [6]–[10]. Similarly, on an even dimensional real manifold M a generalised almost complex structure is given by a reduction of the structure group from $O(n,n)$ to $U(n/2,n/2)$, [6]. A generalised almost complex structure is defined by the existence of a pure spinor which is a section of the exterior bundle $\bigwedge^\bullet T^*M$, [6,11]. Annihilator of pure spinors is maximally isotropic. Moreover, the intersection of $O(n) \times O(n)$ and $U(n/2,n/2)$ gives a reduction to $U(n/2) \times U(n/2)$-structure, [5,6,12]. The structure group reduces to the $SU(n/2,n/2)$ by the existence of a globally defined pure spinor of non-vanishing norm, [12]. It can be further reduced to $SU(n/2) \times SU(n/2)$-structure by the existence of two compatible pure spinors where the compatibility condition is related to their associated generalised almost complex structures compatibility, [12]. Our primary interest is on $SU(3) \times SU(3)$- structures and the associated pure spinors as smooth non-homogeneous differential forms.

The main theme of this thesis is manifolds of generalised G-structure relevant for string compactifications. Superstring theory is a quantum theory of gravity consistent in 10 dimensions, [12]–[14]. There are five consistent superstring theories and the low energy dynamics of massless space-time fields is governed by ten dimensional supergravity theories. The supergravity field equations are nonlinear partial differential equations which can be regarded as a generalisation of field equations of Einstein's theory of general relativity (GR). In GR, the main dynamical field is the Riemannian metric. In type II string theory, in addition to the Riemannian metric one also has a 2-form field B (called the B-field), a scalar field called the dilaton and p-form fields of various degrees called the Ramond-Ramond (RR) fields. Type IIA and IIB are two of the superstring theories which are related with each other with T-duality. The Ramond-Ramond (RR) field strengths are $p = 0,2,4$ degree forms for Type (m) IIA wheras the Ramond-Ramond (RR) field strengths are $p = 1,3,5$ degree forms for Type IIB. We will be referring to a formulation called the democratic formulation where the Hodge duals of these forms are also taking into account so that the RR fields are $p = 0,2,4,6,8,10$ degree forms for Type (m) IIA wheras the RR fields are $p = 1,3,5,7,9$ degree forms for Type IIB, [13,14]. In order to relate string theory with 4 dimensional physical theories one has to consider solutions of the form $M_{1,3} \times Y_6$ corresponding to compactifications to 4 dimensional Minkowski space-time $M_{1,3}$. Here, Y_6 is a compact manifold called the internal manifold. Equivalently, we say that the 10 dimensional theory has been compactified on the 6-dimensional manifold Y_6. In order for the resulting 4-dimensional theory to be realistic the internal manifold must possess certain properties. For example, one usually demands the 4-dimensional theory to exhibit a certain type of symmetry called supersymmetry. For example, requirement of $\mathscr{N} = 2$ supersymmetry in a compactification of type II string theory in the absence of RR p-form field requires the internal manifold to be a Calabi-Yau manifold with 3 complex dimensions [12]–[14]. A Calabi-Yau manifold is a complex Kähler manifold with vanishing first Chern class and it was shown that admits a Ricci-flat metric with $SU(3)$ holonomy.

In the presence of non-vanishing p-form fields in a supersymmetric compactification of Type II string theory, preservation of $\mathscr{N} = 1$ supersymmetry in four dimensions is studied by using tools from generalised geometry defined on the generalised tangent bundle, as was first shown in [15]. Preservation of $\mathscr{N} = 1$ supersymmetry requires that the structure group of the generalised tangent bundle $TM \oplus T^*M$ of the six dimensional internal manifold M is reduced from $SO(6,6)$ to $SU(3) \times SU(3)$. As we discussed above it was given in [16] that this implies existence of two globally defined compatible pure spinors Φ_1 and Φ_2 of non-vanishing norm. These pure spinors can be constructed from the internal spinors arising from the 10 dimensonal Killing spinors generating the supersymmetry transformations in 10 dimensions, [17]. It was shown in [15,18,19] that these pure spinors should satisfy certain first order differential equations, namely supersymmetry equations.

$$d(e^{2A-\phi}e^{B}\wedge\Phi_1) = 0, \tag{1.1}$$

$$d(e^{2A-\phi}e^{B}\wedge\Phi_2) = e^{2A-\phi}dA\wedge e^{B}\wedge\bar{\Phi}_2+\frac{i}{8}e^{3A}e^{B}\wedge\lambda(*_6F). \tag{1.2}$$

Here A is a scalar function, B is a 2-form field, ϕ is a scalar field, $*_6$ is the Hodge duality on the six dimensional internal manifold, F is a non-homogeneous differential form and Φ_1,Φ_2 are pure spinors. All the terms appearing in pure spinor equations will be discussed in chapter 5. We aim to prove that these equations associated with preserved $\mathcal{N}=1$ supersymmetry are covariant under $Pin(d,d)$ transformations. In order to do this, we extend the exterior derivative operator to an $O(d,d)$ covariant differential operator. We also write the Hodge duality operator in a $Pin(d,d)$ covariant way, by utilising the generalised metric. We show the covariance of pure spinor equations (5.1), (5.2) under both constant and non-constant $Pin(d,d)$ transformations.

In this thesis, we also show how to transform pure spinors by Non-Abelian T-duality (NATD) which can be described as a coordinate-dependent $O(d,d)$ transformation as was shown in [20]. $O(10,10)$ matrix associated with the NATD transformation is called T_{NATD}. This transformation acts on the Riemannian metric, the B-field and the dilaton. On the other hand, the spinor fields are transformed by applying a $Pin(d,d)$ transformation, which is generated by S_{NATD}. It is obtained as $\rho(S_{\text{NATD}})=T_{\text{NATD}}$ by considering the double covering homomorphism ρ between $Pin(d,d)$ and $O(d,d)$. It was shown in [20] that NATD is a solution generating transformation for Double Field Theory (DFT). Likewise, we will be able to show in this thesis that NATD is a solution generating transformation for pure spinor equations (5.1), (5.2). Since NATD is a special coordinate dependent $Pin(d,d)$ transformation we will need an extended analysis which will be discussed in Section 5.2.2. It ensures that the solutions of pure spinor equations will remain solutions after NATD transformation due to the special properties of NATD. It is through this analysis that we guarantee in [17] that NAT dual of pure spinors of $\mathcal{N}=1$ vacua still satisfy the differential equations in literature, and therefore, the dual background maintains at least $\mathcal{N}=1$ supersymmetry.

In order to demonstrate our method, we will focus on a specific type of geometries which are known to be solutions of Type IIB supergravity. The geometry we will study will be topologically of the form $M_{1,3} \times \mathscr{M}_3 \times S^3$ so that there is an $SU(2)$ isometry. As we will see in Section 6 that we will be studying the following ansatz for the metric and 5-forms.

$$\begin{aligned} ds^2 &= e^{2A} dx_{1,3}^2 + ds^2(\mathscr{M}_3) + \sum_{i=1}^{3} \left(e^i\right)^2, \\ \mathscr{F}_5 &= \mathscr{F}_2 \wedge e^1 \wedge e^2 \wedge e^3 \\ F_5 &= (1+*)\mathscr{F}_5 = \mathscr{F}_2 \wedge e^1 \wedge e^2 \wedge e^3 - e^{4A} *_3 \mathscr{F}_2 \wedge Vol_4 \end{aligned} \tag{1.3}$$

We will also assume that the geometry admits an $SU(3)$-structure. $SU(3)$-structure can be regarded as a special case of $SU(3) \times SU(3)$ structure and pure spinors associated with $SU(3)$-structure are as follows.

$$\Phi_1 = -\frac{i}{8} e^{i\theta_-} e^A \Omega \ , \quad \Phi_2 = \frac{1}{8} e^{i\theta_+} e^A e^{-iJ} \tag{1.4}$$

Here A is a function that appears in the Riemannian metric and is called the warp factor, J is a real 2-form and Ω is a 3-form satisfying certain conditions. We transform these pure spinors under the NATD transformation along with the transformation of the B-field, metric and the dilaton. The new geometry will automatically solve Type II supergravity equations. New pure spinors will also solve supersymmetry equations given in (5.1), (5.2). We will also be able to show that $SU(3)$ pure spinors are transformed to pure spinors associated with an $SU(2)$ structure, [17].

2. G-STRUCTURES

In this section, we review the construction and properties of G-structures. A G-structure on a differentiable manifold M of dimension n can be described as a reduction of the linear frame bundle $L(M)$ of M to a Lie subgroup G of $GL(n,\mathbb{R})$, [1,2]. Such a reduction is equivalent to the existence of certain geometric structures on M, depending on what the subgroup G is. We will briefly review principal fibre bundles, vector bundles and associated fibre bundles according to [1,2]. Then, we will specifically consider linear frame bundles. We will explain how one can associate vector bundles to principal bundles. Oftentimes, principal bundles and vector bundles provide two different but equivalent approaches.

2.1 Principal Fibre Bundles

To begin, we consider the following description of a principal fiber bundle:

Definition 2.1 [2] Let M be a manifold and G a Lie group. A (differentiable) principal fibre bundle over M (the base manifold) with group G (the structure group) consists of a manifold P (the total manifold) and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right. If $p = pg \Rightarrow g = e$, $(p,g) \in P \times G \to pg = R_g p \in P$ and is associative in the sense that $p(gh) = (pg)h$, for all $g,h \in G$.

(2) M is the quotient space of P by the equivalence relation $\sim$ induced by G, $M = P/G$, $(p' \sim p \Rightarrow \exists g, p' = pg)$ and the canonical projection $\pi : P \to M$ is differentiable;

(3) P is locally trivial, that is, every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$.

That is, there exists a diffeomorphism

$$\psi : \pi^{-1}(U) \to U \times G$$

such that $\psi(p) = (\pi(p), \varphi(p))$ where φ is a mapping $\pi^{-1}(U) \to G$ satisfying $\varphi(pg) = (\varphi(p))g$ for all $p \in \pi^{-1}(U)$ and $g \in G$.

Principal fibre bundle will be denoted by $P(M,G)$. The inverse image of a point $x \in M, \pi^{-1}(x)$, is a closed submanifold of P, which is called the fibre over x. If p is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points pg, $g \in G$ and is called the fibre through p. Considering that the action is free and transitive on every fibre, it would follow that each fibre is diffeomorphic to G. As a result of this, we can consider the fibres as copies of G, [2].

Definition 2.2 [21] A local cross section over an open set $U \subset M$ is a differentiable mapping $\sigma : U \to P$ such that $(\pi \circ \sigma)(x) = x \quad \forall x \in U$.

Remark 2.1 As a way of understanding local cross sections, it is important to understand the fact that they are defined by the locally trivialization mappings, and vice versa, [21].

Proof: [21] First, we will show that a local cross section defines a local trivialization. Let φ_α be a mapping $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G$ such that $\varphi_\alpha(pg) = \varphi_\alpha(p)g$. Then, φ_α can be defined by using σ_α, $\sigma_\alpha(x)\varphi_\alpha(p) = p$, since the action is free then $\varphi_\alpha(p) = e$ for any $p \in P$, where e is the identity element in G. Using $p_2 = pg$, the term $p\,[\varphi_\alpha(p)]^{-1}$ is independent of the choice of a point p. Since for any $p_2 \in P$, we have $p_2\,[\varphi_\alpha\,(p_2)]^{-1} \equiv pg\,[\varphi_\alpha(pg)]^{-1} = pgg^{-1}\,[\varphi_\alpha(p)]^{-1} = \sigma_\alpha(x)$.

Then, ψ_α is defined by

$$\psi_\alpha(p) = (\pi(p), \varphi_\alpha(p)) = (x, \varphi_\alpha(p))$$

Thus, $\psi_\alpha \circ \sigma_\alpha = id$ is satisfied.

Conversely, we will show that a local trivialization map defines a local cross sections. The locally trivialization maps ψ_α define local cross sections σ_α by

$$\sigma_\alpha(x) = \psi_\alpha^{-1}(x,e), \quad \sigma_\alpha(x)g = \psi_\alpha^{-1}(x,g),$$

where e identity element of G. Let us define the identity map as

$$id : U_\alpha \to U_\alpha \times G,$$

$$id : x \mapsto (x,e),$$

and then the local cross section is defined by $\sigma_\alpha = \psi_\alpha^{-1} \circ id$.

Now, we define trivial principal fibre bundles.

Definition 2.3 [22] Given a Lie group G and a manifold M, G acts freely on $P = M \times G$ on the right as follows: for each $g_2 \in G$, R_{g_2} maps $(x, g_1) \in M \times G$ into $(x, g_1 g_2) \in M \times G$. Then the principal fibre bundle $P(M, G)$ is called trivial.

2.1.1 Transition functions

This is the section where we present the transition functions that play an important role in the construction of a principal fibre bundle.

Definition 2.4 [2] A mapping $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ can be defined by

$$\psi_{\beta\alpha}(\pi(p)) = \varphi_\beta(p)\,(\varphi_\alpha(p))^{-1}. \tag{2.1}$$

The family of mappings $\psi_{\beta\alpha}$ are called transition functions of the bundle $P(M, G)$ corresponding to the open covering $\{U_\alpha\}$ of M.

[21] Let us show that the following identity is valid :

$$\psi_\alpha^{-1}(x, e)\varphi_\alpha(p) = p. \tag{2.2}$$

Proof: [21] First, put $\psi_\alpha^{-1}(x, e) = \tilde{p}$ where $\varphi_\alpha(\tilde{p}) = e$ and $\pi(\tilde{p}) = x$. Then, $\tilde{p}\varphi_\alpha(p) = p$ for every $p \in P$. Now, let us write $\varphi_\alpha(\tilde{p}g) = \varphi_\alpha(\tilde{p})g = g$ and since φ_α^{-1} exists we conclude that $\tilde{p}g = \varphi_\alpha^{-1}(g) = p$. Next, it follows that

$$\sigma_\alpha(x)\varphi_\alpha(p) = p = \sigma_\beta(x)\varphi_\beta(p) \tag{2.3}$$

$$\varphi_\alpha(p)\left[\varphi_\beta(p)\right]^{-1} = [\sigma_\alpha(x)]^{-1}\sigma_\beta(x) \tag{2.4}$$

where

$$\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e) = p\,[\varphi_\alpha(p)]^{-1} \quad , \quad \sigma_\beta(x) = \psi_\beta^{-1}(x, e) = p\left[\varphi_\beta(p)\right]^{-1}.$$

Notice that $\varphi_\alpha(p)\left[\varphi_\beta(p)\right]^{-1}$ depends only on $\pi(p) = x \in U_\alpha \cap U_\beta$ not on p. Accordingly, this results in the Definition 2.4.

The importance of the transition functions given in (2.1) arises from the following fact.

Proposition 2.1 [2,21] The transition functions given in (2.1) satisfy the following cocycle property.

$$\psi_{\gamma\alpha}(x) = \psi_{\gamma\beta}(x) \cdot \psi_{\beta\alpha}(x) \quad x \in U_\alpha \cap U_\beta \cap U_\gamma \tag{2.5}$$

Thus, $\psi_{\alpha\beta}(x)^{-1} = g_{\beta\alpha}(x)$ and $\psi_{\alpha\alpha}(x) = id$. The proof can be found in [2].

These functions are important because they connect to local parts of the principal fiber bundle. It is concluded that a principal bundle is trivial if and only if there exists a differentiable mapping $\varphi : P \to G$ such that $\varphi(pg) = \varphi(p)g \ \forall p \in P, \forall g \in G$. Indeed, if a smooth global cross section exists, $\psi(p)$ defines φ by the identification $\psi(p) = (x,g) := (x, \varphi(p))$.

2.2 Vector Bundles

The purpose of this section is to provide some definitions and properties regarding vector bundles.

Definition 2.5 [1] Let M be a manifold. *A vector bundle E* over M is a fibre bundle whose fibres are (real or complex) vector spaces. That is, E is a manifold equipped with a smooth projection $\pi : E \mapsto M$. For each $m \in M$ the fibre $E_m = \pi^{-1}(m)$ has the structure of a vector space, and there is an open neighbourhood U_m of m such that $\pi^{-1}(U_m) \cong U_m \times V$, where V is the fibre of E.

Given a vector bundle one can construct various new bundles like the Whitney sum bundle, tensor product bundle, pull back bundle [3,22]. For our purposes, we give the definition of the tangent bundle, cotangent bundle and the whitney sum bundle because we will need them in Section 4 to consider the whitney sum bundle $TM \oplus T^*M$.

Definition 2.6 [2,23] Let $\{x^1, x^2, ..., x^n\}$ be the local coordinates in a neighborhood of x. The tangent space $T_x(M)$ at the point x is a vector space spanned by the basis $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}\}$. A tangent vector v can be locally written as $v = v^i \frac{\partial}{\partial x^i}$, where $\frac{\partial}{\partial x^i} = d\varphi^{-1}(e^i)$, $e^i = (0, ..., 1, ..., 0)$ with $\varphi : U \subset M \to V \subset \mathbb{R}^n$.

A *tangent vector* at x is the tangent vector at $t = 0$ of some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = x$.

The tangent bundle TM is the disjoint union of the tangent spaces $T_x(M)$, for all $x \in M$.

$$TM = \{(x, v); x \in M, v \in T_x(M)\}$$

Definition 2.7 [2,23] Let $\{x^1, x^2, ..., x^n\}$ be the local coordinates in a neighborhood of x. The cotangent space $T_x^*(M)$ at the point x is a vector space of linear maps

$$\alpha : T_x(M) \to \mathbb{R} \quad v \mapsto \langle \alpha, v \rangle$$

which is also called as space of covectors at x. An assignment of a covector at each point x is called a 1-form. $\{dx^1, dx^2, ..., dx^n\}$ forms a local basis for $T_x^*(M)$. It is the dual basis of $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}\}$; $\langle \frac{\partial}{\partial x^n}, dx^j \rangle = \delta_i^j$.

The cotangent bundle T^*M is the disjoint union of the cotangent spaces $T_x^*(M)$, for all $x \in M$.

$$T^*M = \{(x, w); x \in M, w \in T_x^*(M)\} \tag{2.6}$$

Let us give the definition of G-structures in vector bundles.

Definition 2.8 Let $\pi : E \to M$ be a vector bundle of rank n, and suppose G is a subgroup of $GL(n, \mathbb{R})$. A G-structure on E is a maximal system of smooth local trivializations $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{F}^m$ which cover M and have smooth G-valued transition maps:

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to G.$$

The trivializations (U_α, Φ_α) are called G-compatible.

Definition 2.9 (Whitney sum bundles)

[21,24] Let E_1 and E_2 be two vector bundles with fibres V_1 and V_2 and the structure groups G_1, G_2 over the same base manifold M. The Whitney sum of E_1 and E_2 is defined to be the vector bundle $E_1 \oplus E_2$ by taking the direct sum of the fibres $V_1 \oplus V_2$ and the structure group $G_1 \times G_2$ at every point of the base manifold M. This is given by the restriction of $E_1 \times E_2 \to M \times M$ to the diagonal $M \subset M \times M$. That means the transition functions of $E_1 \oplus E_2$ become the following form:

$$\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix} \tag{2.7}$$

In the next section we will explain the links between vector and principal bundle by showing how to translate from one to the other.

2.2.1 Associated bundles

It is a well-known fact that given a vector bundle one can associate a principal bundle to it. Conversely, given a principal bundle one can associate a vector bundle to it. This will be crucial for us since in Section 4 given a vector bundle $TM \oplus T^*M$ we will associate a principal bundle and we will call it generalised frame bundle in Section 4.3.

Definition 2.10 [2,21,22] Let $P(M,G)$ be a principal bundle and F is a manifold together with a smooth action of the Lie group G. Let us define the left action by

$$g : v \in F \mapsto g\, v. \tag{2.8}$$

Then, a right action can be defined similarly on $P \times F$ by

$$g : (p,v) \mapsto (p,v)g := \left(pg, g^{-1}v\right). \tag{2.9}$$

This action determines an equivalence relation $\sim$ on $P \times F$

$$\left(p', v'\right) \sim (p,v) \Leftrightarrow (p,v)g = \left(p', v'\right) \tag{2.10}$$

Let E be the set of the equivalence classes $\{p,v\}$

$$E := (P \times F)/\sim \equiv P \times_G F. \tag{2.11}$$

The well-defined projection $\pi_E : E \to M$ is given as follows $\pi_E\{p,v\} = \pi(p)$.

We now define the associated bundle to a principal bundle as follows.

Definition 2.11 [2] The quotient space $(P \times F)/G$ is a smooth fiber bundle $E(M,F,G,P)$ called the bundle associated to the principal fibre bundle $P(M,G)$ with fiber F, and is often denoted by $P \times_G F$.

[21] If the fiber F is $\mathbb{R}^n$, with group $G = GL(n,\mathbb{R})$ then the associated bundle $E = E(M,\mathbb{R}^n,GL(n,\mathbb{R}),P)$ is called a real vector bundle associated to the principal bundle $P(M,GL(n,\mathbb{R}))$. Similarly, if the fiber F is $\mathbb{C}^n$ with group $G = GL(n,\mathbb{C})$, then the associated vector bundle E is a complex vector bundle.

Now, we shall give a well known example of the principal fibre bundle and the associated bundle. First, we introduce the followings.

Definition 2.12 [2,21] A frame $u(p) = (X_1, \dots, X_n)$ at a point $p \in M^n$ is a basis of the tangent space T_pM. Let $(e_1, e_2, ..., e_n)$ be a basis for $\mathbb{R}^n$ with $v = r^l e_l \in \mathbb{R}^n$ where $r^l \in \mathbb{R}^n$. Here $X_k = \sum_n X_k^n \frac{\partial}{\partial x^n}$ with non-singular matrices X_k^n. $L(M)$ denotes the set of all linear frames at all points of M.

Example 2.1 [21] $L(M)(M, GL(n,\mathbb{R}^n))$ is a principal bundle over the base manifold M with the structure group $GL(n,\mathbb{R}^n)$ with the map $\pi : L(M) \to M$ and with the fibre $\mathbb{R}^n$. The action of $GL(n,\mathbb{R})$ on $L(M)$ takes a frame into another frame and it is just given by matrix multiplication from the right,

$$(u,A) \mapsto uA \qquad u \in L(M), \quad A \in GL(n,\mathbb{R}). \tag{2.12}$$

$(p, X_k(p))$ can be considered as a typical element of $L(M)$. Let $GL(n,\mathbb{R}^n)$ act on $\mathbb{R}^n$ as above. Then, $L(M) \times \mathbb{R}^n$ consists of the elements of the form: $((p, X_k(p)), r^l)$. Then, one can define

$$((p, X_k(p)), r^l) \mapsto r^l X_k(p) \in T_p(M). \tag{2.13}$$

Hence, the tangent bundle $T(M)$ over M is the vector bundle associated with the principal bundle $L(M)$ with the fibre $\mathbb{R}^n$; $T(M) = L(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$.

We now define the associated principal bundle to a vector bundle which will be significant in Section 4.

Remark 2.2 Let $G \subset GL(n,\mathbb{R})$ be a subgroup, then a G-structure of a vector bundle E is the corresponding G-structure of the associated principal fibre bundle.

Definition 2.13 [21] Assume E is a vector bundle with standard fibre V and structure group G. Then, an associated principal bundle $P(E)$ can be constructed over the same base manifold M with the same transition functions by taking G as fibres.

We now give an example of a principal bundle associated to a vector bundle.

Definition 2.14 [1,4] Let M be a manifold, and $E \to M$ a vector bundle with fibre $\mathbb{R}^k$. Define a manifold F^E by

$$F^E = \{(m, e_1, \dots, e_k) : m \in M \text{ and } (e_1, \dots, e_k) \text{ is a basis for } E_m\}$$

Define $\pi : F^E \to M$ by $\pi : (m, e_1, \dots, e_k) \mapsto m$. For each $A = (A_{ij})$ in $\mathrm{GL}(k,\mathbb{R})$ and $(m, e_1, \dots, e_k)$ in F^E, define $A \cdot (m, e_1, \dots, e_k) = (m, e'_1, \dots, e'_k)$, where $e'_i = \sum_{j=1}^k A_{ij} e_j$.

This gives an action of $\mathrm{GL}(k,\mathbb{R})$ on F^E, which makes F^E into a principal bundle over M, with fibre $\mathrm{GL}(k,\mathbb{R})$. We call F^E the frame bundle of E.

2.3 G-structures

In this thesis, we will be working on the generalised G-structures. It will be defined in terms of a reduction of the structure group of the linear frame bundle. Hence, throughout this section, we are going to discuss some definitions and properties of the reduction of the structure groups. Assume that we have a principal bundle $P(M,G)$ over M, and a principal bundle $Q(N,H)$ on N.

Definition 2.15 [22] A homomorphism of principal bundles consists of a smooth map $\hat{f}: Q \longrightarrow P$ together with a Lie group homomorphism $h: H \longrightarrow G$ such that for all $u \in Q$ and $a \in H$, $\hat{f}$ satisfies

$$\hat{f}(ua) = \hat{f}(u)h(a).$$

This condition implies that $\hat{f}$ maps fibers of Q to fibers of P; hence, there is a smooth map $f: N \longrightarrow M$ such that the following diagram commutes.

$$\begin{array}{ccc} Q & \xrightarrow{\hat{f}} & P \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ N & \xrightarrow{f} & M \end{array} \tag{2.14}$$

If in addition $\hat{f}$ is an embedding of smooth manifolds and h is a group monomorphism then we say that $\hat{f}: Q \to P$ is an embedding of principal bundles. This implies that the map $f: N \longrightarrow M$ is also an embedding. Moreover, let $N = M$ and f be the identity map on M, then $Q(M,H)$ *is called a subbundle or a reduction of the structure group from* G *to the group* H. We also say that $Q(M,H)$ is a reduced subbundle of $P(M,G)$.

The following Proposition is crucial in order to understand a reduction of structure groups.

Proposition 2.2 [2,22] A principal G-bundle $P(M,G)$ can be reduced to a Lie subgroup H if and only if there is an open cover $\{U_\alpha\}$ of M with transition functions $\psi_{\beta\alpha}$ taking their values in H.

Proof: [22] Suppose that the structure group G of $P(M,G)$ can be reduced to a Lie subgroup H. Then there is a principal fibre bundle $Q(M,H)$ with the structure group H, together with a smooth bundle map $f: Q \to P$ such that the diagram commutes and $f(ua) = f(u)a$ for all $a \in H \subset G$.

On the open set $\pi_Q^{-1}(U_\alpha) \subset Q(M,H)$ the maps φ_α^Q and φ_α^P are related by $\varphi_\alpha^Q = \varphi_\alpha^P \circ f$, since $H \subset G$. It follows that for any $v \in P$ there exists an $a \in G$ and $u \in Q$ such that $v = f(ua) = f(u)a$ and using this it leads to $\varphi_\alpha^P(v) = \varphi_\alpha^Q(u)a$.

The transition functions $\psi_{\beta\alpha}^P$ should be

$$\psi_{\beta\alpha}^P(v) = \varphi_\beta^P(v)\left(\varphi_\alpha^P(v)\right)^{-1} = \varphi_\beta^Q(u)a \cdot a^{-1}\left(\varphi_\alpha^Q(u)\right)^{-1} = \varphi_\beta^Q(u)\left(\varphi_\alpha^Q(u)\right)^{-1}$$

That means, the transition functions have their values in H.

Conversely, given transition functions $\psi_{\beta\alpha}^P : U_\alpha \cap U_\beta \to G$ which take their values in the Lie subgroup H, a standard result says that $\psi_{\beta\alpha}^P$ is smooth as a map into H. Thus, there is a principal fibre bundle $Q(M,H)$ with transition functions $\psi_{\beta\alpha}^P$. To construct the bundle map f we define maps $f_\alpha\ \pi_Q^{-1}(U_\alpha) \to \pi_P^{-1}(U_\alpha)$ by putting $f_\alpha = \psi_P^{-1} \circ \psi_Q$. It can be seen that on $U_\alpha \cap U_\beta$, $f_\alpha = f_\beta$. Therefore, there exist $f: Q \to P$ globally. $\square$

2.3.1 Reduction of the structure group of a principal fibre bundle

The section of this chapter discusses the relation between G-structures and some certain geometrical objects that have a significant role to understand the reduction of the structure groups. Let us first describe G-structures.

Definition 2.16 [22] Let $G \subset GL(n,\mathbb{R})$ be a subgroup, then a G-structure on M is a reduction of the structure group of the frame bundle to the subgroup G.

The reduction of the G-structures corresponds to the existence of geometrical objects on M. In general, G-structures can be thought of as a set of classical structures defined on manifolds, some of which resemble tensor fields, in some cases. Now we will examine the most common examples of the G-structures.

2.3.2 SL($n,\mathbb{R}$)-Structure

We discuss the relation between SL($n,\mathbb{R}$)-structures and the volume forms vol according to [3,25]. An action of GL($n,\mathbb{R}$) on $\Lambda^n\mathbb{R}^n$ is induced by the natural action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ as follows:

$$Av = \det(A) \cdot v \tag{2.15}$$

for $A \in \mathrm{GL}(n,\mathbb{R})$ and $v \in \Lambda^n\mathbb{R}^n$. $\Lambda^n\mathbf{R}^n - \{0\} = \mathrm{GL}(n,\mathbb{R})/SL(n,\mathbb{R})$ since the action of the group $GL(n,\mathbb{R})$ is transitive on $\Lambda^n\mathbb{R}^n - \{0\}$ with isotropy subgroup $SL(n,\mathbb{R})$. It follows that the sections of the bundle $L(M)/SL(n,\mathbb{R})$ are in one-to-one correspondence with the volume elements of M. This means that an $SL(n,\mathbb{R})$-structure can essentially be regarded as a volume element on M. It is clear that M admits an $SL(n,\mathbb{R})$-structure if and only if it is orientable. In this regard, it is apparent that M admits an $SL(n,\mathbb{R})$-structure if and only if it is orientable.

2.3.3 $O(n)$-Structure

[2,3,22,25] Let $L(M)$ be the linear frame bundle over an n-dimensional manifold M. Let $(,)$ be the natural inner product and the basis $\{e_1 = (1,0,\ldots,0),\ldots,e_n = (0,\ldots,0,1)\}$ for $\mathbb{R}^n$ which are orthonormal and invariant under the group $O(n)$, and

$$O(n,\mathbb{R}) = \left\{A \in GL(n,\mathbb{R}) | A^{tr}A = 1_n\right\}$$

First, we aim to show that any reduction of the structure group from $GL(n,\mathbb{R})$ to $O(n)$ induces a Riemannian metric g on M. Let $Q(M,O(n))$ be a reduced subbundle of $L(M)$. Each $u \in L(M)$ can be seen as a linear isomorphism of $\mathbb{R}^n$ onto $T_p(M)$ where $p = \pi(u)$, each $u \in Q$ defines an inner product g in $T_p(M)$ by

$$g(X,Y) = \left(u^{-1}X, u^{-1}Y\right) \quad \text{for } X,Y \in T_p(M)$$

Transition functions can be described as follows. First, by using the existence of a metric g an orthonormal frame for T_pM can be constructed via the Gram-Schmidt process. Let $(U_i;x_1,\ldots,x_n)$ and $\left(V_j;y_1,\ldots,y_n\right)$ be two local orthonormal frames on M.

If $x \in U_i \cap V_j$, then

$$V_i = \sum_j H_i^j \frac{\partial}{\partial x_j} = \sum_j K_i^j \frac{\partial}{\partial y_j}$$

where the H_i^j and are K_i^j the components of non-singular orthogonal matrices $\left(H_i^j\right)$ and $\left(K_i^j\right)$ of smooth functions on U, respectively. Since $H_i^j = K_i^j \, (\frac{\partial x_k}{\partial y_i})\Big|_x$, transition functions can be described as

$$\psi_{\alpha\beta}(x) = \psi_i^k(x) = (\frac{\partial x_k}{\partial y_i})\Big|_x \in O(n) \tag{2.16}$$

If $A \in O(n)$ we have

$$\begin{aligned} g_{uA}(X,Y) &= \left\langle (uA)^{-1}X, (uA)^{-1}Y \right\rangle = \left\langle A^{-1}u^{-1}X, A^{-1}u^{-1}Y \right\rangle \\ &= \left\langle u^{-1}X, u^{-1}Y \right\rangle = g_u(X,Y) \end{aligned} \tag{2.17}$$

by the invariance of $\langle \cdot, \cdot \rangle$ under $O(n)$. This shows that g_u is constant along the fibers of $O(M)$, and thus, is a section of $L(M)/O(n)$. Since the natural inner product $(,)$ is invariant under $O(n)$, it follows that $g(X,Y)$ does not depend on $u \in Q$.

Conversely, let M be a manifold equipped with a Riemannian metric g, and Q be the subset of $L(M)$ consisting of orthonormal linear frames $u = (X_1, \ldots, X_n)$ with respect to g. Let $u \in L(M)$ be a linear isomorphism of $\mathbb{R}^n$ onto $T_p(M)$, then $u \in Q$ if and only if $(\xi, \xi') = g\,(u\xi, u\xi')$ for all $\xi, \xi' \in \mathbb{R}^n$.

It can be checked that Q becomes a reduced subbundle of $L(M)$ over M with structure group $O(n)$. The bundle Q will be called the bundle of orthonormal frames over M and will be denoted by $O(M)$. An element of $O(M)$ is an orthonormal frame.

Remark 2.3 Note that, the special orhogonal group $SO(n)$ is defined to be

$$SO(n) = \{A \in O(n) : det(A) = 1\} \tag{2.18}$$

The special orthogonal group can be written as

$$SO(n) = O(n) \cap GL^+(n, \mathbb{R}).$$

Hence, $SO(n)$-structure corresponds to the existence of an oriented Riemannian geometry with a globally defined metric volume form, [22,25].

2.3.4 $GL(n/2,\mathbb{C})$-Structure

We now discuss the relationship between $GL(n/2,\mathbb{C})$-structures and an almost complex structure J, [3,22,25]. $\mathbb{R}^n$ is identified with $\mathbb{C}^{n/2}$ as follows. Let $z_1,...,z_{n/2}$ be the coordinate system for $\mathbb{C}^{n/2}$ where $z_k = x_k + iy_k$ and then $x_1,...,x_{n/2},y_1,...,y_{n/2}$ be the coordinate system for $\mathbb{R}^n$. Now, the multiplication by i in $\mathbb{C}^{n/2}$ is given by the following linear transformation J of $\mathbb{R}^n$.

$$J:\ (x_1,...,x_{n/2},y_1,...,y_{n/2}) \to (y_1,...,y_{n/2},-x_1,...,-x_{n/2}) \tag{2.19}$$

J is called the canonical complex structure and can be written as a matrix form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

In particular $GL(n/2,\mathbb{C})$ can be identified as a subgroup of $GL(n,\mathbb{R})$.

$$GL(n/2,\mathbb{C}) = \{A \in GL(n,\mathbb{R}) | A\,J = J\,A\}\,.$$

This can be seen as follows. Let us take an arbitrary matrix of $GL(n,\mathbb{R})$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.20}$$

where A,B,C,D are $n/2 \times n/2$ matrices. Then,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.21}$$

It follows that $A = D,\ B = -C$. Therefore its matrix form is given by

$$A + iB \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{2.22}$$

Let M_n be an n-dimensional manifold which admits an almost complex structure. It can be seen as a nowhere vanishing fibre-preserving map $J : TM \to TM$ such that $J^2 = -1$. The linear frames $u : \mathbb{R}^n \to T_x(M)$ satisfying $u\ J = J_a\ u$ shows that J_a which is the endomorphism of the T_aM is a section of the associated subbundle $L(M)/GL(n/2,\mathbb{C})$. This is the desired reduced subbundle to have a $GL(n/2,\mathbb{C})$-structure.

Note that one can complexify the tangent space T_xM as follows:

Definition 2.17 The complexified tangent space decomposes to +i and -i eigenspaces of J as follows:

$$T_x^{\mathbb{C}}(M) = T_x^{1,0}(M) \oplus T_x^{0,1}(M) \tag{2.23}$$

where

$$T_{1,0}(M) = \{Z \in T_x^{\mathbb{C}}(M) : \ JZ = iZ\} = \{X - iJX; \ X \in T_x^{\mathbb{C}}(M)\} \tag{2.24}$$

$$T_{0,1}(M) = \{Z \in T_x^{\mathbb{C}}(M) : \ JZ = -iZ\} = \{X + iJX; \ X \in T_x^{\mathbb{C}}(M)\} \tag{2.25}$$

Definition 2.18 (holomorphic coordinates)

On a complex manifold M we can introduce holomorphic coordinates as $z^1, \ldots, z^n, z^j = x^j + iy^j$, a complex local coordinate system of M. Then,

$$dz^j = dx^j + idy^j \qquad d\bar{z}^j = dx^j - idy^j \tag{2.26}$$

Definition 2.19 A form ω of degree $(p,0)$ is said to be holomorphic if $\bar{\partial}\omega = 0$.

If we express ω in terms of $z^1, \ldots, z^n$:

$$\omega = \sum_{1 \le j_1 < \cdots p} f_{j_1 \cdots j_p} dz^{j_1} \wedge \cdots \wedge dz^{j_n},$$

then $\bar{\partial}\omega = 0$ if and only if $\bar{\partial} f_{j_1 \cdots j_n} = 0$.

2.3.5 $U(n/2)$-Structure

We now discuss that there is a one-to-one correspondence between a $U(n/2)$-structure and almost Hermitian structures, [3,22,25]. Let us begin with the description of an almost Hermitian structure.

Definition 2.20 [22] Let M be a manifold equipped with a Riemannian metric g and an almost complex structure J with

$$g(JX, JY) = g(X,Y) \quad X, Y \in \Gamma(TM). \tag{2.27}$$

Then, g is called an *Hermitian metric* and (M, g, J) is called *almost Hermitian manifold.* Furthermore, given a Hermitian metric we define a skew-symmetric non-degenerate real two form which kept invariant by J as follows.

$$\omega(X,Y) = g(X, JY) \quad X, Y \in \Gamma(TM). \tag{2.28}$$

The triple $(J,\ g,\ \omega)$ is called an almost Hermitian structure.

Definition 2.21 The unitary group $U(n/2)$ is defined to be

$$U(n/2) = \left\{A \in GL(n/2,\mathbb{C}) | \bar{A}^{tr} A = I_{n/2}\right\}. \tag{2.29}$$

A manifold with a $U(n/2)$-structure is equipped with an almost complex structure J and a Riemannian metric g since

$$U(n/2) = GL(n/2,\mathbb{C}) \cap O(n).$$

2.3.6 $SU(n/2)$-Structure

There exists a further reduction of the structure group from $U(n/2)$ to $SU(n/2)$ associated to the existence of a globally defined complex $n/2$-form Ω and it implies the existence of an orientation on M. That is given by

$$\frac{1}{n/2!}\omega^{n/2} = \frac{i^{n(n+2)}}{2^{n/2}}\Omega \wedge \bar{\Omega} = \mathrm{vol}_n. \tag{2.30}$$

Therefore, a manifold with an $SU(n/2)$-structure is equipped with an almost complex structure J, an orientation, Hermitian form ω and a Riemannian metric g, [22,25]–[27].

Definition 2.22 [3,22,26] Let g be a Riemannian metric on M, and let (ω,Ω) be respectively a real two-form and a complex $n/2$-form. $SU(n/2)$-structures can be described by (ω,Ω) satisfying the following compatibility condition:

$$\omega \wedge \Omega = 0. \tag{2.31}$$

In chapter 6, the seed background in the specific examples will be assumed to support the $SU(3)$ structure, and we will see directly how the certain differential forms associated to the $SU(3)$ structure are transformed to the certain differential forms associated to an $SU(2)$ structure. Therefore, for the purposes of the examples studied in chapter 6, we will present these structures in detail.

On a 6-dimensional manifold M, reduction of the structure group of the tangent bundle TM to $SU(3)$ is equivalent to existence on M of an invariant real 2-form J and a complex 3-form Ω satisfying the following compatibility conditions, [28]–[31].

$$\begin{aligned} \frac{i}{8}\Omega \wedge \bar{\Omega} &= \frac{1}{3!}J \wedge J \wedge J = \mathrm{vol}_6, \\ J \wedge \Omega &= 0. \end{aligned} \tag{2.32}$$

Similarly, one can also define an $SU(2)$ structure on 4 dimensional manifold M. As we will study in chapter 6, we will need an $SU(2)$ structure on 5 dimensional manifold M. It was first defined in [32] on hypersurfaces of 6 dimensional manifolds endowed with an $SU(3)$-structure. This is a generalisation of Sasakian-Einstein metrics in dimension 5. An $SU(2)$ structure on 5 dimensional M is described through a real and a complex 2-forms plus a nowhere vanishing 1-form on M satisfying certain relations, [32]. Particularly, if these differential forms on a 5 dimensional Riemannian manifold M satisfy certain conditions then M is a Sasaki-Einstein manifold. This is the special case when the 6 dimensional metric cone $M \times \mathbb{R}$ is Kähler and Ricci flat. As we will discuss in Chapter 6 in Example 6.1, Sasaki-Einstein manifolds will be crucial to understanding the examples for the ansatz given in (6.1). In this special case, $SU(2)$ structure on M takes the following form.

$SU(2)$ structure on 5 dimensional manifold M is characterized by the existence of a complex 1-form $z = v + iw$, a real 2-form j and a complex 2-form ω satisfying the following compatibility conditions [28]–[31]:

$$\begin{aligned} \omega \wedge j &= 0, \\ \mathrm{i}_z j = \mathrm{i}_z \omega &= 0, \\ \omega \wedge \bar{\omega} &= 2j \wedge j. \end{aligned} \tag{2.33}$$

Here, $J = j + v \wedge w$ and $\Omega = \omega \wedge z$. Let us show that J, Ω satisfy (2.32). For the first identity, we calculate

$$\begin{aligned} \frac{i}{8}\Omega \wedge \bar{\Omega} &= \frac{1}{2} j \wedge j \wedge v \wedge w \\ \frac{1}{3!} J \wedge J \wedge J &= \frac{1}{2} j \wedge j \wedge v \wedge w. \end{aligned} \tag{2.34}$$

For the second identity, by using (2.33) we show

$$J \wedge \Omega = j \wedge \omega \wedge z + v \wedge w \wedge \omega \wedge z = 0. \tag{2.35}$$

2.4 Connections on Principal Bundles

In the following sections, using a manifold M and a tangent bundle TM, we present connections on the principal fibre bundle and on TM, define torsion as it relates to a connection on TM, and then discuss holonomy groups and its relation with the torsion-free connections on TM.

Definition 2.23 [1,4] Let M be a manifold, and P a principal bunle over M with fibre G, a Lie group. A connection on P is a vector subbundle D of TP called the horizontal subbundle, that is invariant under the G-action on P, and which satisfies $T_pP = C_p \oplus D_p$ for each $p \in P$.

If $\pi(p) = m$, then $\mathrm{d}\pi_p$ maps $T_pP = C_p \oplus D_p$ onto T_mM, and as $C_p = \mathrm{Ker}\,\mathrm{d}\pi_p$, we see that $\mathrm{d}\pi_p$ induces an isomorphism between D_p and T_mM.

Now, we consider how the connection on the reduced subbundle relates to the connection on the linear frame bundle.

Theorem 2.1 [1,4] Let M be a manifold, P a principal bundle over M with fibre G, and D a connection on P. Fix $p \in P$, let $H = \mathrm{Hol}_p(P,D)$, and suppose that H is a closed Lie subgroup of G. Define $Q = \{q \in P : p \sim q\}$. Then Q is a principal subbundle of P with fibre H, and the connection D on P restricts to a connection D' on Q. In other words, P reduces to Q, and the connection D on P reduces to D' on Q.

It is possible to relate the connections between vector bundles and principal bundles as given in Definition 2.11, [1,4]. We now present how to translate the connections on principal bundles to vector bundles.

Definition 2.24 [1,4] Suppose M is a manifold, P a principal bundle over M with fibre G, and D a connection on P. Let ρ be a representation of G on a vector space V, and define E to be the vector bundle $\rho(P)$ over M. If $e \in C^\infty(E)$, then $\pi_D(\mathrm{d}\pi^*(e))$ is a G-invariant section of $V \otimes \pi^*(T^*M)$ over P. Define $\nabla^E e \in C^\infty(E \otimes T^*M)$ to be the unique section of $E \otimes T^*M$ with pull-back $\pi_D(\mathrm{d}\pi^*(e))$ under the natural projection $V \otimes \pi^*(T^*M) \to E$. This defines a connection ∇^E on the vector bundle E over M.

It has been determined that every connection D inside a principal bundle P has been assigned a unique connection ∇^E on the vector bundle $E = \rho(P)$. Suppose $G = \mathrm{GL}(k,\mathbb{R})$ and ρ is the standard representation of G on $\mathbb{R}^k$, so that P is the frame bundle F^E of E, then this means that there is a one-one correspondence between connections on F^E and E. Now, we give the notion of connections on TM compatible with G-structure.

Definition 2.25 [1,4] A connection ∇ on TM is called compatible with the G-structure if the corresponding connection on the linear frame bundle reduces to the connection on the principal subbundle.

We will therefore focus our attention on torsion-free connections in the sense that given a G-structure on a manifold M we investigate the uniqueness of torsion-free connections on TM compatible with G-structure.

3. G-HOLONOMY

Now, we will describe the notion of holonomy group of a connection on a vector bundle rather than on the tangent bundle. We will give some crucial properties.

3.1 Parallel Transport

Definition 3.1 [1,4]Let M be a manifold, $E \to M$ be a vector bundle over M, and ∇^{E} be a connection on E. Let $\gamma : [0,1] \to M$ be a smooth curve in M. Then the pull-back $\gamma^*(E)$ of E to $[0,1]$ is a vector bundle over $[0,1]$ with fibre $E_{\gamma(t)}$ over $t \in [0,1]$, where E_x is the fibre of E over $x \in M$.

Definition 3.2 [1,4] Let s be a smooth section of $\gamma^*(E)$ over $[0,1]$, so that $s(t) \in E_{\gamma(t)}$ for each $[0,1]$. The connection ∇^E pulls back under γ to give a connection on $\gamma^*(E)$ over $[0,1]$. Then, s is said to be parallel if its derivative under this pulled-back connection vanishes, that is $\nabla^E_{\dot\gamma(t)} s(t) = 0$ for all $t \in [0,1]$, where $\dot\gamma(t)$ is $\frac{d}{dt}\gamma(t)$, regarded as a vector in $T_{\gamma(t)}$.

As can be seen from the equation, it is an ordinary differential equation of the first order in $s(t)$,and for any initial value $e \in E_{\gamma(0)}$, then there exists a unique smooth solution s with $s(0) = e$. It is with the help of this idea that we will describe the parallel transport along along γ.

Definition 3.3 [1,4] Let M be a manifold, E be a vector bundle over M, and ∇^E a connection on E. Suppose $\gamma : [0,1] \to M$ is continuous and piecewise-smooth, with $\gamma(0) = x$ and $\gamma(1) = y$, where $x, y \in M$. Then for each $e \in E_x$, there exists a unique continuous smooth section s of $\gamma^*(E)$ satisfying $\nabla^E_{\dot\gamma(t)} s(t) = 0$ for $t \in [0,1]$, with $s(0) = e$. Define $P_\gamma(e) = s(1)$. Then $P_\gamma : E_x \to E_y$ is a well-defined linear map, called *the parallel transport map*.

Now, we define the holonomy group.

Definition 3.4 [1,4] Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Fix a point $x \in M$. We say that γ is a loop based at x if $\gamma : [0,1] \to M$ is piecewise-smooth path with $\gamma(0) = \gamma(1) = x$.

If γ is a loop based at x, then the parallel transport map $P_\gamma : E_x \to E_x$ is an invertible linear map, so that P_γ lies in $\mathrm{GL}(E_x)$, the group of invertible linear transformations of E_x. The holonomy group $Hol_x(\nabla^E)$ of ∇^E based at x is defined to be

$$\mathrm{Hol}_x(\nabla^E) = \{P_\gamma : \gamma \quad \text{is a loop based at } x\} \subset \mathrm{GL}(E_x). \tag{3.1}$$

Remark 3.1 It is called a group since it is a subgroup of $\mathrm{GL}(E_x)$. One of the most important properties of the holonomy group is that it does not depend upon a certain base point, [1,4]. For any $x, y \in M$, $\mathrm{Hol}_x(\nabla^E)$ is isomorphic to $\mathrm{Hol}_y(\nabla^E)$.

Then, it induces the next Proposition.

Proposition 3.1 [1,4] Let M be a manifold, E a vector bundle over M with fibre $\mathbb{R}^n$ and ∇^E a connection on E. For each $x \in M$, the holonomy group $\mathrm{Hol}_x(\nabla^E)$ may be regarded as a subgroup of GL $(n, \mathbb{R})$ defined up to conjugation in $\mathrm{GL}(n, \mathbb{R})$, and in this sense it is independent of the base point x.

Next, it is shown that if M is a simply-connected manifold then Hol (∇^E) is a connected Lie group.

Proposition 3.2 [1,4] Let M be a simply-connected manifold, E vector bundle over M with fibre $\mathbb{R}^n$, and ∇^E a connection on E. Then, Hol (∇^E) is a connected Lie subgroup of GL $(n, \mathbb{R})$.

If M is not simply-connected, the restricted holonomy group $\mathrm{Hol}^0(\nabla^E)$, is defined as follows.

Definition 3.5 [1,4] Let M be a manifold, E a vector bundle over M with fibre $\mathbb{R}^n$, and ∇^E a connection on E. Fix $x \in M$.

A loop γ based at x is called null-homotopic if it can be deformed to the constant loop at x.

Define the restricted holonomy group $\mathrm{Hol}_x^0(\nabla^E)$ of ∇^E to be

$$\mathrm{Hol}_x^0(\nabla^E) = \{P_\gamma : \gamma \textit{ is a null-homotopic loop based at } x\}.$$

Then, $\mathrm{Hol}_x^0\left(\nabla^R\right)$ is a subgroup of $\mathrm{GL}\left(E_x\right)$.

As we remarked above, $\mathrm{Hol}_x^0\left(\nabla^E\right)$ can be seen as a connected Lie subgroup of $\mathrm{GL}(\ n, \mathbb{R})$ defined up to conjugation, and it does not depend on the base point x, $\mathrm{Hol}^0\left(\nabla^E\right) \subseteq \mathrm{GL}(n, \mathbb{R})$.

3.2 G-holonomy and Covariantly Constant Tensors

The holonomy group of a manifold M would be able to determine the constant tensors on M and the constant tensors on M would be able to determine the holonomy group. In other words, studying the holonomy of a connection and studying its constant tensors are two different ways of looking at the same thing. Let us begin with presenting covariantly constant tensors.

Definition 3.6 [1,4] Let M be a manifold, ∇ a connection on M, and S an (r,s) tensor on M, so that $S \in C^\infty\left(\bigotimes^r TM \otimes \bigotimes^s T^*M\right)$ for some r, s. S is said to be a covariantly constant tensor or a parallel tensor if $\nabla S = 0$.

Next, it is shown the one-to-one correspondence between covariantly constant tensors on M and the holonomy group $\mathrm{Hol}(\nabla)$.

Proposition 3.3 [1,4] Let M be a manifold, and ∇ a connection on TM. Fix $x \in M$, and let $H = \mathrm{Hol}_x(\nabla)$. Then, H is a subgroup of $\mathrm{GL}\left(T_xM\right)$.

Let E be the vector bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$ over M. Then the connection ∇ on TM induces a connection ∇^E on E, and H has a natural representation on the fibre E_x of E at x.

Suppose $S \in C^\infty(E)$ is a parallel tensor, so that $\nabla^E S = 0$. Then $S|_x$ is invariant under the action of H on E_x. Conversely, if $S_x \in E_x$ is invariant under the action of H, then, there exists a unique tensor $S \in C^\infty(E)$ such that $\nabla^E S = 0$ and $S|_x = S_x$.

There is also a sense in which this Proposition holds for the case of the tangent bundle that can be viewed as a natural vector bundle that is affixed to a smooth manifold M.

Corollary 3.1 [1,4] Let M be a manifold and ∇ a connection on TM, and fix $x \in M$. Define $G \subset \mathrm{GL}\left(T_xM\right)$ to be the subgroup of $\mathrm{GL}\ \left(T_xM\right)$ that fixes $S|_x$ for all parallel tensors S on M. Then $\mathrm{Hol}_x(\nabla)$ is a subgroup of G.

[1,4] It is shown that if ∇ is a fixed connection on TM, then there is a compatible G-structure P if and only if $Hol(\nabla) \subseteq G$. Now, the most crucial result for us will be stated as follows:

Proposition 3.4 [1,4] Let M be a manifold of dimension n, and G a Lie subgroup of GL $(n,\mathbb{R})$. Then M admits a torsion-free G-structure P if and only if there exists a torsion-free connection ∇ on TM with $Hol(\nabla) = H$, for some subgroup H of G.

According to this, it is clear that torsion-free G structures on a manifold correspond to torsion-free connections ∇ on TM with $Hol(\nabla) \subseteq G$ in a one-to-one correspondence.

4. GEOMETRY OF THE GENERALISED TANGENT BUNDLE: $TM \oplus T^*M$

The tangent bundle TM is the vector bundle associated with the linear frame bundle $L(M)$ as given in Definition 2.6. Similarly, the cotangent bundle T^*M is the vector bundle as given in Definition 2.7. In this section we give some facts which will be useful for the rest of the thesis regarding the geometry of $TM \oplus T^*M$. Replacing the tangent bundle TM of a n dimensional manifold M by $TM \oplus T^*M$ and the Lie bracket on sections of TM by the Courant bracket on sections of $TM \oplus T^*M$ is the main idea of development of the Generalised Geometry framework is first established by Hitchin in [33]. The generalised complex geometry framework combines complex and symplectic geometry on $TM \oplus T^*M$, [6,33]. Generalised tangent bundle $TM \oplus T^*M$ is a prototype of a Courant algebroid which was first defined in [34].

Definition 4.1 [6] The generalised tangent bundle is defined to be the direct sum bundle $TM \oplus T^*M$.

We will consider the principal bundle associated to $TM \oplus T^*M$ where the base manifold is an n-dimensional manifold M. Now, we introduce some important properties of $TM \oplus T^*M$ which we will need in the thesis.

Definition 4.2 [5] There is a natural bilinear form defined as

$$\langle X+\xi, Y+\eta \rangle = \frac{1}{2}(\xi(Y)+\eta(X)) \tag{4.1}$$

where $X, Y \in C^\infty(TM)$ and $\xi, \eta \in C^\infty(T^*M)$.

This bilinear form is symmetric, non-degenerate since $\langle X+\xi, Y+\eta \rangle = 0$, for all $Y+\eta$, implies that $X+\xi = 0$ and $\langle X+\xi, Y+\eta \rangle = \langle Y+\eta, X+\xi \rangle$. Let us define the signature of a bilinear form: For the symmetric bilinear form there exists a basis for which it is represented by a diagonal matrix.

Definition 4.3 [35] (p,q) is called the signature of the bilinear form which is the number of positive and negative eigenvalues of the real symmetric matrix of the symmetric bilinear form with respect to a basis. The bilinear form defined in (4.1) has the matrix representation

$$\eta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}. \tag{4.2}$$

which is of signature (n,n) with respect to the local coordinate basis for $C^\infty(T \oplus T^*)$. It can be diagonalized as follows:

$$\eta = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \tag{4.3}$$

Due to the existence of the bilinear form with signature (n,n) given in (4.1) the structure group of the frame bundle associated to the generalized tangent bundle $TM \oplus T^*M$ is reduced to $O(n,n)$, just like the existence of g reduces the structure group to $O(n)$ as discussed in section 2.3.3.

On the fibers of $TM \oplus T^*M$ there is a natural action of $GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \subset GL(2n,\mathbb{R})$ acting in the following way. The action of $GL(n,\mathbb{R})$ on the linear frame bundle of M takes a frame into another frame and it is just given by matrix multiplication from the right,

$$(u,u^*) \mapsto (uA, u^*(A^{-1})^t) \qquad u \in TM, u^* \in T^*M \quad A \in GL(n,\mathbb{R}). \tag{4.4}$$

The following matrix

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \tag{4.5}$$

acts on $C^\infty(TM \oplus T^*M)$. The structure group acting on the generalised tangent bundle $TM \oplus T^*M$ can be extended from $GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$ to $O(n,n)$ preserving the metric by considering the following transformations: B-transformation which will be given in Definition 4.6 where $B : TM \to T^*M$ and β-transformation which will be given in Definition 4.7 where $\beta : T^*M \to TM$ shortly. Before we discuss these transformations let us introduce some basic facts about $O(n,n)$. Discussion will be on a vector space V but will be translated to an n-dimensional manifold M with the tangent bundle TM.

4.1 Orthogonal Matrices and Transformations

Let V be a vector space of signature (n,n). The orthogonal group $O(n,n)$ is the space of automorphisms of V preserving a bilinear form Q of signature (n,n):

$$O(n,n) = \{A \in Aut(V) : Q(AX,AY) = Q(X,Y), \quad \forall X,Y \in V\} \tag{4.6}$$

With respect to a suitable basis that we have chosen the following matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.7}$$

(4.6) is equivalently written as $A^t \eta A = \eta$ where η is given in (4.2), [20,36]. It leads to the following identities.

$$a^t c + c^t a = 0, \;\; b^t d + d^t b = 0, \;\; a^t d + c^t b = I. \tag{4.8}$$

Note that this is equivalent to $A \eta A^t = \eta$, in this case it leads to the following set of identities.

$$ab^t + ba^t = 0, \;\; cd^t + dc^t = 0, \;\; a^t d + c^t b = I. \tag{4.9}$$

Let $T \in O(n,n)$ be as given in (4.7). This matrix can be embedded in a larger $O(D,D)$ matrix which will be useful understanding Definition 5.1.

$$\hat{T} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \tag{4.10}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are $D \times D$ matrices defined below:

$$\hat{a} = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, \;\; \hat{b} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \;\; \hat{c} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \;\; \hat{d} = \begin{pmatrix} d & 0 \\ 0 & I \end{pmatrix}. \tag{4.11}$$

Definition 4.4 The corresponding orthogonal Lie algebras $so(n,n) = o(n,n)$ are the endomorphisms $A : V \to V$ such that

$$Q(AX,Y) + Q(X,AY) = 0 \tag{4.12}$$

for all $X, Y \in V$.

Due to the existence of the decomposition $V = W \oplus W^*$, the special orthogonal Lie algebra decomposes as follows:

$$so(W \oplus W^*) = \wedge^2(W \oplus W^*) = End(W) \oplus \wedge^2 W^* \oplus \wedge^2 W \tag{4.13}$$

an arbitrary element $Z \in so(n,n)$ can be written as follows, [5,6] :

$$Z = \begin{pmatrix} A & \beta \\ -B & -A^t \end{pmatrix} \tag{4.14}$$

where $A \in End(W)$, $\beta \in \wedge^2 W^*$ and $B \in \wedge^2 W$. One can choose $\{e^1, \cdots, e^n\}$ to form a basis for W and $\{e_1, \cdots, e_n\}$ to form a basis for W^*. Let us analyze exponentiation of these elements which gives $O(n,n)$ elements.

Definition 4.5 [6] $A \in End(W)$ is defined as $A = A_i^j e^i \otimes e_j$. Its action on $W \oplus W^*$ is given as

$$X + \xi \mapsto A(X) - A^t(\xi) \tag{4.15}$$

Note that $A = A_i^j e^i \otimes e_j$ maps $e_i \mapsto e_j$ and $e^j \mapsto -e^i$ as follows:

$$\begin{aligned} e^i \otimes e_j(e_i) &= e^i(e_i)e_j - e_j(e_i)e^i = e_j \\ e^i \otimes e_j(e^j) &= e^i(e^j)e_j - e_j(e^j)e^i = -e^i. \end{aligned} \tag{4.16}$$

Now, the corresponding $O(n,n)$ element is given by exponential map as follows:

$$\begin{pmatrix} e^A & 0 \\ 0 & (e^{A^t})^{-1} \end{pmatrix} \tag{4.17}$$

Definition 4.6 [6] B can be regarded as a skew 2-form $B^* = -B$ and can be written in terms of basis elements as follows:

$$B = \frac{1}{2} B_{ij} e^i \wedge e^j, \;\; B_{ij} = -B_{ji}. \tag{4.18}$$

Note that $e^i \wedge e^j$ maps $e_i \mapsto e^j$ as follows.

$$e^i \wedge e^j(e_i) = e^i(e_i)e^j - e^j(e_i)e^i = e^j. \tag{4.19}$$

Now, we calculate the corresponding $O(n,n)$ element by exponentiation of the Lie algebra element B as given in (4.17). We need to calculate

$$e^B = 1 + B + \frac{1}{2}B^2 + \ldots \tag{4.20}$$

We see that

$$B^2 = \begin{pmatrix} 0 & 0 \\ -B & 0 \end{pmatrix}^2 = 0. \tag{4.21}$$

Therefore,

$$e^B = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \quad , B^{tr} = -B \tag{4.22}$$

e^B is the orthogonal transformation which sends

$$X + \xi \mapsto X + \xi + i_X B. \tag{4.23}$$

Definition 4.7 [6] β can be thought as a skew bivector $\beta^* = -\beta$ and can be written in terms of basis elements as follows:

$$\beta = \frac{1}{2}\beta^{ij} e_i \wedge e_j, \;\; \beta^{ij} = -\beta^{ji}. \tag{4.24}$$

Note that $e_i \wedge e_j$ maps $e^j \mapsto -e_i$ as follows:

$$e_i \wedge e_j(e^j) = e_i(e^j)e_j - e_j(e^j)e_i = -e_i. \tag{4.25}$$

Now, we calculate the corresponding $O(n,n)$ element by exponentiating the Lie algebra element β as given in (4.17). We need to calculate

$$e^{\beta} = 1 + \beta + \frac{1}{2}\beta^2 + \ldots \tag{4.26}$$

We see that

$$\beta^2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}^2 = 0. \tag{4.27}$$

Therefore

$$e^{\beta} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} , \beta^{tr} = -\beta \tag{4.28}$$

e^{β} is the orthogonal transformation which sends

$$X + \xi \mapsto X + \xi + i_{\xi}\beta. \tag{4.29}$$

To sum up, for future purposes we collect the $SO^+(n,n)$ elements connected to the identity obtained by the exponential map as follows:

$$h_B = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} , h_{\beta} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} , h_A = \begin{pmatrix} e^A & 0 \\ 0 & (e^{A^t})^{-1} \end{pmatrix} \tag{4.30}$$

In addition, we have $O(n,n)$ elements h_i^+ (h_i^-) interchanges only $e^i \leftrightarrow e_i$:

$$h_i^{\pm} = \pm \begin{pmatrix} 1 - E_i & \pm E_i \\ \pm E_i & 1 - E_i \end{pmatrix}, \;\; (E_i)_{jk} = \delta_{ij}\delta_{ik}. \tag{4.31}$$

We will take our algebraic work that we have done on V and translate it to an n-dimensional manifold M with the tangent bundle TM. In the next section we will discuss the natural bracket on the sections of the generalised tangent bundle $TM \oplus T^*M$. It was first introduced by T. Courant in [37] in 1990. $TM \oplus T^*M$ is a suitable example of a Courant algebroid, whose definition is given in [34]. More details can be found in [38].

4.2 Courant Bracket

The Courant bracket is defined to be a natural bracket on the sections of the generalised tangent bundle $TM \oplus T^*M$ generalising the Lie bracket.

Definition 4.8 [6,37] The Courant bracket is the natural skew-symmetric bracket defined on smooth sections of $TM \oplus T^*M$, given by

$$[X+\xi, Y+\eta] = [X,Y] + \mathscr{L}_X \eta - \mathscr{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi), \tag{4.32}$$

where $X+\xi, Y+\eta \in C^\infty(TM \oplus T^*M)$.

[6,37] We would like to emphasize that the Courant bracket vanishes on 1-forms and the Courant bracket reduces to the Lie bracket $[X,Y]$ on vector fields, that is if $\pi : TM \oplus T^*M \to TM$ is the natural projection,

$$\pi([A,B]_{Courant}) = [\pi(A), \pi(B)]_{Lie}, \tag{4.33}$$

for any $A, B \in C^\infty(TM \oplus T^*M)$.

It is known that the Lie bracket of smooth vector fields is invariant under diffeomorphisms. This is the only symmetry of the tangent bundle preserving the Lie bracket. On the other hand, the Courant bracket and natural inner product are invariant under diffeomorphisms. In that case, there exists an extra symmetry which is a B-transformation given in Definition 4.6. The Courant bracket commutes with the action of a closed 2-form B. It is presented by the following Proposition.

Proposition 4.1 [5,6,37] The map e^B given in (4.22) is an automorphism of the Courant bracket

$$[e^B(X+\xi), e^B(Y+\eta)] = e^B([X+\xi, Y+\eta])$$

if and only if B is closed, i.e. $dB = 0$.

Proof: [5,6] Let $X+\xi, Y+\eta \in C^\infty(TM \oplus T^*M)$ and let B be a smooth 2-form. Then using the facts the Cartan formula for the Lie derivative of a one form ξ: $\mathscr{L}_X \xi = d(i_X \xi) + i_X d\xi$ and the property $i_Y i_X = -i_X i_Y$ for all $X, Y \in C^\infty(TM)$,

$$
\begin{aligned}
[e^B(X+\xi), e^B(Y+\eta)] &= [X+\xi+i_XB, Y+\eta+i_YB] \\
&= [X+\xi, Y+\eta]+[X, i_YB]+[i_XB, Y] \\
&= [X+\xi, Y+\eta]+L_Xi_YB-\frac{1}{2}di_Xi_YB-L_Yi_XB+\frac{1}{2}di_Yi_XB \\
&= [X+\xi, Y+\eta]+L_Xi_YB-i_YL_XB+i_Yi_XdB \\
&= [X+\xi, Y+\eta]+i_{[X,Y]}B+i_Yi_XdB \\
&= e^B([X+\xi, Y+\eta])+i_Yi_XdB.
\end{aligned}
$$

Thus, e^B is an automorphism of the Courant bracket if and only if the term i_Yi_XdB vanishes for all X, Y, which is the case $dB = 0$. □

By defining Courant bracket an entirely new geometrical structure has been developed which is called the Dirac structure. Here, we need to define the notion of maximally isotropics for arbitrary vector spaces.

Definition 4.9 [11] A subspace $L < V$ is isotropic when $Q(X,Y) = 0$ for all $X, Y \in L$. L is called maximal isotropic when it is of the maximal possible dimension.

Non-degenerate bilinear forms of signature (n,n) are the ones we are most interested in. For such bilinear forms a maximally isotropic subspace is n-dimensional.

Definition 4.10 [33] A maximally isotropic subbundle of $TM \oplus T^*M$ with sections closed under Courant bracket $[C^\infty(L), C^\infty(L)] \subset C^\infty(L)$ is called a Dirac structure.

Dirac structures are important for us due to its relation between pure spinors which will be discussed in section 4.3.2.5.

4.3 Generalised G-Structures

In this section, we will generalise the reduction of the structure group of the frame bundle associated to the generalised tangent bundle $TM \oplus T^*M$ to a suitable Lie subgroup G. One can obtain an associated principal frame bundle to the generalised tangent bundle $TM \oplus T^*M$ with the structure group $O(n,n)$.

Definition 4.11 The generalised frame bundle is the $O(n,n)$ principal frame bundle associated to the generalised tangent bundle $TM \oplus T^*M$.

Let us describe a generalised G-structure.

Definition 4.12 [6] A generalised G-structure is defined to be a reduction of the structure group of the generalised frame bundle associated to the generalised tangent bundle $TM \oplus T^*M$ as given in Definition 2.11.

As we discussed in section 2.3, the reduction of the G-structures corresponds to the existence of geometrical objects on M. In a similar fashion, we will discuss further reductions of the generalised G-structures associated with the existence of new geometrical objects.

4.3.1 $O(n) \times O(n)$-Structure

The structure group $O(n,n)$ can be further reduced to its maximal compact subgroup $O(n) \times O(n)$ by the existence of the generalised metric $\mathscr{H}$ which will be important in Sections 5 and 6 regarding calculations in applications in string theory.

Definition 4.13 [6] A reduction from $O(n,n)$ to its maximal compact subgroup $O(n) \times O(n)$ exists. This is equivalent to choosing an n-dimensional positive definite subbundle $C_+ \subset TM \oplus T^*M$ with respect to the inner product (4.1). Define C_- the negative definite subbundle $C_- \subset TM \oplus T^*M$ with respect to the inner product (4.1) to be the orthogonal complement $C_- = {C_+}^{\perp}$. Hence, we have $TM \oplus T^*M = C_+ \oplus C_-$.

[5] Note that if a Riemannian metric g on M is given one can define the following map.

$$\begin{aligned} g^{\sharp} : TM &\rightarrow T^*M \\ X &\mapsto g^{\sharp}X = \eta \end{aligned} \tag{4.34}$$

where $g(X,Y) = \eta(Y)\ \forall Y \in TM$, $\eta \in T^*M$. Then $i_X(g^{\sharp}X) = g(X,X)$.

C_+ can be described as the following graph:

$$C_+ = \{X + g^{\sharp}X : X \in TM)\} \subset TM \oplus T^*M \tag{4.35}$$

Let $X + \xi \in C_+$ and $\xi = g^{\sharp}X$, then by using (4.1) one can define

$$\langle X+\xi, X+\xi \rangle = i_X\xi = i_X g^{\sharp}X = g(X,X) \tag{4.36}$$

as the positive definite inner product on $TM \oplus T^*M$ restricted to C_+, say $\langle , \rangle\big|_{C_+}$.

Similarly, C_- can be described as the following graph:

$$C_- = \{X - g^\sharp X : X \in TM\} \subset TM \oplus T^*M \tag{4.37}$$

Similarly, one can define

$$\langle X+\xi, X+\xi\rangle = i_X\xi = -i_X g^\sharp X = -g(X,X) \tag{4.38}$$

as the negative definite inner product on $TM \oplus T^*M$ restricted to C_-, say $\langle,\rangle\big|_{C_-}$.

Proposition 4.2 [6] $C_\pm$ are the ±1-eigenbundles of the endomorphism

$$g = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{4.39}$$

with $g^2 = 1$ acting on $TM \oplus T^*M$.

Proof: Let $X+\xi \in C_+$, then we show that it is +1-eigenbundle of g.

$$\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} X \\ \xi \end{pmatrix} = 1.\begin{pmatrix} X \\ \xi \end{pmatrix} \tag{4.40}$$

These equations solved to be $gX = \xi$ and $g^{-1}\xi = X$. Hence, C_+ consists of the elements of the following form $X + gX$.

Similarly, let $X+\xi \in C_-$, then we show it is -1-eigenbundle of g

$$\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} X \\ \xi \end{pmatrix} = -1.\begin{pmatrix} X \\ \xi \end{pmatrix} \tag{4.41}$$

These equations solved to be $gX = -\xi$ and $g^{-1}\xi = -X$. Hence, C_- consists of the elements of the following form $X - gX$. □

[7]–[9,19,39] By applying B-transform e^B as given in (4.22) to g

$$\begin{aligned} e^B : TM \oplus T^*M &\to TM \oplus T^*M \\ X+\xi &\mapsto X+\xi+i_XB \end{aligned} \tag{4.42}$$

we end up with the following

$$\tilde{g} = e^{-B} g e^{B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \tag{4.43}$$

$\tilde{g}$ is the following endomorphism:

$$\tilde{g} : TM \oplus T^*M \to TM \oplus T^*M \quad \tilde{g}^2 = 1. \tag{4.44}$$

As we have discussed for the case g, one can choose a positive definite subbundle $\tilde{C}_+ \subset TM \oplus T^*M$ and the orthogonal complement $\tilde{C}_-$ as the negative definite subbundle to have $C_+ \oplus C_- \subset TM \oplus T^*M$. Let us define these subbundles.

$$\tilde{C}_+ = \{X + (g+B)X : X \in TM\} \subset T \oplus T^* \tag{4.45}$$

Similarly, C_- can be thought as the following graph:

$$\tilde{C}_- = \{X + (g-B)X : X \in T(M)\} \subset TM \oplus T^*M \tag{4.46}$$

Proposition 4.3 $C_\pm$ are the ± 1-eigenbundles of the endomorphism with $\tilde{g}^2 = 1$ acting on $TM \oplus T^*M$.

Proof: [6] Since $\mathscr{H}^2 = 1$, then $\mathscr{H}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Then, it has ± 1 eigenbundles $C_\pm$ corresponding to λ_1, λ_2.

Note that a map $g : TM \to T^*M$ can be described as a metric g and $B : TM \to T^*M$ can be described as a 2-form. We have that

$$B \pm g : TM \to T^*M \tag{4.47}$$

Let us begin with $\tilde{C}_+$ and show that it is $+1$-eigenbundle of $\tilde{g}$ and has elements of the form $(X, (g+B)X)$. Let $X + \xi \in \tilde{C}_+$ we have

$$\begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = 1. \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{4.48}$$

These equations solved to be $\xi = (g+B)X$. Similarly, we show that C_- is -1-eigenbundle of $\tilde{g}$ and has elements of the form $(X, (g-B)X)$. Let $X + \xi \in \tilde{C}_-$ we have

$$\begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = -1. \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{4.49}$$

These equations solved to be $\xi = (g-B)X$.

Definition 4.14 [6] Let the splitting $TM \oplus T^*M = \tilde{C}_+ \oplus \tilde{C}_-$ exists. A generalised metric $\mathscr{H}$ is defined as a positive definite metric on $TM \oplus T^*M$ can be defined as

$$\mathscr{H} = \langle , \rangle\big|_{\tilde{C}_+} - \langle , \rangle\big|_{\tilde{C}_-}. \tag{4.50}$$

Hence, we can conclude the following result.

Proposition 4.4 ([6], Proposition 6.1)

A reduction to $O(n) \times O(n)$ is equivalent to specifying a positive definite metric on $TM \oplus T^*M$ which is the generalised metric $\mathscr{H}^2 = 1$.

We give the important property of the generalised metric in the next Proposition

Proposition 4.5 [39] Let $\mathscr{H}$ be a generalised metric on $TM \oplus T^*M$. Then $\mathscr{H}$ defines a unique pair (g, B), where g is a Riemannian metric on M and B is a 2-form on M. Conversely, any pair (g, B) defines a unique generalized metric.

Proof: [7]–[9,39] We have already given a constructive proof of the second statement. Let us now show that the existence of a generalised metric $\mathscr{H}$ induces a Riemannian metric g and a 2-form B on M. Let the splitting $T \oplus T^* = C_+ \oplus C_-$ exists. The subbundles C_+, C_- are definite and TM and T^*M are isotropic, we have $C_+ \cap TM = C_- \cap T^*M = 0$. Then, C_+, C_- can be thought as the graphs over TM and there exist the following linear maps:

$$P_\pm : T \to T^* \tag{4.51}$$

such that

$$C_\pm = \{t \oplus P_\pm t \,|\, t \in T\}. \tag{4.52}$$

Also, we define the map $Q_+ : T \times T \to \mathbb{R}$ by

$$Q_+(s,t) = (s, P_+(t)). \tag{4.53}$$

Now, one can define a 2-form as the antisymmetric part of (4.53) as follows:

$$B(s,t) = \frac{Q_+(s,t) - Q_+(t,s)}{2}. \tag{4.54}$$

Similarly, one can define a positive definite metric as the symmetric part of (4.53) as follows:

$$g(s,t) = \frac{Q_+(s,t) + Q_+(t,s)}{2}, \;\; s,t \in T. \tag{4.55}$$

Positive definiteness of g can be shown as follows:

$$g(t,t) = (t, P_+ t) = \frac{(t \oplus P_+ t, t \oplus P_+ t)}{2} = \frac{g_+(t \oplus P_+ t, t \oplus P_+ t)}{2} > 0. \tag{4.56}$$

where $t \in T$ is a non-zero element. As we have studied, the generalised metric can be written as a matrix as follows, [7]–[9,19,39]:

$$\mathscr{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{4.57}$$

Conversely, any given a pair (g,B), $\mathscr{H}$ can be defined as given in (4.57) which satisfies Definition 4.14 and $\mathscr{H}$. □

[36] Now, let us review a simpler case when we have a background matrix E of the following form: $E = g + B$ rather than a generalised metric $\mathscr{H}$. Let us present the transformation of $E = g + B$ under an $O(n,n)$ matrix given in (4.7).

$$E' = T(E) = (aE+b)(cE+d)^{-1} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} E \tag{4.58}$$

E' includes the transformed metric and the transformed B-field which are symmetric and antisymmetric parts of E', respectively as given in Proposition 4.5.

$$g' = \frac{E' + E'^t}{2}, \quad B' = \frac{E' - E'^t}{2} \tag{4.59}$$

This is equivalent to the following transformation for $\mathscr{H}$.

$$\mathscr{H}' = T\mathscr{H}T^t. \tag{4.60}$$

We have seen that the existence of a generalised metric reduces the structure group of $TM \oplus T^*M$ from $O(n,n)$ to $O(n) \times O(n)$. In the next subsection we will discuss further reductions of the structure group associated with the existence of a new geometrical object which is the generalised almost complex structure.

4.3.2 $U(n/2,n/2)$-Structure

As we discussed in section 2.3.5 that the existence of an almost complex structure J reduces the structure group $U(n)$. Now, let us begin with introducing a generalised almost complex structure then generalise the reduction of the structure group.

Definition 4.15 [5,6] A generalised almost complex structure on a manifold is an endomorphism $\mathscr{J}$ of $TM \oplus T^*M$ such that

- $\mathscr{J}^2 = -1$,
- $\mathscr{J}^* = -\mathscr{J}$.

The usual complex and symplectic structures can be thought as a generalised complex structure in the following way, [6]. Consider the endomorphism

$$\mathscr{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \tag{4.61}$$

where J is a usual complex structure on V. Then we see that $\mathscr{J}_J^2 = -1$ and $\mathscr{J}_J^* = -\mathscr{J}_J$, i.e. $\mathscr{J}_J$ is a generalised complex structure. Similarly, consider the endomorphism

$$\mathscr{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{4.62}$$

where ω is a usual symplectic structure. Then we see that $\mathscr{J}_\omega^2 = -1$ and $\mathscr{J}_\omega^* = -\mathscr{J}_\omega$, i.e. $\mathscr{J}_\omega$ is a generalised complex structure.

We have seen in section 2.3.5 that the reduction of the structure group to $U(n)$ is equivalent to the existence of an almost complex structure and a Riemannian metric g. Similarly, by the existence of the generalised almost complex structure $\mathscr{J}$, the structure group of $TM \oplus T^*M$ can be further reduced from $O(n,n)$ as follows.

Proposition 4.6 [6] A generalised almost complex structure on $TM \oplus T^*M$ determines a reduction of structure group from $O(n,n)$ to $U(n/2,n/2) = O(n,n) \cap GL(n,\mathbb{C})$.

Here, we would like to introduce the new geometrical objects in defining $U(n/2,n/2)$-structures.

Definition 4.16 [6] A generalised almost complex structure on a real even n-dimensional manifold M is given by the following equivalent data:

- an almost complex structure $\mathscr{J}$ on $TM \oplus T^*M$ which is orthogonal with respect to the natural bilinear form $(,)$ given in (4.1), $\mathscr{J}^* = -\mathscr{J}$ and $\mathscr{J}^* = -1$. i.e. a reduction of structure for the $O(n,n)$-bundle $T \oplus T^*$ to the group $U(n/2,n/2)$,

- a maximal isotropic sub-bundle of the complexified generalised tangent bundle $L < (TM \oplus T^*M) \otimes \mathbb{C}$ with $L \cap \overline{L} = 0$,

- A pure spinor line sub-bundle $U < \bigwedge^\bullet T^* \otimes \mathbb{C}$, called the canonical line bundle, satisfying $(\varphi, \overline{\varphi}) \neq 0$ at each point $x \in M$ for any generator $\varphi \in U_x$.

In order to understand the last part, we will examine the notion of spinors which will be defined in Definition 4.26. In order to do so we need to begin with Clifford algebras. Then we will study pure spinors.

4.3.2.1 Clifford algebras

In this section, we will give the definition and some key properties of the Clifford algebra. The significance of Clifford algebras arises from the fact that the orthogonal Lie algebra is contained in the Clifford algebra as a subalgebra. Then, a representation of the orthogonal Lie algebra can be constructed through Clifford algebra. We will work on the linear algebra on vector spaces V in the first place. The reason we are doing is later we will extend all the data to $T_xM \oplus T_x^*M$.

We begin with defining the Clifford algebra as follows.

Definition 4.17 [11] Given a symmetric bilinear form Q on a vector space V, the Clifford algebra $Cl(V,Q)$ is an associative algebra with unit 1, which contains and is generated by V, with

$$v.v = Q(v,v)\ 1\ in\ \ Cl(V,Q).$$

The operation . is called the Clifford algebra product. Note that, using polarization

$$Q(v,w) = \frac{Q(v+w,v+w) - Q(v,v) - Q(w,w)}{2} \tag{4.63}$$

one can write the Clifford algebra product as follows:

$$\{v,w\} \equiv v.w + w.v = 2Q(v,w). \tag{4.64}$$

Definition 4.18 [40] The Clifford algebra is a Z_2 graded algebra and it decomposes as $Cl = Cl^{\text{even}} \oplus Cl^{\text{odd}}$. Cl^{even} is called *even subspace* and its elements are spanned by products of an even number of elements in V whereas Cl^{odd} is called the *odd subspace* and its elements are spanned by products of an odd number of elements of V. The space Cl^{even} is not only a subspace but establishes a subalgebra.

We now discuss the Clifford algebra generators, [41]. First, let us define the inclusion map as follows:

Definition 4.19

$$i : V \to Cl(V,Q) \tag{4.65}$$

The image of e_i under (4.65) is defined to be Γ^i satisfying (4.64):

$$\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2Q(e^i,e^j)1. \tag{4.66}$$

We introduce the following $Cl(V,Q)$ elements.

Definition 4.20 [41] One can define the following element.

$$\Gamma^{i_1 i_2 \ldots i_n} = \frac{1}{n!}\sum_{\sigma}(-1)^{\sigma}\Gamma^{i_{\sigma(1)}}\Gamma^{i_{\sigma(2)}}\ldots\Gamma^{i_{\sigma(n)}}, \tag{4.67}$$

where $(-1)^{\sigma}$ is the sign of the permutation σ of $\{1,2,\ldots,n\}$.

For instance

$$\Gamma^{i_1 i_2} = \frac{1}{2}(\Gamma^{i_1}\Gamma^{i_2} - \Gamma^{i_2}\Gamma^{i_1}) \tag{4.68}$$

$$\Gamma^{i_1 i_2 i_3} = \frac{1}{6}(\Gamma^{i_1}\Gamma^{i_2}\Gamma^{i_3} - \Gamma^{i_1}\Gamma^{i_3}\Gamma^{i_2} + \Gamma^{i_2}\Gamma^{i_3}\Gamma^{i_1} - \Gamma^{i_2}\Gamma^{i_1}\Gamma^{i_3} + \Gamma^{i_3}\Gamma^{i_1}\Gamma^{i_2} - \Gamma^{i_3}\Gamma^{i_2}\Gamma^{i_1}). \tag{4.69}$$

Hence, $\{1,\Gamma^{i_1},\Gamma^{i_1 i_2},\ldots,\Gamma^{i_1 i_2\ldots i_n} : \ i_1 < i_2 < \ldots < i_n\}$ forms a $1+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n} = (1+1)^n = 2^{dimV}$ dimensional basis for $Cl(V)$.

On the other hand, the exterior algebra $\bigwedge^{\bullet} V$ where $\bigwedge^{\bullet} V = \bigwedge^0 V \oplus \bigwedge^1 V \oplus \cdots \oplus \bigwedge^n V$ has $1+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n} = (1+1)^n = 2^n$ dimensional basis: $\{1, e^{i_1}, e^{i_1}\wedge e^{i_2},\ldots, e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_n}\}$. Hence, $Cl(V,Q)$ and $\bigwedge^{\bullet} V$ are 2^n dimensional and one can map the generators of $Cl(V,Q)$ to the basis elements of $\wedge V^{\bullet}$. This can be seen as follows:

$$1 \mapsto 1 \tag{4.70}$$

$$e^i \mapsto \Gamma^i \tag{4.71}$$

$$\frac{1}{n!}\alpha_{i_1,\ldots,i_n} e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_n} \mapsto \alpha_{i_1,\ldots,i_n}\Gamma^{i_1 i_2\ldots i_n}. \tag{4.72}$$

It leads to the following result.

Lemma 4.1 [41] There is a vector space isomorphism between $\wedge V^{\bullet} \cong Cl(V,Q)$.

There is an embedding of $\mathfrak{so}(Q)$ in Cl^{even}.

We will state it and give the proof as follows.

Lemma 4.2 [11] $\text{so}(Q)$ embeds as a Lie subalgebra of $Cl(V,Q)^{\text{even}}$.

Proof: [11] We begin with examining the isomorphism of bivectors $\bigwedge^2 V$ with $\mathfrak{so}(Q)$. The isomorphism is given by

$$\varphi : \bigwedge^2 V \cong \text{so}(Q) \subset \text{End}(V), \tag{4.73}$$

$$a\wedge b \mapsto \varphi_{a\wedge b}$$

for a and b in V, where $\varphi_{a\wedge b}$ is defined by

$$\varphi_{a\wedge b}(v) = 2(Q(b,v)a - Q(a,v)b) \tag{4.74}$$

First, let us show that $\varphi_{a\wedge b}$ is in $\mathfrak{so}(Q)$. We need to show that

$$Q(\varphi_{a\wedge b}(v), w) + Q(v, \varphi_{a\wedge b}(w)) = 0$$

$$\begin{aligned}
Q(\varphi_{a\wedge b}(v), w) + Q(v, \varphi_{a\wedge b}(w)) &= 2Q(Q(b,v)a - Q(a,v)b, w + 2Q(v, Q(b,w)a - Q(a,w)b)) \\
&= 2Q(Q(b,v)a, w) - 2Q(Q(a,v)b, w) \\
&+ 2Q(Q(b,w)a, v) - 2Q(Q(a,w)b, v) \\
&= 2Q(b,v)Q(a,w) - 2Q(a,v)Q(b,w) \\
&+ 2Q(b,w)Q(a,v) - 2Q(a,w)Q(b,v) \\
&= 0
\end{aligned}$$

The natural basis of $\bigwedge^2 V$, $e^1 \wedge e^{n+j}$ maps to some scalar factor of the the natural basis of $\mathrm{End}(V)$ which is given by $2\left(E_{i,j} - E_{n+j,n+i}\right)$. Here $E_{i,j}$ is the endomorphism of V carrying e^j to e^i and e^i to $-e^j$ and killing e^k for all $k \neq j$. Hence, the map is an isomorphism. Since this is a Lie algebra isomorphism, we need to examine the bracket on $\bigwedge^2 V$ to make an analogy to the bracket in the Clifford algebra.

$$
\begin{aligned}
[\varphi_{a\wedge b}, \varphi_{c\wedge d}](v) &= \varphi_{a\wedge b}\circ\varphi_{c\wedge d}(v) - \varphi_{c\wedge d}\circ\varphi_{a\wedge b}(v) \\
&= 2\varphi_{a\wedge b}(Q(d,v)c - Q(c,v)d) \\
&- 2\varphi_{c\wedge d}(Q(b,v)a - Q(a,v)b) \\
&= 4Q(d,v)(Q(b,c)a - Q(a,c)b) \\
&- 4Q(c,v)(Q(b,d)a - Q(a,d)b) \\
&- 4Q(b,v)(Q(d,a)c - Q(c,a)d) \\
&+ 4Q(a,v)(Q(d,b)c - Q(c,b)d) \\
&= 4Q(b,c)(Q(d,v)a - Q(a,v)d) \\
&- 4Q(b,d)(Q(c,v)a - Q(a,v)c) \\
&- 4Q(a,d)(Q(b,v)c - Q(c,v)b) \\
&+ 4Q(a,c)(Q(b,v)d - Q(d,v)b) \\
&= 2Q(b,c)\varphi_{a\wedge d}(v) - 2Q(b,d)\varphi_{a\wedge c}(v) \\
&- 2Q(a,d)\varphi_{c\wedge b}(v) + 2Q(a,c)\varphi_{d\wedge b}(v)
\end{aligned}
$$

It is concluded that an explicit formula for the bracket on $\bigwedge^2 V$ becomes

$$
\begin{aligned}
[a\wedge b, c\wedge d] = \quad & 2Q(b,c)a\wedge d - 2Q(b,d)a\wedge c \\
& -2Q(a,d)c\wedge b + 2Q(a,c)d\wedge b.
\end{aligned}
$$

Besides, the Clifford algebra bracket takes the form of

$$
\begin{aligned}
[a\cdot b, c\cdot d] &= a\cdot b\cdot c\cdot d - c\cdot d\cdot a\cdot b \\
&= (2Q(b,c)a\cdot d - a\cdot c\cdot b\cdot d) - (2Q(a,d)c\cdot b - c\cdot a\cdot d\cdot b) \\
&= 2Q(b,c)a\cdot c - (2Q(b,d)a\cdot c - a\cdot c\cdot d\cdot b) \\
&= 2Q(a,d)c\cdot b + (2Q(a,c)\cdot d\cdot b - a\cdot c\cdot d\cdot b) \\
&= 2Q(b,c)a\cdot d - 2Q(b,d)a\cdot c - 2Q(a,d)c\cdot b + 2Q(a,c)\cdot d\cdot b.
\end{aligned}
$$

Eventually, the a map of Lie algebras $\psi : \bigwedge^2 V \to Cl(V,Q)$ can be defined by

$$
\psi(a\wedge b) = \frac{1}{2}(a\cdot b - b\cdot a) = a\cdot b - Q(a,b). \tag{4.75}
$$

If we look at the basis elements, $\mathbf{e}^i \wedge \mathbf{e}^j$ maps to $\{1, e^{i_1}e^{i_2}...e^{i_k}\}$, where $i_1 < i_2 < ... < i_n$. The map $\psi \circ \varphi^{-1} : \mathrm{so}(Q) \to Cl(V,Q)^{\mathrm{even}}$ is the desired embedding map. This ends the proof. □

Now, we analyze the action of $Cl(V,Q)$ on $\bigwedge^{\bullet} V$. This means that for every element in the Clifford algebra $Cl(V,Q)$ there exists an operator that acts on $\bigwedge^{\bullet} V$. First, we need the following map.

Lemma 4.3 The linear map

$$\pi : V \to End(\bigwedge^{\bullet} V) \tag{4.76}$$

$$\pi(x)\alpha = x \wedge \alpha + i_{x^\flat}\alpha \tag{4.77}$$

where $x \in V$, $\alpha \in \bigwedge^{\bullet} V$ and $\flat : V \to V^*$ $B(x,y) = x^\flat(y)$ satisfy (4.64).

Proof:

$$\begin{aligned} \pi^2(x)\alpha &= \pi(x)(x \wedge \alpha + i_{x^\flat}\alpha) \\ &= x \wedge i_{x^\flat}\alpha + i_{x^\flat}(x \wedge \alpha) \\ &= (i_{x^\flat}x)\alpha \\ &= 2Q(x, x^\flat)\alpha. \end{aligned} \tag{4.78}$$

Therefore, π satisfies (4.64).

By using the fact that the Clifford algebra $Cl(V,Q)$ is a universal algebra, one can extend the map π to the following map.

$$\pi : Cl(V,Q) \to End(\bigwedge^{\bullet} V) \tag{4.79}$$

This map clearly satisfies (4.64), that means this is a representation of the Clifford algebra $Cl(V,Q)$ with the carrier space $\bigwedge^{\bullet} V$. This representation is not irreducible. One way to introduce an irreducible representation when Q is of signature (n,n) is to split the vector space V into n-dimensional maximal isotropic subspaces $V = W \oplus W'$ and define the irreducible representation on the subspace W which we will introduce in section 4.3.2.2. This is important for us since we work on the generalised tangent bundle $TM \oplus T^*M$ with the bilinear form (4.1) of (n,n) signature.

Let us introduce some operations in $Cl(V,Q)$ which will be useful for further constructions.

Definition 4.21 [11] The main automorphism $\tau : Cl \longrightarrow Cl$ and the main involution $\alpha : Cl \longrightarrow Cl$ maps are determined by

$$\tau(v_1 \cdot \ldots \cdot v_k) = v_k \cdot \ldots \cdot v_1 \quad (4.80)$$

$$\alpha(v_1 \cdot \ldots \cdot v_k) = (-1)^k v_1 \cdot \ldots \cdot v_k \quad (4.81)$$

for $v_1, \ldots, v_k$ in V.

Next, we introduce the following operation which is the composite of the main automorphism and the main involution.

Definition 4.22 [11] The Clifford algebra $Cl(V,Q)$ has an anti-involution or conjugation operation $x \longmapsto x^*$ determined by

$$(v_1 \cdot \ldots \cdot v_k)^* = (-1)^k v_k \cdot \ldots \cdot v_1 \quad (4.82)$$

for any $v_1, \ldots, v_k$ in V.

We now introduce Pinor group forms a closed subgroup of the group of units in the Clifford algebra as follows.

Definition 4.23 [11] The Pinor group can be defined as follows.

$$\mathrm{Pin}(Q) = \{x \in Cl(V,Q) : x \cdot x^* = 1 \text{ } and \text{ } x \cdot V \cdot x^* \subset V\}. \quad (4.83)$$

Furthermore, we introduce Spinor group forms a closed subgroup of the group of units in the even part of the Clifford algebra as follows.

Definition 4.24 [11] The Spinor group can be defined as follows.

$$\mathrm{Spin}(Q) = \{x \in Cl(V,Q)^{even} : x \cdot x^* = \pm 1 \text{ } and \text{ } x \cdot V \cdot x^* \subset V\}. \quad (4.84)$$

Every element of the Spinor group establishes an endomorphism of V as follows:

Proposition 4.7 [11] For $x \in \mathrm{Spin}(Q), \rho(x)$ is in $\mathrm{SO}(Q)$. The mapping

$$\rho : \mathrm{Spin}(Q) \rightarrow \mathrm{SO}(Q) \quad (4.85)$$

$$\rho(X) : v \mapsto x.v.x^{-1} \quad (4.86)$$

is a homomorphism, making $\mathrm{Spin}(Q)$ a connected two-sheeted covering of $\mathrm{SO}(Q)$. The kernel of ρ is {1,-1}.

There exists a natural isomorphism of the corresponding Lie algebras which is obtained by the exponentiation of ρ;

$$\begin{aligned}\rho' &= \frac{d}{dt}\rho(e^{tx})|_{t=0} \;\; x \in spin(Q) \\ &= \frac{d}{dt}(e^{t(xv^*x)})|_{t=0} \\ &= \frac{d}{dt}(e^{tx}ve^{-tx})|_{t=0} \\ &= xv - xv = [x,v].\end{aligned} \tag{4.87}$$

It is presented as follows:

Proposition 4.8 [11] There is also an isomorphism of the Lie algebra of the group $\mathrm{Spin}(Q)$ and the Lie algebra $\mathfrak{so}(Q)$ which is given by

$$\rho' : spin(Q) \longrightarrow so(Q) \tag{4.88}$$

$$\rho'(x)(v) = [x,v]. \tag{4.89}$$

4.3.2.2 Special case: split signature

As we study on the generalised tangent bundle $TM \oplus T^*M$ of signature (n,n) we specifically study the split signature case of Lemma 4.3.

[11] Let W be a maximally isotropic subspace of dimension n and W' be the orthogonal complement of W with respect to the bilinear form Q. Then, there exists the decomposition $V = W \oplus W'$. In this case, one can write $Spin(Q) \cong Spin(n,n)$ and $\mathfrak{so}(Q) = \mathfrak{so}(n,n)$.

In this case, we can choose $\{e^1, \cdots, e^n\}$ to form a basis for W and $\{e_1, \cdots, e_n\}$ to form a basis for W'. Hence, $e^M = \{e^1, \cdots, e^n, e_1, \cdots, e_n\} = \{e^i, e_i\}$ forms a basis of V. These satisfy

$$Q(e_i, e_j) = Q(e^i, e^j) = 0, \quad Q(e^i, e_j) = \delta_j{}^i \quad \forall i, j. \tag{4.90}$$

We will work with the following Clifford algebra basis elements given in Definition 4.19

$$\psi^M \equiv \frac{1}{\sqrt{2}}\Gamma^M \tag{4.91}$$

with $\Gamma^M = (\Gamma_i, \Gamma^i)$ since ψ^M satisfies (4.64) so that $i_{\psi^m}\psi^n = \delta_m{}^n$. Namely,

$$\begin{aligned} \psi^i\psi^j + \psi^j\psi^i &= 0 \\ \psi_i\psi_j + \psi_j\psi_i &= 0 \\ \psi^i\psi_i + \psi_i\psi^i &= 1 \\ \psi^i\psi_j + \psi_i\psi^j &= 0. \end{aligned} \tag{4.92}$$

In this special case of split signature, we would like to introduce a Pinor group element which we will use for calculations in our application in chapter 5 and 6.

Definition 4.25 [42] The Pinor group $Pin(n,n)$ element C is called the charge conjugation matrix. The charge conjugation matrix is an appropriate name for this element of the Pinor group, since it satisfies the same Gamma matrix relations as the standard charge conjugation matrix in quantum field theory, [43]. It is equivalent to C^+ and C^- for even and odd dimensions respectively, where

$$C = C^+ \equiv \Lambda_1^+ \dots \Lambda_n^+, \tag{4.93}$$

$$C = C^- \equiv \Lambda_1^- \dots \Lambda_n^-. \tag{4.94}$$

Here

$$\Lambda_i^\pm = (\psi^i \mp \psi_i). \tag{4.95}$$

Indeed, $Pin(n,n)$ elements $\Lambda_i^\pm$ are obtained via (4.85),

$$\rho(\Lambda_i^\pm) = h_i^\pm$$

where $O(n,n)$ elements h_i is given as (4.31).

Similar to the $O(n,n)$ elements h_i, $\Lambda_i^\pm$ maps $\psi_i \leftrightarrow \psi^i$ in the following manner, [42] .

$$\Lambda_i.e^M.(\Lambda_i)^{-1} = \begin{cases} -e_i & \text{if } e^M = e^i \\ -e^i & \text{if } e^M = e_i\,. \\ -e^M & \text{otherwise} \end{cases} \tag{4.96}$$

Let $e^M = e^i$, then we write it in terms of ψ^M and calculate from the Clifford commutation relations (4.92)

$$\begin{aligned} (\psi^i - \psi_i)\psi^i(\psi_i - \psi^i) &= -\psi_i\psi^i\,(\psi_i - \psi^i) \\ &= -\psi_i\psi^i\psi_i \\ &= -\psi_i\,(1 - \psi_i\psi^i) \\ &= -\psi_i. \end{aligned} \tag{4.97}$$

Let $e^M = e_i$, then we write

$$\begin{aligned}(\psi^i - \psi_i)\psi_i(\psi_i - \psi^i) &= -\psi^i\psi_i\,(\psi_i - \psi^i) \\ &= -\psi^i\psi_i\psi^i \\ &= -(1-\psi_i\psi^i)\psi^i \\ &= -\psi^i.\end{aligned} \tag{4.98}$$

The other possibility is nothing but a linear combination of the first two cases. Hence, we show that (4.99) is valid. Therefore, one can write the identities for C as follows:
Lemma 4.4

$$\begin{aligned}C_+\psi_i(C_+)^{-1} &= -(-1)^n\psi^i \\ C_+\psi^i(C_+)^{-1} &= -(-1)^n\psi_i\,, \\ C_-\psi_i(C_-)^{-1} &= \quad(-1)^n\psi^i \\ C_-\psi^i(C_-)^{-1} &= \quad(-1)^n\psi_i\end{aligned}$$

Hence, it leads to

$$C\,\psi_i C^{-1} = \psi^i\,, \qquad C\,\psi^i C^{-1} = \psi_i\,. \tag{4.99}$$

Furthermore, it is possible to define the action of a dagger operator in the Clifford algebra as

$$S^\dagger = C\tau(S)C^{-1} \tag{4.100}$$

where $\tau(S) = \pm S^{-1}$ where τ is given as Definition 4.21.

4.3.2.3 Irreducible spin representations

As we have discussed in Lemma 4.3 that the representation given in (4.79) is not irreducible. Now, in the special case of the split signature, we reconstruct the representation and discuss that this is an irreducible representation.

Lemma 4.5 [11] The decomposition $V = W \oplus W'$ determines an isomorphism of algebras $Cl(V,Q) \cong End(\bigwedge^\bullet W)$ where $\bigwedge^\bullet W = \bigwedge^0 W \oplus \cdots \oplus \bigwedge^n W$.

Proof: [11,43] Using the fact that $Cl(V,Q)$ can be identified as V given in Definition 4.17, the isomorphism can be constructed between V and $E = \mathrm{End}(\bigwedge^\bullet W)$. Let us construct the maps

$$l : W \to E \quad \textit{such that}$$

$$l(w)\alpha = w \wedge \alpha \tag{4.101}$$

where $\alpha \in \bigwedge^{\bullet} W$, and

$$l' : W' \to E \quad \textit{such that}$$

$$l'(w')\alpha = \mathrm{i}_{(w')^{\natural}}\,\alpha \tag{4.102}$$

where

$$(w')^{\natural}(w) = 2Q(w,w'). \tag{4.103}$$

Then, the isomorphism map become

$$V = W \oplus W' \to End(\overset{\bullet}{\bigwedge} W)$$

$$w + w' \longmapsto l(w) + l'(w')$$

$$w + w' \longmapsto w \wedge \alpha + \mathrm{i}_{(w')^{\natural}}\,\alpha \tag{4.104}$$

We need to show that $l(w)^2 = 0$, and $l'(w')^2 = 0$.

First,

$$l(w)^2(\alpha) = l(l(w)(\alpha) = l(w \wedge \alpha) = w \wedge w \wedge \alpha = 0.$$

Second,

$$l'\left(w'\right)^2(\alpha) = l'(\mathrm{i}_{(w')^{\natural}}\,\alpha) = \mathrm{i}_{(w')^{\natural}}(\mathrm{i}_{(w')^{\natural}}\,\alpha) = 0.$$

Furthermore,

$$l(w) \circ l'\left(w'\right) + l'\left(w'\right) \circ l(w) = 2Q\left(w,w'\right)I \tag{4.105}$$

for any $w \in W, w' \in W'$. Since,

$$\begin{aligned} l(\mathrm{i}_{(w')^{\natural}}\,\alpha) + l'(w \wedge \alpha) &= w \wedge \mathrm{i}_{(w')^{\natural}}\,\alpha + \mathrm{i}_{(w')^{\natural}}(w \wedge \alpha) && (4.106)\\ &= (\mathrm{i}_{(w')^{\natural}}\,w)(\alpha) && (4.107)\\ &= 2Q(w,w')\alpha I. && (4.108) \end{aligned}$$

Furthermore, if one takes a look at what happens to the basis of the resulting map, it becomes evident that it is an isomorphism.

Note that in the special case of the split signature (n,n) Lemma 4.1 can be visualized as follows:

$$e^M \mapsto \Gamma^M \tag{4.109}$$

so on a differential form α

$$\Gamma^i.\alpha = e^i \wedge \alpha, \qquad \Gamma_i.\alpha = i_{e_i}\alpha. \tag{4.110}$$

[11] The map given in (4.104) is an irreducible representation of the Clifford algebra Cl with the carrier space the exterior algebra $\bigwedge^\bullet W = \bigwedge^0 W \oplus \cdots \oplus \bigwedge^n W$. This is also a representation of the orthogonal Lie algebra $\mathfrak{so}(Q) = \mathfrak{so}(n,n)$ with the carrier space the exterior algebra $\bigwedge^\bullet W = \bigwedge^0 W \oplus \cdots \oplus \bigwedge^n W$, due to the fact that $\mathfrak{so}(Q)$ embeds in Cl^{even}.

[11,43] According to the decomposition $\wedge W = \bigwedge^{\text{even}} W \oplus \bigwedge^{\text{odd}} W$ and Lemma 4.5 there exists an isomorphism

$$Cl(V,Q)^{\text{even}} \cong \text{End}\left(\bigwedge^{\text{even}} W\right) \oplus \text{End}\left(\bigwedge^{\text{odd}} W\right). \tag{4.111}$$

Next, using Lemma 4.2 there exists an embedding of Lie algebras:

$$\mathfrak{so}(Q) \subset Cl(V,Q)^{\text{even}} \cong \mathfrak{gl}\left(\bigwedge^{\text{even}} W\right) \oplus \mathfrak{gl}\left(\bigwedge^{\text{odd}} W\right). \tag{4.112}$$

Therefore, the representation of $\mathfrak{so}(Q)$ decomposes as

$$S = S^+ \oplus S^- \tag{4.113}$$

where

$$S^+ = \Lambda^{\text{even}} W \textit{ and } S^- = \Lambda^{\text{odd}} W. \tag{4.114}$$

Definition 4.26 [11,43] The representation of the special orthogonal algebra $\mathfrak{so}(Q)$ embedded in the Clifford algebra with the carrier space $\bigwedge^\bullet W$ in (4.104) is called the spin representation. Elements of the spin representation are called spinors.

The subspaces S^+, *and* S^- are kept invariant under the action of $\mathfrak{so}(n,n)$ and are called *half spin representations*, and their elements are called *chiral spinors*.

By the quadratic form Q, one can identify the subspace W' with W^*. When we request that $W = T_x^*M$ we end up with $V = T_x^*M \oplus T_xM$. According to what we described on $V = W \oplus W^*$, the generalised tangent bundle is endowed with the same canonical bilinear form and orientation as those described on $V = W \oplus W^*$. Hence, the structure group of $TM \oplus T^*M$ can be seen as $SO(n,n)$. By the fact that the $SO(n,n)$ bundle $TM \oplus T^*M$ on an orientable manifold always carries a $Spin(n,n)$-structure one can extend all the linear algebra that we have studied can be applied to whole bundle $TM \oplus T^*M$. Hence, sections of $\bigwedge^\bullet T^*M$ which are exterior forms can be identified as $Spin(n,n)$ spinor fields.

Now, we will discuss the natural inner product on the space of spinors.

Definition 4.27 Mukai pairing is the natural inner product on the Clifford module $\wedge^\bullet W$ and described as follows. $\langle\ ,\ \rangle : S \otimes S \to \wedge^n W$:

$$\langle \Phi_1, \Phi_2 \rangle = (\tau(\Phi_1) \wedge \Phi_2)_{\text{top}} = \sum_j (-1)^j (\Phi_1^{2j} \wedge \Phi_2^{n-2j} + \Phi_1^{2j+1} \wedge \Phi_2^{n-2j-1}), \quad \Phi_1, \Phi_2 \in \wedge^\bullet W, \tag{4.115}$$

here $()_{\text{top}}$ denotes the top degree component of the form and the superscript k denotes the k-form component of the form.

Mukai pairing is symmetric in dimensions $n \equiv 0,1$ (mod 4) and is skew-symmetric otherwise:

$$\langle \Phi_1, \Phi_2 \rangle = (-1)^{n(n-1)/2} \langle \Phi_2, \Phi_1 \rangle. \tag{4.116}$$

See [43] for details.

Mukai pairing has the following property.

Proposition 4.9 [6,10]

$$\langle v\Phi_1, v\Phi_2 \rangle = Q(v,v) \langle \Phi_1, \Phi_2 \rangle, \tag{4.117}$$

where $v \in W \oplus W^*$ $dimW = dimW^* = n.$

Proof: [6,10] Let Φ_1, Φ_2 be spinors in other words differential forms of degree k and of degree $n-k$ respectively, and $v = X + \xi$. First, let us calculate

$$\tau(v\Phi_1) = \tau(i_X \Phi_1 + \xi \wedge \Phi_1) = (-1)^{k-1} i_X \tau(\Phi_1) + \tau(\Phi_1) \wedge \xi \tag{4.118}$$

Then, we calculate the Mukai pairing as given in Definition 4.27:

$$\langle v\Phi_1, v\Phi_2\rangle = (\tau(v\Phi_1), v\Phi_2)_{\text{top}} = (-1)^{k-1} i_X \tau(\Phi_1)\wedge\xi\wedge\Phi_2 + \tau(\Phi_1)\wedge\xi\wedge i_X\Phi_2. \tag{4.119}$$

Note that $\tau(\Phi_1)\wedge\xi\wedge i_X\Phi_2$ is of degree $n+1$ hence it vanishes. Then, we take the interior product of this form with X as follows:

$$\begin{aligned} i_X(\tau(\Phi_1)\wedge\xi\wedge\Phi_2) &= (-1)^k\tau(\Phi_1)\wedge\xi(X)\wedge\Phi_2 \\ &\quad +(i_X\tau(\Phi_1)\wedge\xi\wedge\Phi_2 + (-1)^{k+1}\tau(\Phi_1)\wedge\xi\wedge i_X\Phi_2) \\ &= (-1)^k\xi(X)\langle\Phi_1,\Phi_2\rangle - (-1)^k\langle v\Phi_1, v\Phi_2\rangle \\ &= 0. \end{aligned} \tag{4.120}$$

It leads to the following

$$\begin{aligned} \langle v\Phi_1, v\Phi_2\rangle &= \xi(X)\langle\Phi_1,\Phi_2\rangle \\ &= Q(v,v)\langle\Phi_1,\Phi_2\rangle. \end{aligned} \tag{4.121}$$

[6,17] Since $Q(v,v) = \pm 1$, when $v \in Spin(n,n)$, (4.117) implies that Mukai pairing has an important property related to the action of the Spin group given in (4.84) , [6]:

$$\langle S\Phi_1, S\Phi_2\rangle = \pm\langle\Phi_1,\Phi_2\rangle, \;\; S \in Spin(n,n). \tag{4.122}$$

In the special case when $v \in Spin^+(n,n)$ we have $Q(v,v) = +1$, so Mukai pairing is invariant under the connected component to identity, $\text{Spin}^+(n,n)$.

When we take the charge conjugation elements C given in Definition 4.25 instead of S, we will have the following form. Using $Q(\Lambda_i,\Lambda_i) = -i_{\psi_i}\psi^i = -1$, repeated use of (4.122) gives

$$< C_n\chi_1, C_n\chi_2 >= (-1)^n < \chi_1, \chi_2 > \tag{4.123}$$

This will be important in calculations of section 5.2.1.

Now, let us see the relation between bilinear forms on $\bigwedge^n W$ and the Mukai pairing.

Definition 4.28 [43] One can describe a non-degenerate bilinear form on $\bigwedge^n W$ due to the existence of an inner product on T^*, :

$$(\Phi_1,\Phi_2) = \Phi_1\wedge *\Phi_2 = \sum_j \Phi_1^j \wedge *\Phi_2^j \tag{4.124}$$

where $*$ is the Here Hodge star is taken with respect to the inner product on W.

Now, we give the relation between this bilinear form and the Mukai pairing.

Lemma 4.6 [43] The bilinear form (4.124) is related to the Mukai pairing in the following way:

$$(\Phi_1,\Phi_2) = \langle \Phi_1, C^{-1}\Phi_2 \rangle = (\tau(\Phi_1)\wedge C^{-1}\Phi_2)_{\text{top}}, \tag{4.125}$$

where the charge conjugation matrix presented in (4.93, 4.94) should be written in terms of an orthonormal basis with respect to the inner product on W.

We will give an example to exemplify Lemma 4.6 as follows:

Example 4.1 Let $\Phi_1 = \Phi_0 + \Phi_2 + \Phi_4$ be a non-homogeneous differential forms, namely a sum of all possible 0-forms, 2-forms and 4-forms respectively. Here $\Phi_2 = \Phi_{ij}\psi^i \wedge \psi^j$ and $\Phi_4 = \Phi_{ijkl}\psi^i \wedge \psi^j \wedge \psi^k \wedge \psi^l$. First we calculate $\Phi_1 \wedge *\Phi_1$ with respect to the flat metric on $\mathscr{R}^4$.

$$\Phi_1 \wedge *\Phi_1 = ({\Phi_0}^2 + {\Phi_{12}}^2 + {\Phi_{13}}^2 + {\Phi_{14}}^2 + {\Phi_{23}}^2 + {\Phi_{24}}^2 + {\Phi_{34}}^2 + {\Phi_{1234}}^2)\psi^1 \wedge \psi^2 \wedge \psi^3 \wedge \psi^4. \tag{4.126}$$

On the other hand, we calculate $(\tau(\Phi_1)\wedge C^{-1}\Phi_1)_{\text{top}}$. First, we have

$$\tau(\Phi_1) = \Phi_0 - \Phi_2 + \Phi_4. \tag{4.127}$$

Next, we have

$$C^{-1}\Phi_1 = C\Phi_1 = C^{-}\Phi_1 = (\psi^1 - \psi_1)(\psi^2 - \psi_2)(\psi^3 - \psi_3)(\psi^4 - \psi_4)\Phi_1 \tag{4.128}$$

given in Definition 4.25. We use the relations given in (4.92) and get

$$\begin{aligned}(\tau(\Phi_1)\wedge C^{-1}\Phi_1)_{\text{top}} \;=\; & {\Phi_0}^2\psi^1 \wedge \psi^2 \wedge \psi^3 \wedge \psi^4 + \\ & ({\Phi_{12}}^2 + {\Phi_{13}}^2 + {\Phi_{14}}^2 + {\Phi_{23}}^2)\psi^1 \wedge \psi^2 \wedge \psi^3 \wedge \psi^4 \\ & + {\Phi_{1234}}^2\psi^1 \wedge \psi^2 \wedge \psi^3 \wedge \psi^4.\end{aligned} \tag{4.129}$$

Hence, we verify that (4.125) holds.

4.3.2.4 Spinorial action of $Spin(n,n)$ on a spinor field Φ

Since we have defined the spinor fields, we discuss the spinorial action of $Spin(n,n)$ on spinor fields Φ.

Definition 4.29 Let B is given as Definition 4.6. It maps $e^i \wedge e^j$ maps $\partial_i \mapsto e^j$ and its image in the Clifford algebra is given as $\frac{1}{2}B_{ij}e^je^i$. Since, e^je^i maps $e^i \wedge e^j$ and $e_i \mapsto e^j$. Its spinorial action on Φ is

$$B \cdot \Phi = \frac{1}{2}B_{ij}e^j \wedge (e^i \wedge \Phi) = -B \wedge \Phi. \tag{4.130}$$

By exponentiating we get the spinorial action of $Spin(n,n)$ element as follows:

$$S_B : \ \Phi \longmapsto e^{-B} \wedge \varphi = (1 - B + \frac{1}{2}B \wedge B - \ldots) \wedge \Phi. \tag{4.131}$$

This elements is given as $\rho(S_B) = h_B$ where ρ is as given in (4.85), h_B is given as (4.30).

Definition 4.30 Let β be as given in Definition 4.7. It maps $e_i \wedge e_j$ and $e^j \mapsto -e_i$. Its image in Clifford algebra is given as

$$\frac{1}{2}\beta^{ij}e^je^i \tag{4.132}$$

since it maps $e^j \mapsto -e_i$. Its spinorial action on Φ is

$$\beta \cdot \Phi = \frac{1}{2}\beta^{ij}i_{e_j}(i_{e_i}\Phi) = i_\beta \Phi. \tag{4.133}$$

Similarly, by exponentiating we get the spinorial action of $Spin(n,n)$ element as follows:

$$S_\beta : \ \Phi \longmapsto e^\beta \Phi = (1 + i_\beta + \frac{1}{2}i_\beta^2 + \cdots)\Phi. \tag{4.134}$$

This elements is given as $\rho(S_\beta) = h_\beta$ where ρ is as given in (4.85), h_β is given as (4.30).

Definition 4.31 Let A be given as in Definition 4.5. It maps $e_i \mapsto e_j$ and $e^j \mapsto -e^i$. Its image in the Clifford algebra is given as $\frac{1}{2}A_i^j(e_je^i - e^ie_j)$, since it maps $e_i \mapsto e_j$ and $e^j \mapsto -e^i$.

Its spinorial action on Φ is given as

$$\begin{aligned} A\cdot\Phi &= \frac{1}{2}A_i^j(e_je^i-e^ie_j)\cdot\Phi && (4.135)\\ &= \frac{1}{2}A_i^j(i_{e_j}(e^i\wedge\Phi)-e^i\wedge i_{e_j}\Phi) && (4.136)\\ &= \frac{1}{2}A_i^j(\delta_j^i\Phi-e^i\wedge i_{e_j}\Phi-e^i\wedge i_{e_j}\Phi) && (4.137)\\ &= \frac{1}{2}A_i^j\delta_j^i\Phi-A_i^je^i\wedge i_{e_j}\Phi && (4.138)\\ &= \frac{1}{2}(TrA)\Phi-A^*\Phi, && (4.139)\end{aligned}$$

where $\Phi\mapsto -A^*\Phi=-A_i^je^i\wedge i_{e_j}\Phi$.

The spinorial action of $Spin(n,n)$ is given as follows:

$$S_A\cdot\Phi=\frac{1}{\sqrt{\det e^A}}(e^{A*})\Phi.$$

Here, $e^{\frac{1}{2}(TrA)}=\sqrt{\det e^A}$. This elements is given as $\rho(S_A)=h_A$ where ρ is as given in (4.85), h_A is given as (4.30).

Note that, $h_i^\pm$ given in (4.31) corresponds to $\Lambda_i^\pm$ satisfying $\rho(\Lambda_i^\pm)=h_i^\pm$.

4.3.2.5 Pure spinors

There is a special type of spinors which are called pure spinors. We will deal with the pure spinors and their transformations in the section 6. These pure spinors must satisfy certain differential equations. Here is a brief introduction to pure spinors for the calculations regarding them.

The significance of pure spinors is their annihilator is not only isotropic but maximally isotropic. It is useful to note that null spaces are isotropic in a sense that if $v,w\in L_\Phi$, then

$$2Q(v,w)\Phi=(vw+wv)\cdot\Phi=0, \qquad (4.140)$$

implying that $Q(v,w)=0\ \forall\ v,w\in L_\Phi$.

Definition 4.32 [6] Let Φ be any nonzero spinor. Then, its *null space* $L_\Phi<W\oplus W^*$ is described as follows:

$$L_\Phi=\{v\in W\oplus W^*\ :\ v\cdot\Phi=0\}, \qquad (4.141)$$

where $v = w + w'$ acts on Φ as given in (4.104)

L_Φ depends equivariantly on Φ under the spin representation:

$$L_{g \cdot \Phi} = \rho(g) L_\Phi \quad \forall g \in Spin(W \oplus W^*). \tag{4.142}$$

Definition 4.33 A spinor Φ is called *pure* when L_Φ is maximally isotropic.

Theorem 4.1 [6] Every maximal isotropic in $W \oplus W^*$ corresponds to a pure spinor line generated by

$$\Phi_L = \exp(B + i\omega)\Omega, \tag{4.143}$$

where B, ω are real 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ for some linearly independent complex 1-forms $(\theta_1, \ldots, \theta_k)$. The integer k is called the *type* of the maximal isotropic. The maximal isotropic is of real index zero if and only if

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0, \tag{4.144}$$

or in other words

- $(\theta_1, \ldots, \theta_k, \overline{\theta}_1, \ldots, \overline{\theta}_k)$ are linearly independent, and
- ω is nondegenerate when restricted to the real $(n-k)$-dimensional subspace $\Delta \leq V$ defined by $\Delta = Ker(\Omega \wedge \overline{\Omega})$.

Proof: [12] We begin with the two extreme cases of a pure spinor. First of them is $\Phi = 1$. In this case, the null space is given by all contraction operators as follows: $\text{Span}\{\text{i}_m\}$. It is maximally isotropic due to the dimension $m = 2n$.

Second case is when $\Phi = \text{vol}$. In this case, the null space is given by all the wedge operators; $\text{Span}\{\text{d}x^m \wedge\}$. It is also maximally isotropic due to the dimension $m = 2n$.

Now, let us merge the two cases. Let us assume that a k-form $\Phi_k = \text{d}x^1 \wedge \ldots \wedge \text{d}x^k$ which can be treated as a $Cliff(2n, 2n)$ spinor. Its null space becomes the following d-dimensional space; $\text{Span}\left(\{\text{d}x^1 \wedge, \ldots \text{d}x^k \wedge\} \cup \{\text{i}_{k+1}, \ldots \text{i}_m\}\right)$. That means, it is maximally isotropic. Hence, Φ_k is also a pure spinor.

Now, let us consider more general case when Φ_k is any k-form. This case consists of two legs; one of them is when Φ_k is decomposable and the other one Φ_k is non-decomposable.

Assume that Φ is a decomposable k-form

$$\Phi_k = \theta_1 \wedge \ldots \wedge \theta_k \tag{4.145}$$

As in the previous case, its null space is the union of the wedge product of the k one-forms and any contraction that is orthogonal to the one-forms:

$$\mathrm{Span}\left(\{\theta_1\wedge, \ldots, \theta_k\wedge\} \cup \{v \mid \mathrm{i}_v\,\theta_1 = \cdots = \mathrm{i}_v\,\theta_k = 0\}\right) \tag{4.146}$$

where $\mathrm{i}_v \equiv v^m \iota_m$. It is also d-dimensional, namely maximally isotropic. Hence, Φ_k in (4.145) is a pure spinor.

Let us see what happens when Φ_k is not decomposable. In this case, the null space (4.146) becomes a set of linear combinations of wedge products and contractions;

$$Span\left(\{\iota_m + B_{mn}\,\mathrm{d}x^n\wedge,\ 1 \le m \le 2n\}\right). \tag{4.147}$$

where $B = B_{mn}\,\mathrm{d}x^m \wedge\,\mathrm{d}x^n$ is a complex 2-form.

We have already shown in section 6.5 that null spaces are isotropic. It follows that

$$\left\{\mathrm{i}_m + B_{mn}\,\mathrm{d}x^n\wedge, \mathrm{i}_p + B_{pq}\,\mathrm{d}x^q\wedge\right\} = \delta^q_m B_{pq} + \delta^n_p B_{mn} = B_{pm} + B_{mp} = 0. \tag{4.148}$$

It leads to $B_{pm} = -B_{mp}$, namely B is an antisymmetric 2-form. Moreover, it has the maximum possible dimension $2n$, namely the null space of the non-decomposable $Clif(2n,2n)$ spinor Φ_k is maximally isotropic. Hence, it should correspond to a pure spinor.

As we have discussed in Definition 4.26 the chiral spinors for calculation purposes we prefer positive chirality pure spinor $\Phi \in \Lambda^{\text{even}}\,T^*$.

Let the non-homogeneous form $\Phi = \Phi_0 + \Phi_2 + \cdots \Phi_d$ is given, where Φ_k is a k-form, Φ_0 is a constant.

Φ must be annihilated by the action of null space element:

$$\iota_m \Phi + B_{mn}\,\mathrm{d}x^n \wedge \Phi = 0. \tag{4.149}$$

First, collecting 1-forms we get

$$B_{mn}\,\mathrm{d}x^n \Phi_0 + \mathrm{i}_m\,\Phi_2 = 0. \tag{4.150}$$

This equation is solved uniquely as follows.

$$\imath_m \Phi_2 = -B_{mn}\, dx^n \Phi_0 \tag{4.151}$$

$$\begin{aligned} \Phi_2 &= -\frac{1}{2} B_{mn}\, dx^m \wedge dx^n \Phi_0 \\ &= -B\Phi_0. \end{aligned} \tag{4.152}$$

Second, collecting 3-forms we get

$$B_{mn}\, dx^n \wedge \Phi_2 + \mathrm{i}_m \Phi_4 = 0. \tag{4.153}$$

This equation is solved as follows.

$$\begin{aligned} \mathrm{i}_m \Phi_4 &= -B_{mn}\, dx^n \Phi_2 \\ &= \frac{1}{2} B_{mn} B_{pq}\, dx^n \wedge dx^p \wedge dx^q\, \Phi_0 \end{aligned} \tag{4.154}$$

$$\Phi_4 = \frac{1}{2} B \wedge B\, \Phi_0. \tag{4.155}$$

Finally, one can obtain

$$\Phi = \left(1 - B + \frac{1}{2} B \wedge B - \frac{1}{6} B \wedge B \wedge B + \ldots \right) \Phi_0 = e^{-B}\, \Phi_0. \tag{4.156}$$

We conclude that, (4.147) is also the null space of the pure spinor $\Phi = e^{-B}$.

Now, let us merge the two cases:

$$\Phi = e^{-B} \wedge \theta_1 \wedge \ldots \wedge \theta_k. \tag{4.157}$$

This gives the most generic form of a pure spinor given in (4.143) when we replace $-B$ with $B + i\omega$ in order to be consistent with the literature.

This concludes the analysis related to the almost complex structures and pure spinors. We will now describe certain examples of generalised complex structures and pure spinors in detail, [6].

Example 4.2 <u>Symplectic type $(k = 0)$</u>

A symplectic structure determines the generalised complex structure, a maximal isotropic subspace and a spinor line as follows: The generalised complex structure $\mathscr{J}_\omega$ is as given in (4.62). The maximal isotropic

$$L = \{X - i\omega(X)\ :\ X \in V \otimes \mathbb{C}\} \tag{4.158}$$

annihilates a spinor line generated by

$$\varphi_L = e^{i\omega}, \tag{4.159}$$

Since,

$$\begin{aligned}
(X - i\omega(X))e^{i\omega} &= (X - i\omega(X))(1 + i\omega - \frac{1}{2}\omega\wedge\omega - \frac{i}{6}\omega\wedge\omega\wedge\omega + ...) && (4.160)\\
&= \iota_X 1 + i\iota_X\omega - \frac{1}{2}\iota_X(\omega\wedge\omega) - \frac{i}{6}\iota_X(\omega\wedge\omega\wedge\omega) && (4.161)\\
&\quad -i(\iota_X\omega + i\iota_X(\omega\wedge\omega) - \frac{1}{2}\iota_X(\omega\wedge\omega) - \frac{i}{6}\iota_X(\omega\wedge\omega\wedge\omega))\\
&= 0.
\end{aligned}$$

Let us show L is the +i-eigenspace of $\mathscr{J}_\omega$.

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}\begin{pmatrix} X \\ -i\omega(X) \end{pmatrix} = \begin{pmatrix} iX \\ \omega(X) \end{pmatrix} = i\begin{pmatrix} X \\ -i\omega(X) \end{pmatrix} \tag{4.162}$$

[6] This generalised complex structure is of type $k = 0$, here k is the codimension of the projection of L to $V\otimes\mathbb{C}$. Furthermore, a transformation by a B-field can be achieved. Since, B-transformation preserves projections to $V\otimes\mathbb{C}$, then it also preserves type, one can obtain another generalised complex structure of type $k = 0$ after the transformation.

$$\mathscr{J}'_\omega = e^{-B}\mathscr{J}_\omega e^B = \begin{pmatrix} -\omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & B\omega^{-1} \end{pmatrix}, \tag{4.163}$$

B-transformation also generates new maximal isotropic and a pure spinor as follows.

$$e^{-B}(L) = \{X - (B + i\omega)(X) \;:\; X \in V\otimes\mathbb{C}\} \tag{4.164}$$

$$\varphi_{e^{-B}L} = e^{B+i\omega}. \tag{4.165}$$

This is called a *B-symplectic structure*. It is concluded that any generalised complex structure of type $k = 0$ is a B-field transform of a symplectic structure.

Complex type $(k = n)$

A complex structure determines the generalised complex structure, a maximal isotropic subspace and a spinor line as follows: The generalised complex structure is given as in (4.61).

A maximal isotropic and a spinor line generated by φ_L are given as follows.

$$L = V_{0,1}\oplus V^*_{1,0} \tag{4.166}$$

consists of $\{\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, ..., \frac{\partial}{\partial \bar{z}_n}, dz_1, dz_2, ...dz_n\}$ and annihilates

$$\varphi_L = \Omega^{n,0}. \tag{4.167}$$

Let us show L is the +i-eigenspace of $\mathscr{J}_J$.

$$\begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \bar{z}_i} \\ dz_i \end{pmatrix} = i \begin{pmatrix} \frac{\partial}{\partial \bar{z}_i} \\ dz_i \end{pmatrix} \tag{4.168}$$

Furthermore, a transformation by a B-field can be achieved. We obtain another generalised complex structure of type $k = n$ after the transformation.

$$\mathscr{J}_J' = e^{-B} \mathscr{J}_J e^B = \begin{pmatrix} -J & 0 \\ BJ + J^*B & J^* \end{pmatrix}, \tag{4.169}$$

B-transformation also generates new maximal isotropic and a pure spinor as follows.

$$e^{-B}(L) = \{X + \xi - i_X B \ : \ X + \xi \in V_{0,1} \oplus V_{1,0}^*\}, \tag{4.170}$$

$$\varphi_{e^{-B}L} = e^B \Omega^{n,0}. \tag{4.171}$$

Now, we will continue the reduction of the generalised G-structures. We will study further reductions from $U(n/2, n/2)$ in the next sections.

4.3.3 $U(n/2) \times U(n/2)$-structure

It is possible to reduce the structure group $U(n/2, n/2)$ that we discussed in section 4.3.2 to the maximal compact subgroup $U(n/2) \times U(n/2)$-structure by the existence of two compatible generalised almost complex structures.

Since $\mathscr{H}^2 = 1$ and $\mathscr{H}\mathscr{J} = \mathscr{J}\mathscr{H}$, $(\mathscr{H}\mathscr{J})^2 = -1$ and using $\mathscr{H}$ is symmetric and $\mathscr{J}^* = -\mathscr{J}$ then $(\mathscr{H}\mathscr{J})^* = -(\mathscr{H}\mathscr{J})$. One can notice that $\mathscr{H}\mathscr{J}_1 = \mathscr{J}_2$ is a new generalised almost complex structure. Let us introduce the compatibility condition of the generalised almost complex structures.

Definition 4.34 [12] Two generalised almost complex structures are called compatible if they commute

$$[\mathscr{J}_1, \mathscr{J}_2] = 0 \tag{4.172}$$

and if

$$\mathscr{H} = -\mathscr{J}_1 \mathscr{J}_2 \tag{4.173}$$

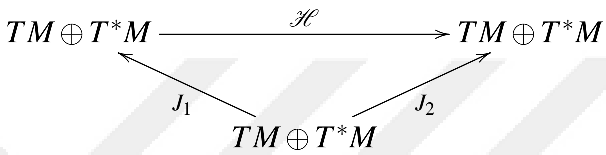
is a positive definite metric on the generalised tangent bundle $TM \oplus T^*M$, namely the generalised metric.

Remark 4.1 Here, the bracket given in (4.172) is the Lie bracket, and by multiplying $-\mathscr{J}_1$ with $\mathscr{J}_2$ we mean the composition map of $\mathscr{J}_1^{-1} = -\mathscr{J}_1$ with $\mathscr{J}_2$.

Hence, it leads to the following Proposition.

Proposition 4.10 ([6], Proposition 6.2)

A reduction to $U(n/2) \times U(n/2)$ is equivalent to the existence of two generalised almost complex structures $\mathscr{J}_1, \mathscr{J}_2$ as well as a positive definite metric $\mathscr{H}$ satisfying $\mathscr{H}^2 = 1$, which are related by the following commuting diagram:



Note that the conditions on $\mathscr{H}$ are equivalent to requiring that $\mathscr{J}_1$ and $\mathscr{J}_2$ commute and that $-\mathscr{J}_1\mathscr{J}_2$ is positive definite.

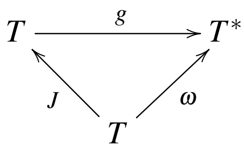
Note that

$$\mathscr{H}^2 = -\mathscr{J}_1\mathscr{J}_2(-\mathscr{J}_1\mathscr{J}_2) = \mathscr{J}_1{}^2\mathscr{J}_2{}^2 = -1(-1) = 1. \tag{4.174}$$

As it has presented in the Definition 4.13 that, the existence of a positive definite metric G satisfying $\mathscr{H}^2 = 1$ ensures that the structure group reduces to $O(n) \times O(n)$. Alternatively, a reduction to $U(n/2) \times U(n/2)$ can be seen as

$$U(n/2, n/2) \cap (O(n) \times O(n)) = U(n/2) \times U(n/2). \tag{4.175}$$

Example 4.3 [6] Let (g, J, ω) be a usual Kähler structure on a manifold. It means that g is a Riemannian metric g, J is a complex structure and ω is a symplectic structure with the following commuting diagram.



As we have expressed in (4.61) and (4.62), there exists generalised almost complex structures corresponding to complex and symplectic structures. First, we need to show that $\mathscr{J}_J, \mathscr{J}_\omega$ commute, i.e. $[\mathscr{J}_1, \mathscr{J}_2]$.

$$\mathscr{J}_J \mathscr{J}_\omega = \begin{pmatrix} 0 & -J\omega^{-1} \\ -J^*\omega & 0 \end{pmatrix}$$

and

$$\mathscr{J}_\omega \mathscr{J}_J = \begin{pmatrix} 0 & \omega^{-1}J^* \\ \omega J & 0 \end{pmatrix}$$

Please note that

$$\begin{aligned} -J\omega^{-1} &= J^*\omega^{-1} = \omega^{-1}J^* \\ -J^*\omega &= J\omega = \omega J. \end{aligned} \tag{4.176}$$

Hence, $[\mathscr{J}_1, \mathscr{J}_2]$ is satisfied.

Next, we will show that $\mathscr{H}$ is a positive definite metric on $T \oplus T^*$.

$$\mathscr{H} = -\mathscr{J}_J \mathscr{J}_\omega = \begin{pmatrix} 0 & J\omega^{-1} \\ J^*\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

Please note that $g = J^*\omega$ and $g^{-1} = (J^*\omega)^{-1} = \omega^{-1}J^{*-1} = J\omega$. $(\mathscr{J}_J, \mathscr{J}_\omega)$ defines a generalised Kähler structure.

4.3.4 $SU(n/2) \times SU(n/2)$-structure

We begin with defining the reduction of the structure group to $SU(n/2, n/2)$ as follows:
Definition 4.35 [10] A reduction to $SU(n/2, n/2)$ is equivalent to the existence of a nowhere vanishing globally defined pure spinor Φ such that $\langle \Phi, \bar{\Phi} \rangle \neq 0$ at all points.

It is possible to reduce the structure group $SU(n/2, n/2)$ to the maximal compact subgroup $SU(n/2) \times SU(n/2)$-structure by the existence of two compatible pure spinors. First, Let us describe the norm $\| \Phi \|$ of a pure spinor Φ, [18,19]. It is described through the Mukai pairing $<,>$, described in Definition 4.27.
Definition 4.36

$$< \Phi, \bar{\Phi} >= -i \| \Phi \|^2 \, vol \tag{4.177}$$

where *vol* is the volume form determined by the Riemannian metric.

Let us introduce this compatibility conditions, [12]. Two pure spinors Φ_1, Φ_2 are called compatible if their associated generalised almost complex structures $\mathscr{J}_{\Phi_1}, \mathscr{J}_{\Phi_2}$ are compatible in the sense of Definition 4.34 and if these pure spinors have equal norm. Hence,

$$\langle \Phi_1, \bar{\Phi}_1 \rangle = \langle \Phi_2, \bar{\Phi}_2 \rangle, \tag{4.178}$$

where $\langle , \rangle$ is Mukai pairing defined in Definition 4.27.

4.3.4.1 $SU(3) \times SU(3)$-structure

We restrict to the case when $n = 6$ since there will be an assumption in Chapter 6 that the seed background supports an $SU(3)$ structure as a special case of a $SU(3) \times SU(3)$ structure, and we will be able to perform a special transformation to explicitly show the pure spinors associated to the $SU(3)$ structure and the pure spinors associated to the $SU(2)$ structure are related by our calculation methods.

Definition 4.37 [12,18,19] The structure group of the generalised tangent bundle $TM \oplus T^*M$ of the 6-dimensional manifold M reduces to $SU(3) \times SU(3)$ if there exists two compatible globally defined $SU(3) \times SU(3)$ pure spinors Φ_1 and Φ_2 of non-vanishing equal norm.

The explicit form of the $SU(3) \times SU(3)$ pure spinors can be given as below:

Definition 4.38 [18,19,44] ($SU(3) \times SU(3)$ pure spinors:)

$$\Phi_1 = \frac{a\bar{b}}{8} e^{z\bar{z}/2} \left[c_1 e^{-ij} - ic_2 \omega \right], \tag{4.179}$$

$$\Phi_2 = -\frac{ab}{8} z \wedge \left[c_2 e^{-ij} + ic_1 \omega \right] \tag{4.180}$$

where c_1, c_2 are complex functions on M. Here $z = v + iw$ is a complex 1-form, j is a real 2-form and ω is a complex 2-form.

We also emphasize that the pure spinors given in (4.179) is of the form of a generic pure spinor given in Theorem 4.1 in equation (4.143). The $SU(3)$ structure given in (2.32) is actually a special case of the $SU(3) \times SU(3)$ structure, and the spinor that is used to describe the $SU(3)$ structure is also a special case of the (4.179).

Definition 4.39 [18,19,44] $SU(3)$ pure spinors:

$$\Phi_1 = -i \, \frac{ab}{8} \Omega, \qquad \Phi_2 = \frac{a\bar{b}}{8} e^{-iJ} \tag{4.181}$$

where J is real 2-form and Ω a complex 3-form given in (2.32).

Comparing to (4.179) and (4.180) we have $J = j + v \wedge w$, $\Omega = \omega \wedge (v + iw)$ and $c_1 = 1$, $c_2 = 0$.

We would like to emphasize that the first condition comes from the fact that pure spinors given in (4.179) have non-vanishing norm. Since J and Ω are non-degenerate differential forms, we end up with the following identity:

$$\langle \Phi_1, \bar{\Phi}_1 \rangle = \langle \Phi_2, \bar{\Phi}_2 \rangle = \frac{i}{8} \Omega \wedge \bar{\Omega} = \frac{1}{3!} J \wedge J \wedge J \neq 0. \tag{4.182}$$

Again, $SU(2)$ structure can be regarded as a special case of $SU(3) \times SU(3)$ structure [5,6,33] and the form of the pure spinor describing the $SU(2)$ structure is a special case of (4.179). The corresponding pure spinors are [18,19,45]:

Definition 4.40 [18,19,44] $SU(2)$ pure spinors:

$$\Phi_1 = -\frac{ab}{8} e^{-ij} \wedge (v + iw) \ , \qquad \Phi_2 = -i \frac{a\bar{b}}{8} e^{-iv \wedge w} \wedge \omega. \tag{4.183}$$

Here $z = v + iw$ is a complex 1-form, j and ω are 2-forms that describe an $SU(2)$ structure given in (2.33).

Comparing to (4.179) and (4.180) we have $c_1 = 0$ and $c_2 = 1$.

We would like to emphasize that the last condition comes from the fact that pure spinors given in (4.179) have non-vanishing norm. In this case, since j and ω are non-degenerate we have

$$\langle \Phi_1, \bar{\Phi}_1 \rangle = \langle \Phi_2, \bar{\Phi}_2 \rangle = \omega \wedge \bar{\omega} = 2j \wedge j \neq 0. \tag{4.184}$$

In this special case of generic $SU(3) \times SU(3)$ structure, $SU(2)$ structure can be described as intersection of two different $SU(3)$ structures.

Note that there exist two pairs of spinors generate two different $SU(3)$ structures that can be decomposed through the following forms, [19,44]:

$$\begin{aligned} J_1 &= j + v \wedge w \ , \qquad \Omega_2 = \omega \wedge (v + iw) \\ J_2 &= j - v \wedge w \ , \qquad \Omega_2 = \omega \wedge (v - iw) \end{aligned} \tag{4.185}$$

where j, ω, v and w are the $SU(2)$ structure defining forms given in (2.33).

5. SUPERSYMMETRY EQUATIONS AS PURE SPINOR EQUATIONS

Superstring theory is a quantum theory of gravity consistent in 10 dimensions. The field equations which describe the dynamics of supergravity (low energy approximation) are nonlinear partial differential equations which can be regarded as a generalisation of field equations of Einstein's theory of general relativity (GR). In GR, the main dynamical field is the Riemannian metric. In string theory, in addition to the Riemannian metric one also has a B-field, the dilaton and the Ramond-Ramond (RR) fields which are p-form fields. The Ramond-Ramond (RR) field strengths are 0,2,4 degree forms for Type (m) IIA wheras the Ramond-Ramond (RR) field strengths are 1,3,5 degree forms for Type IIB. In the democratic formulation, the Hodge duals of these forms are also taking into account so that the RR fields are 0,2,4,6,8,10 degree forms for Type (m) IIA wheras the RR fields are 1,3,5,7,9 degree forms for Type IIB, [13,14]. In order to relate string theory with 4 dimensional physical theories one has to consider solutions of the form $M_{1,3} \times Y_6$ corresponding to compactifications to 4 dimensional Minkowski space-time $M_{1,3}$. Here, Y_6 is a compact manifold called the internal manifold. Equivalently, we say that the 10 dimensional theory has been compactified on the 6-dimensional manifold Y_6. In order for the resulting 4-dimensional theory to be realistic the internal manifold must possess certain properties. For example, one usually demands the 4-dimensional theory to exhibit a certain type of symmetry called supersymmetry. For example, requirement of $\mathscr{N} = 2$ supersymmetry in a compactification of type II string theory in the absence of RR p-form field requires the internal manifold to be a Calabi-Yau manifold with 3 complex dimensions [12]–[14]. A Calabi-Yau manifold is a complex Kähler manifold with vanishing first Chern class and it was shown that it admits a Ricci-flat metric with $SU(3)$ holonomy. When the RR p-forms are non-vanishing the in the internal manifold is not a Calabi-Yau manifold anymore. However, a manifold of $SU(3)$-structure is still a consistent internal manifold. Recently, it has been understood that [15] for a supersymmetric compactification with non-vanishing p-forms, the internal manifold

Y_6 must have a generalised $SU(3)\times SU(3)$-structure of which $SU(3)$-structure is a special case.

The conditions to be obeyed by the internal space in a supersymmetric compactification of Type II string theory can be neatly described within the framework of generalised complex geometry [5,6]. The following fact was first shown in the seminal paper [18].
Lemma 5.1 [18,19,48] Demanding that the four dimensional solution preserves at least $\mathscr{N}=1$ supersymmetry implies that the structure group of the generalised tangent bundle $TM\oplus T^*M$ of the six dimensional internal manifold M is reduced from $SO(6,6)$ to $SU(3)\times SU(3)$. This topological condition on the internal manifold implies the existence of two globally defined pure spinors Φ_1 and Φ_2 of non-vanishing norm which are compatible in the sense of (4.178).

These $Cliff(6,6)$ spinors can be constructed from the internal spinors arising from the 10 dimensonal Killing spinors generating the supersymmetry transformations in 10 dimensions. A $Cliff(6,6)$ spinor can be mapped to a non-homogenous differential form (a polyform) through the Clifford map given in Lemma 4.3.2.3.

Lemma 5.2 It was shown in [18,19,48] that the Killing spinor equations coming from supersymmetry variations is equivalent to the following differential equations for the two pure spinors:

$$d(e^{2A-\phi}e^B\wedge\Phi_1) = 0, \tag{5.1}$$

$$d(e^{2A-\phi}e^B\wedge\Phi_2) = e^{2A-\phi}dA\wedge e^B\wedge\bar{\Phi}_2+\frac{i}{8}e^{3A}e^B\wedge\lambda(*_6F). \tag{5.2}$$

Remark 5.1 Note that for our purposes, we have presented the equations in a form where the B field appears explicitly, rather than writing them in terms of the differential operator $d_H = d+H\wedge$ as was originally done in [18,19,48].

The necessary and sufficient conditions for $\mathscr{N}=1$ supersymmetry are as follows, [19]:

$$(d+H\wedge)(e^{2A-\phi}\Phi_1) = 0, \tag{5.3}$$

$$(d+H\wedge)(e^{2A-\phi}\Phi_2) = e^{2A-\phi}dA\wedge\bar{\Phi}_2+\frac{i}{8}e^{3A}*\lambda(F) \tag{5.4}$$

and have norms $||\Phi_{1,2}||^2=\frac{e^{2A}}{8}$, where H is a closed 3-form $H=dB$ which has a significance in applications in physics. The equivalence of (5.1), (5.2) with (5.3), (5.4) arises from the identity $d+H\wedge = d(1+B\wedge) = d\ e^B$ since $B\wedge B=0$.

Let us first explain the ingredients in the pure spinor equations. In the equations above, A is the warp factor that appears in the compactification ansatz of the form $M_{1,3} \times Y_6$ where $M_{1,3}$ is four dimensional Minkowski space-time that we observe and Y_6 is a 6-dimensional internal manifold.

The metric is of the following form

$$ds^2 = e^{2A(y)} dx_{3,1}^2 + g_{mn} dy^m dy^n, \ \ m,n = 1,\cdots,6. \tag{5.5}$$

ϕ is the dilaton field and $*_6$ is the Hodge duality on the six dimensional internal manifold.

F is related to the polyform $F^{(10)}$ that encodes the Ramond-Ramond (RR) fields in the democratic formulation of supergravity.

$$F^{(10)} = F + \mathrm{vol}_4 \wedge *_6(\lambda F). \tag{5.6}$$

Here, $F = F_0 + F_2 + F_4 + F_6$ for Type IIA and $F = F_1 + F_3 + F_5$ for Type IIB, and they are internal forms having components only along the six dimensional internal space. Also,

$$\lambda(A_n) \equiv (-1)^{Int[n/2]} A_n = (-1)^{n(n-1)/2} A_n \tag{5.7}$$

for an n-form A_n. A $Spin(d,d)$[1] spinor F has positive chirality for Type IIA and is of negative chirality for Type IIB. The chirality of the pure spinor Φ_1 is the same as that of the RR fluxes and the pure spinor Φ_2 has opposite chirality.

In the next two sections, we aim to show the covariance of the pure spinor equations given in (5.1), (5.2) under $Pin(d,d)$ transformations.

5.1 Covariance of Pure Spinor Equations under $Pin(d,d)$

5.1.1 Constant $Pin(d,d)$ transformation

We will consider the transformation of the pure spinor equations under a constant $O(d,d)$ matrix T and the corresponding $Pin(d,d)$ matrix P with $\rho(P) = T$, where ρ is the double covering homomorphism given in (4.85).

[1] Henceforth, we will assume that the metric is of signature (d,d) rather than (n,n).

Theorem 5.1 [17,49] The pure spinor equations (5.1) and (5.2) are covariant under the following constant $Pin(d,d)$ transformations.

$$
\begin{aligned}
\Phi &\rightarrow \Phi' = \sqrt{G}\, P.\Phi = \sqrt{G}\, e^{-B'} P e^{B} \Phi && (5.8)\\
F &\rightarrow F' = P.F = e^{-B'} P e^{B} F && (5.9)\\
\mathscr{H} &\rightarrow \mathscr{H}' = T\ \mathscr{H}(g,B)\ T^t && (5.10)\\
e^{\phi} &\rightarrow e^{\phi'} = \sqrt{G} e^{\phi} && (5.11)\\
A &\rightarrow A && (5.12)
\end{aligned}
$$

where $T \in O(d,d)$ and $P \in Pin(d,d)$.

Before we prove this theorem we investigate the transformations of each field $\Phi, F, \mathscr{H}, \phi$, A and the operator $*$ in detail.

We begin with the transformation of the metric g under an $O(D,D)$ matrix T as follows:
Lemma 5.3 [20] The metric g transforms under an $O(D,D)$ matrix $\hat{T}$ as follows:

$$g' = \frac{1}{(\hat{c}E+\hat{d})^t}\, g\, \frac{1}{(\hat{c}E+\hat{d})}. \tag{5.13}$$

$$\frac{det g'}{det g} = det((\hat{c}E+\hat{d})^{-1})^2. \tag{5.14}$$

Proof: Here note that the transformation of the metric and the B-field is as given in (4.59). We compute

$$
\begin{aligned}
g' &= \frac{E'+E'^t}{2} \qquad (5.15)\\
&= \frac{1}{2}\Big(\frac{(\hat{a}E+\hat{b})}{(\hat{c}E+\hat{d})} + \frac{(\hat{a}E+\hat{b})^t}{(\hat{c}E+\hat{d})^t}\Big)\\
&= \frac{1}{2}\Big(\frac{1}{(\hat{c}E+\hat{d})^t}(\hat{a}E+\hat{b})^t\ (\hat{c}E+\hat{d})\frac{1}{(\hat{c}E+\hat{d})} + \frac{1}{(\hat{c}E+\hat{d})^t}(\hat{c}E+\hat{d})^t\ (\hat{a}E+\hat{b})\frac{1}{(\hat{c}E+\hat{d})}\Big)\\
&= \frac{1}{(\hat{c}E+\hat{d})^t}\frac{E+E^t}{2}\frac{1}{(\hat{c}E+\hat{d})}.
\end{aligned}
$$

Here, we use the identities given in (4.8).

On the other hand we will analyze the transformation of the volume form $vol = *_d 1$. This follows immediately from the transformation of the metric g which can be read off from the symmetric part of E' in (4.58).

Lemma 5.4 The transformation of the volume form is given by the transformation of the metric given in (5.13) as follows:

$$*_d 1 = \sqrt{detg} dy^1 \wedge \cdots dy^d \rightarrow *'_d 1 = \sqrt{detg'} dy^1 \wedge \cdots dy^d = G\sqrt{detg} dy^1 \wedge \cdots dy^d = G *_d 1,$$

as given in (5.14) here we have

$$G \equiv \frac{\sqrt{detg'}}{\sqrt{detg}} = (det(cE+d)^{-1})^2. \tag{5.16}$$

Let us consider the transformation of the $\mathscr{H}$ under an arbitrary $O(D,D)$ matrix h. Taking into account that $\mathscr{H}$ is associated with the metric g and the field B, there exists a transformation of $\mathscr{H}$ implied by the transformation of E given in (4.58). As a starting point, we take into consideration the case when E is created from the identity background I by the action of h, [36]. Suppose that for any E we have a corresponding transformation $h_E \in O(D,D)$ as follows:

$$E = h_E(I). \tag{5.17}$$

To define h_E , we assign a vielbein A and using the fact that g is symmetric we write the symmetric element as $AA^t = g$. In the explicit expression for h_E as in (4.7), we are able to determine that if we use A and B we obtain the following result:

$$h_E = \begin{pmatrix} A & B(A^t)^{-1} \\ 0 & (A^t)^{-1} \end{pmatrix}. \tag{5.18}$$

We show that h_E is an element of $O(D,D)$ using (4.9). We replace $a \rightarrow A$, $b \rightarrow B(A^t)^{-1}$, $c = 0$ and $d \rightarrow (A^t)^{-1}$ in (4.7). We see that (4.9) holds when $B = -B^t$. This result matches with the Definition 4.6.

Now, let us show that h_E satisfies (5.17).

$$\begin{aligned} h_E(I) &= (AI + B(A^t)^{-1})(0 \cdot I + (A^t)^{-1})^{-1} \\ &= (A + B(A^t)^{-1})A^t \\ &= AA^t + B \\ &= g + B = E \end{aligned} \tag{5.19}$$

We have demonstrated that the transformation (5.18) is explicitly designed to generate a background E from the identity background, [36]. Now, we calculate the product of matrices $h_E h_E^t$ as follows:

$$\begin{aligned} h_E h_E^t &= \begin{pmatrix} A & B(A^t)^{-1} \\ 0 & (A^t)^{-1} \end{pmatrix} \begin{pmatrix} A^t & 0 \\ -A^{-1}B & A^{-1} \end{pmatrix} \\ &= \begin{pmatrix} AA^t + B(A^t)^{-1}A^{-1}B^t & B(A^t)^{-1}A^{-1} \\ (A^t)^{-1}A^{-1}B^t & (A^t)^{-1}A^{-1} \end{pmatrix} \\ &= \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}. \end{aligned} \tag{5.20}$$

This is the form of $\mathscr{H}(E)$ given in (4.57). We conclude that $\mathscr{H}(E) = h_E h_E^t$. Now, suppose that E' is a transformation of E by h which equivalent to (4.58). We have $E' = h\,E = hh_E(I) = h_{E'}(I)$.

This implies the following transformation of $\mathscr{H}$ related to $E' = (g', B')$.

$$\begin{aligned} \mathscr{H}'(E') &= h_{E'}\, h_{E'}^t = h\, h_E\, (h\, h_E\,)^t \\ &= h\, h_E h_E^t\, h^t = h\, (h_E\, h_E^t)\, h^t = h\mathscr{H}(E)\, h^t. \end{aligned} \tag{5.21}$$

Hence, it is concluded that the transformation given in (4.58) is equivalent to the following transformation of $\mathscr{H}$, [36].

$$\mathscr{H}'(E') = \mathscr{H}'(g', B') = h\, \mathscr{H}(g, B)\, h^t. \tag{5.22}$$

This proves (5.10).

Remark 5.2 [17] We observe that the pure spinors transform under $Pin(d,d)$ in basically the same way as the RR fields transform. However, there is a slight change in (5.8) which makes sure that the norms of the pure spinors are kept invariant (up to a sign). The norm of a pure spinor Φ is defined via the Mukai pairing is as given in (4.177). As discussed in (4.117), the Mukai pairing as given in Definition 4.27 has the following transformation property under the action of certain elements P of $Pin(d,d)$:

$$< P\Phi_1, P\Phi_2 > = \pm < \Phi_1, \Phi_2 >, \tag{5.23}$$

where either $P \in Spin(d,d)$ or is of the form $P = C_n S$ or $P = SC_n$ with $S \in Spin(d,d)$ and C_n is as in Definition 4.25. The NATD matrix (5.63) is obviously of this form.

The transformation of the generalised dilaton field $e^{-2d} = \sqrt{\det g}\; e^{-2\phi}$ is invariant under $O(d,d)$, that is $e^{-2d'} = e^{-2d}$ so that:

$$e^{-2\phi'}\sqrt{\det g'} = e^{-2\phi}\sqrt{\det g}. \tag{5.24}$$

Therefore, we conclude that in (5.11) the dilaton field ϕ transforms exactly with the same $\sqrt{G}$ factor.

Now all we have to do is to figure out the transformation of the term involving Hodge duality on the right hand side of equation (5.2).

Lemma 5.5 $*\lambda(F)$ can be written as

$$*\lambda(F) = -C_d^{-1}S_g^{-1}F. \tag{5.25}$$

Here $S_g^{-1} = S_{g^{-1}}$ is the $Spin(d,d)$ element that projects onto the following $SO(d,d)$ element

$$h_{g^{-1}} \equiv \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \tag{5.26}$$

under the double covering homomorphism ρ given in (4.85) that is, $\rho(S_{g^{-1}}) = h_{g^{-1}}$. Note that the equation (5.25) is valid in all even dimensions due to the properties of the charge conjugation matrix C given in Definition 4.25 with (4.95). In odd dimensions, the definition of $\mathscr{K}$ involves $(\psi^i + \psi_i)$, rather than the $(\psi^i - \psi_i)$ in (4.3.2.2). Since we deal with in particular for $*_6$ with $d = 6$, (5.25) is valid. Note that S_g becomes the identity when the metric is Euclidean which was given in Example 4.1.

It is useful to write $C_d S_{g^{-1}}$ as $e^{-B}\mathscr{K}_d e^B$ where $\mathscr{K}_d = C_d^{-1}\mathbb{S}$ and $\mathbb{S} \equiv S_B^\dagger S_{g^{-1}} S_B$ is the $Spin(d,d)$ element that projects onto the generalised metric $\mathscr{H}$; $\rho(\mathbb{S}) = \mathscr{H}$, $\rho(S_B^\dagger S_{g^{-1}} S_B) = h_B^t h_g^{-1} h_B = \mathscr{H}$.

Here we have used that $S_B = e^{-B}$ and using (4.100) we have $S_B^\dagger = C_d S_{-B} C_d^{-1} = C_d e^B C_d^{-1}$. Indeed,

$$\begin{aligned} e^{-B}\mathscr{K}_d e^B &= e^{-B}C_d^{-1}\mathbb{S}e^B = e^{-B}C_d^{-1}S_B^\dagger S_{g^{-1}} S_B e^B &(5.27)\\ &= e^{-B}C_d^{-1}C_d e^B C_d^{-1} S_{g^{-1}} e^{-B} e^B &(5.28)\\ &= C_d^{-1}S_{g^{-1}}. &(5.29)\end{aligned}$$

Rewriting (5.25) for $d = 6$ and in terms of $\mathscr{K}_d$, we have the desired identity:

$$*_6\lambda(F) = -e^{-B}\mathscr{K}_6 e^B F. \tag{5.30}$$

$\square$

As given in (4.72) we have an identification between dx_i with Γ^i, then we take $d = \Gamma^i \partial_i$. Let α be a k-form:

$$\alpha = \alpha_I dx^I = \alpha_I dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \tag{5.31}$$

Then, exterior derivative of α be a k+1-form:

$$d\alpha = \frac{\partial \alpha_I}{\partial x^i} dx^i \wedge dx^I \tag{5.32}$$

We have the Clifford algebra element corresponding to α:

$$\gamma = \alpha_I \Gamma^{i_1} \Gamma^{i_2} \ldots \Gamma^{i_1} \tag{5.33}$$

If we take the exterior derivative of γ, we have

$$d\gamma = \Gamma^i (\partial_i \alpha_I) \Gamma^{i_1} \Gamma^{i_2} \ldots \Gamma^{i_k} \tag{5.34}$$

Then, as we give in (4.72) we verify that the Clifford algebra element corresponding to (5.32) is (5.34).

It can be extended to the following form in order to write the equations in an $O(d,d)$ covariant way.

$$d + \tilde{d} \equiv \frac{1}{2} \Gamma^M \partial_M = \frac{1}{2} (\Gamma^i \partial_i + \Gamma_i \tilde{\partial}^i) = \psi^i \partial_i + \psi_i \tilde{\partial}^i. \tag{5.35}$$

Note that we will take $\tilde{\partial}^i = 0$. Here, the gamma matrices are as defined in Definition 4.19.

Therefore, we rewrite the pure spinor equations (5.1), (5.2) by replacing the exterior derivative operator d with $\Gamma^M \partial_M$ and using (5.30) as follows:

Lemma 5.6 The pure spinor equations (5.1), (5.2) can be written in the following form:

$$\Gamma^M \partial_M (e^{2A-\phi} e^B \wedge \Phi_1) = 0, \tag{5.36}$$

$$\Gamma^M \partial_M (e^{2A-\phi} e^B \wedge \Phi_2) = e^{2A-\phi} \Gamma^M \partial_M A \wedge e^B \wedge \bar{\Phi}_2 \mp \frac{i}{8} e^{3A} \mathcal{K}_6 e^B F. \tag{5.37}$$

In its most basic form, Double Field Theory (DFT) constitutes the idea of incorporating $O(d,d)$ T-duality, a distinctive symmetry of string theory, into a field theory defined on a double configuration space, which could be interpreted as a symmetry of the field theory. As well as the standard space-time coordinates, the doubled space also includes the dual coordinates, which are associated with the winding excitations of closed string theory on backgrounds, [43]. This will allow us to extend the exterior derivative operator d formally to an $O(d,d)$ covariant derivative operator which also acts along the dual coordinates. This extension is only formal since one also imposes an $O(d,d)$ covariant constraint which ensures that all fields depend on the standard coordinates only. [17] The presence of the Lie group $O(d,d)$, which is the global symmetry group of DFT, makes it possible to describe the transformation under NATD of the Type II supergravity fields as a transformation in DFT. There is a democratic formulation of Type II supergravity which is the starting point for describing p-form fields. More precisely, one rewrites the supergravity fields in terms of the DFT fields $\mathscr{H}, d, \chi$, where $\mathscr{H}$ is the generalized metric that encodes the metric and the B-field, d is the generalized dilaton field and χ is the spinor field that packages the modified RR fields of Type II supergravity in the democratic formulation. These fields, being solutions of Type II supergravity also solve the DFT equations in the supergravity frame. As it is assumed that the isometry is respected by all the fields in the background, it is possible to go to a non-holonomic frame so that the DFT fields, when written with respect to such a frame, are independent of the isometry coordinates. We refer to such fields as *untwisted fields* as in [20,50]. Plugging the initial DFT fields in the field equations of DFT (of both the NS-NS sector and RR sector of Type II supergravity), one sees that the untwisted DFT fields satisfy the field equations of Gauged Double Field Theory (GDFT) which is a deformation of DFT, obtained from a Scherk-Schwarz reduction and the deformation is determined entirely by the fluxes associated with the Scherk-Schwarz twist matrix with geometric fluxes associated with isometry.

Remark 5.3 Note that the equations (5.36), (5.37) reduce to equations (5.1) and (5.2) in the supergravity frame where fields do not depend on the winding type coordinates so that $\tilde{\partial}^i = 0$, that is $\Gamma^M \partial_M = \Gamma^i \partial_i = d$. The upper sign in the last term of (5.37) is for Type IIB and the lower sign is for Type IIA. This is because in six dimensions $*_6 \lambda = \lambda *_6$ for odd degree forms, whereas $*_6 \lambda = -\lambda *_6$ for even degree forms.

We will discuss all the terms appearing in (5.36),(5.37) in order to show the action of a constant $Pin(d,d)$ transformation commute with these terms. First, we need to give the transformation of the term $e^{3A}\mathscr{K}_6 e^B F$ which appears in (5.37) under $P \in Pin(d,d)$ as follows. This is in fact a part of the proof of Theorem 5.1.

Lemma 5.7 [17] The transformation of the term $e^{3A}\mathscr{K}_6 e^B F$ under $P \in Pin(d,d)$ is

$$\begin{aligned} e^{3A'}\mathscr{K}'_d e^{B'} F' &= e^{3A'}(P\mathscr{K}_d P^{-1})e^{B'}(e^{-B'}Pe^B F) \\ &= P\left(e^{3A}\mathscr{K}_d e^B F\right). \end{aligned} \tag{5.38}$$

Proof: [17] We know that $F^{(10)}$ in (5.6) transforms as in (5.9). Let us discuss what this implies for the transformation of the internal forms F. We have

$$\begin{aligned} F^{(10)'} &= e^{-B'}Pe^B(F - \mathrm{vol}_4 \wedge e^{-B}\mathscr{K}_6 e^B F) \\ &= e^{-B'}Pe^B F - \mathrm{vol}_4 \wedge e^{-B'}P\mathscr{K}_6 e^B F. \end{aligned} \tag{5.39}$$

where we have used (5.30) and the fact that vol_4, being an even form, commutes with all elements of $Pin(d,d)$ and we use the fact that A is invariant under $Pin(d,d)$. To rewrite (5.39) in the form (5.6) we first take F' given in (5.9) which is again an internal form, as all the $Pin(d,d)$ operators on the left hand side have actions only on the internal space and then use the fact that under $P \in Pin(d,d)$ the field $\mathscr{K}_d$ transforms as

$$\mathscr{K}_d \to P.\mathscr{K}_d = \mathscr{K}'_d = P\mathscr{K}_d P^{-1}. \tag{5.40}$$

Inserting a $P^{-1}P$ after $\mathscr{K}$ in the second term of the right hand side of (5.39) and using (5.30) and (5.9), we obtain

$$\begin{aligned} F^{(10)'} &= F' - \mathrm{vol}_4 \wedge e^{-B'}\mathscr{K}'_6 e^{B'} F' \\ &= F' + \mathrm{vol}_4 \wedge \tilde{*}_6 \lambda(F'). \end{aligned} \tag{5.41}$$

Note that F' has components only along the six dimensional deformed space and the Hodge duality is taken with respect to the metric after the $O(d,d)$ transformation. This shows us that not only the polyform $F^{(10)}$ that encodes the RR fields in the democratic formulation, but also the internal polyform F that appears in the pure spinor equations (5.1,5.2) transform in the expected way as given in (5.9).

Using the transformation properties (5.40) and (5.9), (5.8), (5.11) we prove that (5.38) holds. We next discuss whether or not the generalized exterior derivative operator $\Gamma^M\partial_M$ commutes with the action of $Pin(d,d)$. We first start with $Spin(d,d)$ and we have the following:

Lemma 5.8 [17]

$$\Gamma^M \partial_M (S\,\chi) = S(\Gamma^M \partial_M \chi), \quad S \in Spin(d,d) \tag{5.42}$$

for any spinor field χ.

Proof: [17] Using the fact that h is the $SO(d,d)$ element that satisfies $\rho(S^{-1}) = h$, it leads to the following identity as given in (4.85):

$$(h^{-1})^M_A \Gamma^A = S^{-1} \Gamma^M S, \tag{5.43}$$

we see that for constant $S \in Spin(d,d)$:

$$\begin{aligned} \Gamma^M \partial_M (S\,\chi) &= \Gamma^M S\, \partial_M \chi \\ &= S(h^{-1})^M_A \Gamma^A \partial_M \chi \\ &= S\Gamma^A (h^{-1})^M_A \partial_M \chi. \end{aligned} \tag{5.44}$$

Then, if we have

$$(h^{-1})^M_A \partial_M \chi = \partial_A \chi \tag{5.45}$$

the commutation relation (5.42) holds

$$S\Gamma^A ((h^{-1})^M_A \partial_M)\chi = S\Gamma^A \partial_A \chi = S(\Gamma^M \partial_M \chi). \tag{5.46}$$

Note that we would have

$$(h^{-1})^M_A \partial_M \chi(hX) = \partial'_A \chi(X') \tag{5.47}$$

since one also transforms $X \to X' = hX$. However in all the examples we will be looking at, the transformation generated by P will act only along the coordinates on which the pure spinors will not depend, so that we will always have $X' = X$ and hence $\partial'_A \chi = \partial_A \chi$. The equation (5.42) holds as desired, as long as the condition (5.45) is satisfied.

By using (5.8), (5.11), (5.40), (5.9) and the invariance of A we prove that the pure spinor equations (5.36), (5.37) are covariant under $Spin(d,d)$ transformations.

Remark 5.4 Note that there is no sign flip in front of the last term on the right hand side of (5.37), since $Spin(d,d)$ transformations takes a solution of Type IIA/IIB to a solution also of Type IIA/IIB. However, a $Pin(d,d)$ transformation which involves odd number of reflections maps a solution of Type IIA to a solution of Type IIB and vice versa, and hence the sign of the aforementioned term in (5.37) flips after the transformation. Despite this, the pure spinor equations (5.36, 5.37) are still covariant, since for such P, the differential operator $d = \Gamma^M \partial_M$ and P anti-commutes, as we will now discuss.

We will be looking at the $Pin(d,d)$ elements that can be written as a product of $Spin(d,d)$ elements and Λ_i given in Definition 4.25 in (4.95) satisfying (4.96). Because the NATD matrix is of this form. Our discussions here can be straightforwardly extended so as to include the $Pin(d,d)$ elements which also involve the elements Λ_i^+ given in (4.95), but we refrain from doing that in order to avoid equations cluttered with pluses and minuses. Now, we are ready to discuss the proof of the Theorem 5.1 at the beginning of this section.

Lemma 5.9 [17] The action of P and the exterior derivative operator $d = \Gamma^M \partial_M$ on the $Clif(d,d)$ spinors $\Phi_{1,2}$ and F commutes if P involves an even number of Λ_is and they anti-commute otherwise.

Proof: [17] We use the equations (5.8),(5.11) and the fact that A is invariant we get

$$e^{2A'-\phi'} e^{B'} \wedge \Phi'_{1,2} = \frac{1}{\sqrt{G}} e^{2A-\phi} e^{B'} \wedge (\sqrt{G} e^{-B'} P e^{-B} \, \Phi_{1,2}) = P\left(e^{2A-\phi} e^B \wedge \Phi_{1,2}\right) \tag{5.48}$$

Then, we will show the commutation relations between d and the action of P. Due to the relations (4.96), we see that

$$d(\Lambda_i \chi) = \Gamma^M \partial_M (\Lambda_i \chi) = -\Lambda_i \Gamma^M \partial_M \chi = -\Lambda_i d\chi, \tag{5.49}$$

provided that $^M \neq \ ^i$ or $^M \neq \ _i$. We show the covariance of the pure spinor equations (5.1) and (5.2) under $P \in Pin(d,d)$. This proves Theorem 5.1. □

[17] As discussed above, this condition is automatically satisfied for Abelian T-duality, due to the existence of d commuting isometries. This makes it possible to choose a coordinate system such that none of the fields depend on the coordinates along which the constant $O(d,d)/Pin(d,d)$ transformation acts, and hence the desired

commutation or anti-commutation relations hold. Therefore, we conclude that the pure spinor equations are covariant under Abelian T-duality. As for NATD, (5.45) is also satisfied with a convenient choice of coordinates (again due to existence of isometries), but we still need to discuss the situation with non-constant P, since the NATD matrix (5.62) is not constant as has been assumed above. This discussion will be carried out in the next section.

5.1.2 Non-constant $Pin(d,d)$ transformation

In this subsection, we extend the discussion in the previous subsection to the case where the $Pin(d,d)$ transformation and hence the corresponding $O(d,d)$ transformation depends on some of the internal coordinates. This is important, as the NATD transformation is known to be generated by such coordinate dependent $Pin(d,d)$ transformations. The transformation properties summarized in Theorem 5.1 are obviously still valid, even when $P \in Pin(d,d)$ is coordinate dependent. However, one has to be more careful in discussing the commutation of the exterior derivative operator d and the action of P, as now d also acts on P.

Let us first discuss the case when the $Pin(d,d)$ matrix does in fact lie in the subgroup $Spin^+(d,d)$, $P = S \in Spin^+(d,d)$.

Lemma 5.10 [17]

$$\Gamma^M \partial_M (S\chi) = S(\Gamma^A \nabla_A \chi), \tag{5.50}$$

where

$$\nabla_A = \partial_A + \frac{1}{12} f_{ABC} \Gamma^B \Gamma^C - \frac{1}{2} f_A. \tag{5.51}$$

Proof: [17] We begin with

$$\begin{aligned} \Gamma^M \partial_M (S\, \chi(X)) &= \Gamma^M (\partial_M\, S)\chi(X) + \Gamma^M\, S(\partial_M\, \chi(X)) \\ &= \left\{\Gamma^M S(S^{-1}\partial_M S) + \Gamma^M S\, \partial_M\right\} \chi(X) \\ &= \left\{\Gamma^M S\, (S^{-1}\partial_M S + \partial_M)\right\} \chi(X) \\ &= S\, \Gamma^A\, (h^{-1})^M_A \left(S^{-1}\partial_M S + \partial_M\right) \chi(X), \end{aligned} \tag{5.52}$$

where $\rho(S^{-1}) = h$ and in passing to the last line, we have used (5.43).

To calculate the term $\Gamma^A\,(h^{-1})^M_{\ A} S^{-1}\partial_M S$ in (5.52) we will use an important identity that follows from the fact that the Lie algebras of $SO(d,d)$ and $Spin(d,d)$ are isomorphic:

$$\begin{aligned}\Gamma^A\,(h^{-1})^M_{\ \ A} S^{-1}\partial_M\, S &= \frac{1}{4}\Omega_{ABC}\Gamma^A\,\Gamma^B\,\Gamma^C \\ &= \frac{1}{12} f_{ABC}\Gamma^A\,\Gamma^B\,\Gamma^C\,\chi(X) - \frac{1}{2} f_B\Gamma^B\,\chi(X).\end{aligned} \tag{5.53}$$

Here, f_{ABC} are the fluxes associated with the matrix S, $f_{ABC} = 3\Omega_{[ABC]}$, [20,43,50].

$$\Omega_{ABC} = -(U^{-1})^M_{\ A}\partial_M (U^{-1})^N_{\ B} U^D_{\ N}\eta_{CD}$$

Note that Ω_{ABC} are antisymmetric in the last two indices: $\Omega_{ABC} = -\Omega_{ACB}$.

$$f_A = -\partial_M (U^{-1})^M_{\ A} = \Omega^C_{\ AC} \tag{5.54}$$

Now, we again assume that the transformation matrix S is such that (5.45) is obeyed. We emphasize again that this condition is trivially satisfied if the field χ does not depend on the coordinates along which S and hence h acts nontrivially. This is indeed the case for NATD and is guaranteed by the fact that NATD acts along isometry directions. Then, we prove that (5.50) holds.

Let us now discuss what happens when P involves odd number of Λ_i factors, so that P does not lie in the $Spin(d,d)$ subgroup.

Lemma 5.11 [17] Assume that P is of the form $P = C_n S$, where $S \in Spin^+(d,d)$ and C_n is as in Definition 4.25 with n odd. Then,

$$\Gamma^M \partial_M (P\chi) = -P(\Gamma^A \nabla_A \chi), \tag{5.55}$$

where ∇ is as in (5.51).

Proof: [17] Equation (5.43) is valid for all $Pin(d,d)$ elements, so we have

$$(h_1 \cdots h_n U)^M_{\ A}\Gamma^A = P^{-1}\Gamma^M P, \tag{5.56}$$

where U is the $SO^+(d,d)$ element that satisfies $\rho(S) = U$, and h_i satisfy $\rho(h_i) = \Lambda_i$ and are given in (4.31). When n is odd, it can be easily seen that $h_1 \cdots h_n = -J^d_n$, where J^d_n is the $O(d,d)$ matrix obtained by embedding the $O(n,n)$ matrix

$$J_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \tag{5.57}$$

in $O(d,d)$ in the usual way. Therefore, we have

$$\Gamma^M P = -P\Gamma^A\ (J_n^d U)^M{}_A. \tag{5.58}$$

We recalculate (5.52) for $P = C_n S$ and using (5.56) and (5.58) as follows:

$$\begin{aligned}
\Gamma^M \partial_M (P\,\chi(X)) &= \Gamma^M (\partial_M\, P)\chi(X) + \Gamma^M\, P(\partial_M\, \chi(X)) \\
&= \left\{\Gamma^M P(P^{-1}\partial_M P) + \Gamma^M S\, \partial_M\right\}\chi(X) \\
&= \left\{\Gamma^M P\,(P^{-1}\partial_M P + \partial_M)\right\}\chi(X) \\
&= -P\,\Gamma^A\,(J_n^d U)^M_A\left(\partial_M + S^{-1}\partial_M S\right)\chi(X),
\end{aligned} \tag{5.59}$$

where we have also used $P^{-1}\partial_M P = S^{-1}\partial_M S$ for $P = C_n S$. Therefore, we prove that (5.55) holds where ∇ in (5.51) with fluxes $f'_{ABC} = (J_n^d)^D_A f_{DBC}$ with f being the fluxes associated with the $Spin^+(d,d)$ matrix S.

Collecting the results in (5.48,5.38,5.42, 5.50) and (5.55), we conclude the following:

Theorem 5.2 [17] The fields *after* the transformation generated by the non-constant $P \in Pin(d,d)$ satisfy the supersymmetry equations (5.36,5.37) if and only if the fields *before* the transformation satisfy the following equations, which can be regarded as a deformation of those in (5.36,5.37) determined by the fluxes associated with P.

$$\Gamma^M \nabla_M (e^{2A-\phi} e^B \wedge \Phi_1) = 0, \tag{5.60}$$

$$\Gamma^M \nabla_M (e^{2A-\phi} e^B \wedge \Phi_2) = e^{2A-\phi}\Gamma^M \partial_M A \wedge e^B \wedge \bar{\Phi}_2 \mp (-1)^n \frac{i}{8} e^{3A} \mathscr{K}_6 e^B F. \tag{5.61}$$

Here, n is the number of Λ_i factors that appear in the definition of $P = C_n S, S \in Spin^+(d,d)$.

Before that, we will analyze Non-Abelian T-duality in the following section.

5.2 Non-Abelian T-duality

Non-Abelian T-duality (NATD) is a generalization of T-duality with non-Abelian isometries. It can be regarded as both a solution generating transformation for supergravity and an $O(10,10)$ transformation. We will study the $O(10,10)$ matrix associated with the NATD transformation which is obtained by embedding an $O(3,3)$ matrix in $O(10,10)$ as given in (4.11).

Let us introduce the NATD matrix embedded in $O(10,10)$.

Definition 5.1 Let T_{NATD} be an $O(n,n)$ matrix embedded in $O(10,10)$ and is given by

$$T_{\text{NATD}} = \begin{pmatrix} 0 & 1 \\ 1 & \theta_{ij} \end{pmatrix} \tag{5.62}$$

where $\theta_{ij} = v_k C_{ij}^k$, and $i,j = 1,2,...,n$ Here, v_k are coordinates of the NAT dual background, and $C_{ij}{}^k$ are the structure constants of the n dimensional Lie algebra of the isometry group G, so $i,j,k = 1,\ldots,n$.

5.2.1 $O(d,d)$ and $Pin(d,d)$ transformation rules

Now, we will analyze the transformation rules for metric, B-field, dilaton and spinor fields under an $O(D,D,\mathbb{R})$ matrix T given in (5.62). It was shown in [20] that the $O(D,D,\mathbb{R})$ transformation given in (5.62) is indeed a solution generating transformation. We will analyze the transformation of $Spin(d,d)$ fields under NATD. Therefore, we will need the $Pin(n,n)$ matrix S_{NATD} determined by the Lie group homomorphism ρ given in (4.85),

$$S_{\text{NATD}} = CS_\theta = S_\beta C. \tag{5.63}$$

We show that

$$\rho(S_{\text{NATD}}) = \rho(CS_\theta) = \rho(C)\rho(S_\theta) = J\, h_\theta = (T_{\text{NATD}}) \tag{5.64}$$

$$T_{\text{NATD}} = \begin{pmatrix} 0 & 1 \\ 1 & \theta_{IJ} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \theta_{IJ} \\ 0 & 1 \end{pmatrix} = J\, h_\theta \tag{5.65}$$

where S_θ and S_β are the $Spin^+(10,10)$ obtained via $SO^+(10,10)$ matrix and the charge conjugation matrix C given in Definition 4.25, with the property $\rho(C) = J$. S_θ and S_β corresponds to B-transformations and β-transformations given in Definition 4.6, Definition 4.7 with $\theta_{ij} = v_K C_{ij}{}^k$ and $\beta_{ij} = v_k C_{ij}{}^k$, respectively.

In chapter 6, we will study a special geometry endowed with an $SU(2)$ isometry. For the appropriate basis system it is possible to write the structure constants of the algebra $\mathfrak{su}(2)$ of the special unitary group $SU(2)$ as $C_{ij}{}^k = \varepsilon_{ij}{}^k$. Then, the spinorial action of S_θ with $C_{ij}{}^k = \varepsilon_{ij}{}^k$ is as given (4.131) and it can be calculated as follows:

$$\begin{aligned} S_\theta.\alpha &= e^{-\theta}\wedge\alpha = (1-\theta+\frac{1}{2}\theta\wedge\theta-\frac{1}{6}\theta\wedge\theta\wedge\theta+...)\wedge\alpha \\ &= (1 - v_3\, dv^2\wedge dv^1 - v_2\, dv^1\wedge dv^3 - v_1\, dv^3\wedge dv^2)\wedge\alpha. \end{aligned} \tag{5.66}$$

Here, passing to the second line we use the fact that $\theta \wedge \theta = 0$. The transformation rule for p-forms is calculated with $P = S_{\text{NATD}}$ as follows:

$$F' = e^{-B'} S_{\text{NATD}} e^{B} F. \tag{5.67}$$

To clearly explain the rules for transformation, it is best to introduce coordinates that will make the isometry symmetry manifest where the isometry group is G and that will help elucidate the rules. We begin with the metric in the following form, [20,50]:

$$\begin{aligned}
ds^2 &= G_{\mu\nu} dx^\mu dx^\nu && (5.68)\\
&= G_{mn} dx^m dx^n + 2G_{mi} dx^m d\theta^i + G_{ij} d\theta^i d\theta^j && (5.69)\\
&= G_{mn} dx^m dx^n + 2G_{mI} dx^m \sigma^I + G_{IJ} \sigma^I \sigma^J && (5.70)\\
&= G_{\alpha\beta} \sigma^\alpha \sigma^\beta, && (5.71)
\end{aligned}$$

where $\theta^i, i = 1, \cdots, d$ are coordinates for G and $\sigma^\alpha = \delta^\alpha_m dx^m$ and $\sigma^I, I = 1, \cdots, dimG$ are the left invariant 1-forms $\sigma^I = l^I{}_i d\theta^i$ on G. They can be defined through the Maurer-Cartan form

$$g^{-1} dg = \sigma^I T_I \tag{5.72}$$

with T_I forming a basis for the Lie algebra $\mathscr{G}$ of G. Similarly, the B-field can be written as follows:

$$\begin{aligned}
B &= \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu && (5.73)\\
&= \frac{1}{2} B_{mn} dx^m \wedge dx^n + B_{mI} dx^m \wedge \sigma^I + \frac{1}{2} B_{IJ} \sigma^I \wedge \sigma^J. && (5.74)
\end{aligned}$$

As long as the fields have a θ dependence all the θ dependencies will be encoded in $l^I{}_i$, due to the isometry group G action. Therefore, all the θ dependence of the fields in (5.70) and (5.74) are encoded in $l^I{}_i$, [20,50].

Similarly, p-form flux can be written as

$$F = \sum_p \left(F^{(p)}(x) + F_I^{(p-1)}(x) \sigma^I + \frac{1}{2} F_{IJ}^{(p-2)}(x) \sigma^I \wedge \sigma^J + \cdots + F^{(p-d)} \sigma^1 \wedge \cdots \wedge \sigma^d \right) \tag{5.75}$$

Here each $p-$form is decomposed according to how many legs it has along the G directions.

Definition 5.2 [20,36] Due to the existence of the isometries, there exists the following $O(10,10)$ matrix.

$$L = \begin{pmatrix} l^t & 0 \\ 0 & l^{-1} \end{pmatrix}. \tag{5.76}$$

Here, l is the $GL(10)$ matrix obtained by embedding the $GL(d)$ matrix l_d as given in (4.11) with components

$$(l_d)^I{}_i = l^I{}_i \tag{5.77}$$

such that $(l_d)^I{}_m = l^a_i = 0$, $(l_d)^a{}_m = \delta^a_m$. Here l^I_i are components of the left invariant 1-forms $\sigma^I = l^I{}_i d\theta^i$ on G defined from the Maurer-Cartan form $g^{-1}dg = \sigma^I T_I$ with T_I forming a basis for the Lie algebra $\mathscr{G}$ of the isometry group G. It is obvious that $L \in O(10,10)$: $l^t\, l^{-1} = 1$.

Lemma 5.12 [20,36] The generalised metric associated with a suergravity background g, B with isometries is called the untwisted fields $\mathscr{H}(x)$ and can be factorized as follows:

$$\mathscr{H}(x,\theta) = L(\theta)\mathscr{H}(x)L^T(\theta). \tag{5.78}$$

Untwisted field $F(x)$ can be factorized as follows:

$$F(x,\theta) = e^{-B(x,\theta)}S_L(\theta)e^{B(x)}F(x). \tag{5.79}$$

Furthermore, one can introduce the field $\mathscr{K} = C_d^{-1}\mathbb{S}$ where C_d is as given in Definition 4.25 and $\mathbb{S}$ is the element in $Spin^-(d,d)$ that projects onto $\mathscr{H}$ under the double covering homomorphism ρ given in (4.85) between $Pin(d,d)$ and $O(d,d)$, that is $\rho(\mathbb{S}) = \mathscr{H}$. The field $\mathscr{K}(x)$ can be factorized as follows:

$$\mathscr{K}'(x,\theta) = S_L(\theta)\mathscr{K}(x)(S_L)^{-1}(\theta). \tag{5.80}$$

Proof: The background matrix $E = g + B$ can be transformed as follows:

$$E(x,\nu) = L(\theta).E(x). \tag{5.81}$$

We can rewrite (5.76) as in (4.58) as follows:

$$E(x,\theta) = (l^T(\theta)E(x))(l^{-1}(\theta))^{-1} = l^T(\theta)E(x)l(\theta). \tag{5.82}$$

By using (5.22), (5.81) can equivalently be written in terms of separated (x,θ) coordinates as in (5.78).

Now, in order to prove (5.79) we write (5.75), as follows, [20]:

$$F(x,\theta) = S_L(\theta)F(x), \tag{5.83}$$

where $F(x,\theta)$ is the spinor field that encodes the components of the field strengths written with respect to the coordinate basis $(dx^1,\cdots,dx^{(10-d)},d\theta^1,\cdots,d\theta^d)$.

We first prove the following identiy:

$$S_L^{-1}(\theta)e^{B(x,\theta)}S_L(\theta) = e^{B(x)}. \tag{5.84}$$

Using (5.82) we write $B(x,\theta) = l^T B(x) l$. Then,

$$h_{B(x,\theta)} = L h_{B(x)} L^{-1} \tag{5.85}$$

holds where

$$h_B = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}. \tag{5.86}$$

We rewrite it as

$$L^{-1} h_{B(x,\theta)} L = h_{B(x)}. \tag{5.87}$$

Then, using the homomorphism ρ given in (4.85) and $\rho(e^B) = h_B$ and $\rho(S_L) = L$ we have

$$\rho(S_L^{-1})\rho(e^{B(x,\theta)})\rho(S_L) = \rho(e^{B(x)}). \tag{5.88}$$

We rewrite it as

$$\rho(S_L^{-1} e^{B(x,\theta)} S_L) = \rho(e^{B(x)}). \tag{5.89}$$

Then, it gives (5.84). Now, we use (5.84) to show that (5.83) is equivalent to (5.95). We write the right side of (5.95) as follows:

$$\begin{aligned} e^{-B(x,\theta)}S_L(\theta)e^{B(x)}F(x) &= e^{-B(x,\theta)}S_L(\theta)(S_L^{-1}e^{B(x,\theta)}S_L)F(x) \\ &= S_L(\theta)F(x). \end{aligned} \tag{5.90}$$

Now, we will prove the (5.80). We have $\mathscr{K} \equiv C^{-1}\mathbb{S}$. Therefore, we multiply (5.96) by C^{-1} from left and using (4.100) for $(S_L^{-1})^\dagger$ as follows:

$$C^{-1}\mathbb{S}(x,\theta) = C^{-1}(C\, S_L(\theta)C^{-1}\mathbb{S}(x)(S_L)^{-1}(\theta)). \tag{5.91}$$

We end up with

$$\mathscr{K}(x,\theta) = S_L(\theta)\mathscr{K}(x)S_L^{-1}(\theta). \tag{5.92}$$

This proves (5.80). □

We replace L with T_{NATD} as given in (5.62) as (4.10) with the rule (4.58) and S_L with S_{NATD}. In this case, all the fields will depend on x and ν due to the (5.62). We collect the transformation rules for untwisted fields as follows:

Definition 5.3 [20] The NAT dual of the untwisted fields $\mathscr{H}', d', \chi'$ are found by acting on the untwisted fields $\mathscr{H}(x), d(x), \chi(x)$ by the $O(d,d)$ matrix (5.62) as below:

$$\begin{aligned} \mathscr{H}'(x,\nu) &= T_{\text{NATD}}(\nu)\mathscr{H}(x)(T_{\text{NATD}})^t(\nu) && (5.93)\\ \mathscr{K}'(x,\nu) &= S_{\text{NATD}}(\nu)\mathscr{K}(x)(S_{\text{NATD}})^{-1}(\nu) && (5.94)\\ F'(x,\nu) &= e^{-B'(x,\nu)}S_{\text{NATD}}(\nu)e^{B(x)}F(x). && (5.95) \end{aligned}$$

Definition 5.4 The field $\mathbb{S}$ can also be written in terms of the separated coordinates (x,θ) as follows, [20]:

$$\mathbb{S}(x,\theta) = (S_{\text{NATD}}^{-1})^{\dagger}(\theta)\mathbb{S}(x)(S_{\text{NATD}})^{-1}(\theta), \tag{5.96}$$

where $\rho(S_{\text{NATD}}) = T_{\text{NATD}}$, and ρ is the double covering homomorphism given in (4.85).

We also assume that the pure spinors associated with the background respect the isometry, it leads to the following.

Lemma 5.13 [17] The transformation rule for both pure spinors is given as follows.

$$\Phi(X,\theta) = \sqrt{\det l}\; S_{\text{NATD}}(\theta)\;.\;\Phi(X) = \sqrt{\det l}\; e^{-B'(X,\theta)}S_{\text{NATD}}(\theta)e^{B(X)}\Phi. \tag{5.97}$$

The fact that the NAT dual fields can be written in terms of DFT fields as in (5.93-5.95) makes it possible to prove that NATD is a solution generating transformation for the field equations of Type II supergravity. This was shown in [20].

5.2.2 Invariance of pure spinor equations under NATD

[17] The transformation under NATD of the fields in the NS-NS sector can be performed via the the action of the matrix T_{NATD} given in (5.62). Accordingly, the transformation of the RR fields can be performed via the projected element S_{NATD} under the double covering homomorphism between $Pin(d,d)$ and $O(d,d)$, see equations (5.93)-(5.95). An important point that should be stressed here is that S_{NATD} and T_{NATD} act on the so-called *untwisted fields* $g(X), B(X), \phi(X), \Phi(X)$ and $F(X)$. These untwisted fields depend only on $10 - dimG$ coordinates, where $dimG$ is the dimension of the non-Abelian isometry group G and is related to the background fields $g(X,\theta), B(X,\theta), \phi(X,\theta), \Phi(X,\theta)$ and $F(X,\theta)$ exactly as in (5.93)-(5.95), where we replace the NATD coordinates ν with the space-time coordinates θ associated with the isometry directions. We replace T_{NATD} with L given in (5.76) and S_{NATD} with S_L where $S_L \in Spin^+(d,d)$ is such that $\rho(S_L) = L$.

Now suppose that the background we start with preserves at least $\mathcal{N} = 1$ supersymmetry so that the pure spinor equations (5.1) and (5.2) are satisfied.

Lemma 5.14 [17] The pure spinor equations (5.1) and (5.2) will remain invariant under the NATD.

Proof: [17] The untwisted fields which have no dependence on the isometry directions satisfy the deformed pure spinor equations (5.60) and (5.61) where the deformation is determined by the flux associated with the matrices L and S_L. According to [20] this is just geometric flux with $f_{ij}^{\ k} = C_{ij}^{\ k}$. We act on these untwisted fields with the NATD matrices (5.62) and (5.94) with the transformation rules given in (5.93-5.95) to generate the NAT dual background. The resulting fields satisfy the field equations of Type II supergravity as was shown in [20] by embedding these equations in DFT. We need to check supersymmetry of the dual background. Therefore, we also transform the untwisted pure spinors of the initial background that is, the pure spinors $\Phi(X)$ in (5.97) rather than $\Phi(X,\theta)$ as in (5.8) with $P = S_{\text{NATD}}$. Now we have to check whether these new pure spinors $\Phi(X,\nu)$ still satisfy the supersymmetry equations (5.1) and (5.2). As discussed in section 5.1.2 in Theorem 5.2, this is equivalent to checking whether the untwisted pure spinors satisfy the deformed supersymmetry equations (5.60) and (5.61), where the deformation is determined by the flux associated with the NATD matrix S_{NATD}. Due to the special form of the NATD matrix: $S_{\text{NATD}} = C_n S_\theta$, the associated flux yields geometric flux with $f_{ij}^{\;\;k} = C_{ij}^{\;\;k}$ as was shown in [20], and we already know that the untwisted pure spinors satisfy these deformed equations due to the existence of isometry respected by the initial background and the pure spinors associated with it. This completes the proof that a background that preserves $\mathcal{N} = 1$ supersymmetry will still be supersymmetric after NATD.

6. APPLICATIONS

In this section, we will demonstrate how the NATD transformation formulas work by looking at a specific class of Type IIB backgrounds, which were first studied in [16]. The topology of the background we will study is $M_{1,3} \times \mathscr{M}_3 \times S^3$ so that there is an $SU(2)$ isometry associated with S^3, which can be utilized to perform NATD. There exists also an $SU(3)$ structure in this ansatz.

We have already discussed in section 4.3.2.3 that the pure spinors associated to the $SU(3)$ structure can be treated as non-homogeneous differential forms. We will perform the Non-Abelian T-duality (NATD) transformation which is a solution generating transformation as was shown in [20]. We aim to transform pure spinors under NATD with the formulas given in section 5.2.1 and show that they still obey the supersymmetry conditions. We will transform $SU(3)$ pure spinors. The resultant pure spinors become a constant multiple of $SU(2)$ pure spinors.

The ansatz for the metric and the 5-form flux is

$$\begin{aligned} ds^2 &= e^{2A}dx_{1,3}^2 + ds^2(\mathscr{M}_3) + \sum_{i=1}^{3}\left(e^i\right)^2, \\ \mathscr{F}_5 &= \mathscr{F}_2 \wedge e^1 \wedge e^2 \wedge e^3 \\ F_5 &= (1+*)\mathscr{F}_5 = \mathscr{F}_2 \wedge e^1 \wedge e^2 \wedge e^3 - e^{4A} *_3 \mathscr{F}_2 \wedge Vol_4 \end{aligned} \tag{6.1}$$

and $F_1 = F_3 = B = \phi = 0$. $\mathscr{F}_2$ is a 2-form, $*_3$ is the Hodge star operator on $\mathscr{M}_3$, and A is the warp factor. It is a function which has dependence only on the coordinates of $\mathscr{M}_3$. S^3 is assumed to be fibered over $\mathscr{M}_3$ and hence the vielbeins e^i on S^3 have the form

$$e^i = \lambda_i(\sigma_i + \mathscr{A}_i). \tag{6.2}$$

Here, $\mathscr{A}_i$ are 1-forms on $\mathscr{M}_3$ and λ_i are functions on $\mathscr{M}_3$. The forms σ^i are left invariant 1-forms for the isometry group $SU(2)$ so that $d\sigma^i = \frac{1}{2}\varepsilon^i_{jk}\sigma^j \wedge \sigma^k$. We denote the left invariant vector fields L_i, so $i_{L_i}\sigma^j = \delta_i^j$. [16] Let us define a set of undetermined frame fields h^i so that

$$ds^2(\mathscr{M}_3) = \sum_{i=1}^{3} (h^i)^2. \tag{6.3}$$

Another assumption that is made in [16] is that this geometry preserves at least $\mathscr{N} = 1$ supersymmetry in four dimensions in the form of an $SU(3)$ structure characterized by the following 2-form J and 3-form Ω which are given by means of a vielbein e^i and frame fields h^i:

$$J = h^3 \wedge e^3 + e^1 \wedge e^2 + h^1 \wedge h^2 \,, \quad \Omega = (h^3 + ie^3) \wedge (e^1 + ie^2) \wedge (h^1 + ih^2)\,. \tag{6.4}$$

Obviously, they satisfy $J \wedge \Omega = 0$ and $\frac{i}{8}\Omega \wedge \bar{\Omega} = \frac{1}{3!} J \wedge J \wedge J = \mathrm{vol}_6$.

As we discussed in section 2.3.6, $SU(3)$ structure can be regarded as a special case of $SU(3) \times SU(3)$ structure with associated pure spinors . In our case, setting $\Phi_1 = \Phi_-$ and $\Phi_2 = \Phi_+$ we have

$$\Phi_+ = \frac{1}{8} e^{i\theta_+} e^A e^{-iJ} \,, \quad \Phi_- = -\frac{i}{8} e^{i\theta_-} e^A \Omega \tag{6.5}$$

Due to assumption of preservation of $\mathscr{N} = 1$ supersymmetry, these pure spinors must satisfy the pure spinor equations. As shown in [16], this forces $\theta_+ = \frac{\pi}{2}$ and $\mathscr{A}_1 = \mathscr{A}_2 = 0$. The possible values for θ_- for different geometries is given. One can see that it is of the general form of a general $SU(3)$ pure spinor given in (4.181) with $a = e^{i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ and $b = e^{i\theta_-/2} e^{-i\theta_+/2} e^{A/2}$, which satisfy $|a|^2 = |b|^2 = e^A$.

Example 6.1 The ansatz (6.1) is general enough to cover many examples important for AdS/CFT duality, notably $AdS_5 \times T^{1,1}$, $AdS_5 \times Y^{p,q}$ and $AdS_5 \times S^5$. $T^{1,1}$ is a well-known example of a 5-dimensional Sasaki-Einstein manifold $T^{1,1} = S^2 \times S^3$ which is called conifold, [51]. $Y^{p,q}$ is also topologically $S^2 \times S^3$ for p, q are relatively prime, [52]. A Sasaki-Einstein manifold can be examined in the framework of Generalised Complex Geometry. It is a Riemannian manifold which is Sasakian and Einstein. A Riemannian manifold (M, g) is called *Sasakian* if and only if its metric cone $(C(M), g)$ is Kähler. It can be seen as the odd dimensional version of Kähler geometry, which is a combination of complex, symplectic, and Riemannian geometry. Sasaki-Einstein manifolds are sandwiched between two Kählerian geometries, [22]. The detailed description of how these backgrounds fall within this general ansatz can be found in Appendix B of [16]. For example, for $T^{1,1}$ background the required values are as follows:

$$A = \log r, \quad \mathscr{A}_3 = \cos\theta d\varphi, \quad \theta_- = 0,$$

$$\lambda_1 = \lambda_2 = \frac{1}{\sqrt{6}},\ \lambda_3 = \frac{1}{3}, \quad h^1 = \frac{1}{\sqrt{6}}\sin\theta d\varphi,\ h^2 = \frac{1}{\sqrt{6}}d\theta,\ h^3 = \frac{dr}{r}\cdot$$

On the other hand, the required values for the $AdS_5 \times S^5$ background are:

$$A = \log 2R, \quad \mathscr{A}_3 = 0, \quad \theta_- = \beta,\ \lambda_1 = \lambda_2 = \lambda_3 = \cos\alpha,$$

$$h^1 = 2\frac{R\cos\alpha d\alpha + \sin\alpha dR}{R},\ h^2 = 2\sin\alpha d\beta,\ h^3 = 2\frac{\cos\alpha dR - R\sin\alpha d\alpha}{R}\cdot$$

Now, we perform the NATD transformation of the background described by the ansatz (6.1). We begin with the transformation of the metric and the B-field. For this we use the $O(6,6)$ matrix T_{NATD} given in (5.62).

Then we read off the transformed metric and the transformed B-field from the symmetric and antisymmetric parts of E', respectively as given in (4.59). This gives

$$\begin{aligned}
ds'^2 &= e^{2A}dx_{1,3}^2 + ds^2(\mathscr{M}_3) + \frac{1}{\Delta}\Bigg((v_i v_j + \frac{\lambda_1^2\lambda_2^2\lambda_3^2}{\lambda_{(i)}^2}\delta_{(i)j})\, dv_i\, dv_j - 2\lambda_3^2\lambda_2^2 v_2\, dv_1\, \mathscr{A}_3 \\
&\quad +2\lambda_3^2\lambda_1^2 v_1\, dv_2\, \mathscr{A}_3 + (\lambda_3^2\Delta - 4\lambda_3^4(\lambda_1^2\lambda_2^2 + v_3^2))\, \mathscr{A}_3\, \mathscr{A}_3 \Bigg) \\
B' &= -\frac{1}{\Delta}\Bigg(\frac{1}{2}\varepsilon_{ijk} v_i \lambda_i^2\, dv_j \wedge dv_k + \lambda_3^2 v_3 v_1\, dv_1 \wedge \mathscr{A}_3 \\
&\quad +\lambda_3^2 v_3 v_2\, dv_2 \wedge \mathscr{A}_3 + (\lambda_3^2 v_3^2 + \lambda_1^2\lambda_2^2\lambda_3^2)\, dv_3 \wedge \mathscr{A}_3\Bigg), \\
\Delta &= G^{-1} = \lambda_1^2\lambda_2^2\lambda_3^2 + \lambda_1^2 v_1^2 + \lambda_2^2 v_2^2 + \lambda_3^2 v_3^2 \\
e^{-2\phi'} &= \Delta \qquad (6.6)
\end{aligned}$$

We obtain the same results obtained in [16] except for a sign difference in the B-field.

Next, we perform the NATD transformation of the RR flux F_5 from the transformation rule (5.9), with S_{NATD}. It is convenient to write the spinor field F that packages the RR fluxes as a non-homogeneous differential form as in (5.75).

This non-homogeneous differential form maps to a Clifford algebra element in the usual way as given in Lemma 4.5 where we identify the left invariant 1-forms σ^i given in (5.72) with the Clifford algebra elements ψ^i, for $i = 1,2,3$ as we have did in (4.72).

It has the following form:

$$F = \sum_p \left(F^{(p)} + F_i^{(p-1)} \psi^i + \frac{1}{2} F_{ij}^{(p-2)} \psi^i \psi^j + F^{(p-3)} \psi^1 \psi^2 \psi^3 \right) \tag{6.7}$$

Then, the spinorial action of ψ^i on F is given by wedge product, whereas the spinorial action of ψ_i is given by contraction as we have studied in section 4.3.2.4.

Since there is no B-field, we will first calculate the action of CS_θ on differential forms then apply $e^{-B'}$. The action of S_θ on a non-homogeneous differential form Φ is calculated as in (5.66) given by wedge product.

On the other hand, the action of C given in Definition 4.25 can be calculated with the rules given in (4.92) where C is given as follows:

$$C = (\psi^1 - \psi_1)(\psi^2 - \psi_2)(\psi^3 - \psi_3). \tag{6.8}$$

We calculate the following NATD transformed RR flux as follows:

$$\begin{aligned} F_5' &= e^{-B'} CS_\theta \, F_5 \\ &= \lambda_1\lambda_2\lambda_3 \, \mathscr{F}_2 - \lambda_1\lambda_2\lambda_3 \, B' \wedge \mathscr{F}_2 - \lambda_1\lambda_2\lambda_3 \, dv^3 \wedge \mathscr{A}_3 \wedge \mathscr{F}_2 + e^{4A} Vol_4 \wedge v^i dv^i *_3 \mathscr{F}_2 \\ &\quad - e^{4A} Vol_4 \wedge dv^1 \wedge dv^2 \wedge dv^3 \; *_3 \mathscr{F}_2 - B' \wedge e^{4A} Vol_4 \wedge dv^i \; *_3 \mathscr{F}_2 \end{aligned} \tag{6.9}$$

where the Hodge duality $*_3$ is taken with respect to the transformed metric. After identifying ψ^i with dv_i this polyform packages all the RR fluxes of the NAT dual background which we read off to be, [17]:

$$F_2' = \lambda_1\lambda_2\lambda_3 \mathscr{F}_2, \tag{6.10}$$

$$F_4' = (-B' + \mathscr{A}_3 \wedge dv_3) \wedge F_2', \tag{6.11}$$

$$F_6' = *_{10} F_4' = e^{4A} Vol_4 \wedge v_i \, dv_i \wedge *_3 \mathscr{F}_2, \tag{6.12}$$

$$F_8' = - *_{10} F_2' = -B' \wedge F_6' + e^{4A} Vol_4 \wedge *_3 \mathscr{F}_2 \wedge dv_1 \wedge dv_2 \wedge dv_3. \tag{6.13}$$

Remark 6.1 These agree with the results obtained in [16] up to sign differences in B', and the 6- and 8-forms due to differences in conventions.

Finally, we will apply the NATD transformation rule with $P = S_{\text{NATD}}$ to the $SU(3)$ pure spinors given in (6.5) and obtain the NAT-dual pure spinors Φ'_+ and Φ'_-. In obtaining the results there, we first calculate Φ'_-. We begin with $S_\theta\Phi_-$ by using (5.66):

$$\begin{aligned}
S_\theta\Phi_- &= \Phi_- + v_1\, dv^2\wedge dv^3\wedge\Phi_- + v_2\, dv^3\wedge dv^2\wedge\Phi_- + v_3\, dv^1\wedge dv^2\wedge\Phi_- \\
&= \Phi_- - \frac{i}{8}e^{i\theta_-}e^A\{\lambda_1 v_1\, dv^2\wedge dv^3\wedge h^3\wedge dv^1\wedge h^1 + i\,\lambda_1 v_1\, dv^2\wedge dv^3\wedge h^3\wedge dv^1\wedge h^2 \\
&\quad -\lambda_1\lambda_3 v_1\, dv^2\wedge dv^3\wedge\mathscr{A}_3\wedge dv^1\wedge h^2 + i\,\lambda_2 v_2\, dv^3\wedge dv^1\wedge h^3\wedge dv^2\wedge h^1 \\
&\quad - \lambda_2 v_2\, dv^3\wedge dv^1\wedge h^3\wedge dv^2\wedge h^2 - i\,\lambda_2\lambda_3 v_2\, dv^3\wedge dv^1\wedge\mathscr{A}_3\wedge dv^2\wedge h^2\}.
\end{aligned}$$

Applying $\sqrt{G}\ e^{-B'}C$ to $S_\theta\Phi_-$ where C is as given in (6.8), we obtain Φ'_-, whose explicit form is given as follows:

$$\begin{aligned}
\Phi'_- &= \frac{-i}{8\sqrt{\Delta}}e^{i\theta_-}e^A\Big(-\lambda_2\lambda_3\, dv_1\wedge h^1 - i\,\lambda_1\lambda_3\, dv_2\wedge h^1 \\
&\quad +(v_1\lambda_1 + i\, v_2\lambda_2)\, h^1\wedge h^3 + i\,\lambda_2\lambda_3\, h^2\wedge dv_1 \\
&\quad -\lambda_1\lambda_3\, h^2\wedge dv_2 - (i\, v_1\lambda_1 - v_2\lambda_2)\, h^2\wedge h^3 - (i\,\lambda_2\lambda_3 v_2 + \lambda_1\lambda_3 v_1)\, h^2\wedge\mathscr{A}_3 \\
&\quad +\frac{1}{\Delta}(\lambda_1\lambda_3^2 v_1 v_3 + i\,\lambda_2\lambda_3^2 v_2 v_3)\, h^3\wedge dv_1\wedge h^1\wedge dv_2 \\
&\quad +\frac{1}{\Delta}(i\,\lambda_1\lambda_3^2 v_1 v_3 - \lambda_2\lambda_3^2 v_2 v_3)\, h^3\wedge dv_1\wedge h^2\wedge dv_2 \\
&\quad +\frac{1}{\Delta}(i\,\lambda_2^3 v_2^2 + \lambda_1\lambda_2^2 v_2 v_1 - i\,\lambda_2\Delta)\, h^3\wedge dv_3\wedge h^1\wedge dv_1 \\
&\quad +\frac{1}{\Delta}(\lambda_1\Delta - i\,\lambda_1\lambda_2 v_2 v_1 - \lambda_1^3 v_1^2)\, h^3\wedge dv_3\wedge h^1\wedge dv_2 \\
&\quad +\frac{1}{\Delta}(\lambda_2\Delta + i\,\lambda_1\lambda_2^2 v_2 v_1 - \lambda_2^3 v_2^2)\, h^3\wedge dv_3\wedge h^2\wedge dv_1 \\
&\quad +\frac{1}{\Delta}(i\,\lambda_1\Delta + \lambda_2\lambda_1^2 v_2 v_1 - i\,\lambda_1^3 v_1^2)\, h^3\wedge dv_3\wedge h^2\wedge dv_2 \\
&\quad -\frac{1}{\Delta}\lambda_1\lambda_2\lambda_3(\lambda_1 v_1 + i\,\lambda_2 v_2)\, dv_1\wedge dv_2\wedge dv_3\wedge h^1 \\
&\quad +\frac{1}{\Delta}\lambda_1\lambda_2\lambda_3(\lambda_2 v_2 - i\,\lambda_1 v_1)\, dv_1\wedge dv_2\wedge dv_3\wedge h^2 \\
&\quad +\frac{1}{\Delta}(i\,\lambda_1\lambda_3^2 v_1 v_2 v_3 - \lambda_2\lambda_3^2 v_2^2 v_3)\, h^3\wedge dv_2\wedge h^2\wedge\mathscr{A}_3 \\
&\quad +\frac{1}{\Delta}(i\,\lambda_1\lambda_3^2 v_1^2 v_3 - \lambda_2\lambda_3^2 v_1 v_2 v_3)\, h^3\wedge dv_1\wedge h^2\wedge\mathscr{A}_3 \\
&\quad -\frac{1}{\Delta}(\lambda_1^2\lambda_2^3\lambda_3^2 v_2 + \lambda_2\lambda_3^2 v_2 v_3^2 - i\,\lambda_2^2\lambda_1^3\lambda_3^2 v_1 - i\,\lambda_1\lambda_3^2 v_1 v_3^2)\, h^3\wedge dv_3\wedge h^2\wedge\mathscr{A}_3 \\
&\quad +\frac{1}{\Delta}(\lambda_1\lambda_2^2\lambda_3 v_1 v_2 - i\,\lambda_1^2\lambda_2\lambda_3 v_1^2)\, dv_1\wedge dv_3\wedge h^2\wedge\mathscr{A}_3 \\
&\quad +\frac{i}{\Delta}(\lambda_1\lambda_2^2\lambda_3 v_2^2 - \lambda_1^2\lambda_2\lambda_3 v_1 v_2)\, dv_2\wedge dv_3\wedge h^2\wedge\mathscr{A}_3\Big).
\end{aligned} \tag{6.14}$$

Note that after the transformation ψ^i are identified with dv_i.

Next we calculate Φ'_+. We begin with $S_\theta \Phi_+$ by using (5.66):

$$\begin{aligned}
S_\theta \Phi_+ &= \Phi_+ + v_1\, dv^2 \wedge dv^3 \wedge \Phi_+ + v_2\, dv^3 \wedge dv^2 \wedge \Phi_+ + v_3\, dv^1 \wedge dv^2 \wedge \Phi_+ \\
&= \Phi_+ + \frac{1}{8} e^{i\theta_+} e^A \{ v_1\, dv^2 \wedge dv^3 - i\,\lambda_3 v_1\, dv^2 \wedge dv^3 \wedge h^3 \wedge \mathscr{A}_3 - i\, v_1\, dv^2 \wedge dv^3 \wedge h^1 \wedge h^2 \\
&\quad + v_2\, dv^3 \wedge dv^1 - i\,\lambda_3 v_2\, dv^3 \wedge dv^1 \wedge h^3 \wedge \mathscr{A}_3 - i\, v_2\, dv^3 \wedge dv^1 \wedge h^1 \wedge h^2 \\
&\quad + v_3\, dv^1 \wedge dv^2 - i\,\lambda_3 v_3\, dv^1 \wedge dv^2 \wedge h^3 \wedge \mathscr{A}_3 - i\,\lambda_3 v_3\, dv^1 \wedge dv^2 \wedge h^3 \wedge dv^3 \\
&\quad - i\, v_3\, dv^1 \wedge dv^2 \wedge h^1 \wedge h^2 + \lambda_3 v_3\, dv^1 \wedge dv^2 \wedge dv^3 \wedge h^1 \wedge h^2 \wedge h^3 \}.
\end{aligned}$$

Applying $\sqrt{G}\, e^{-B'} C$ to $S_\theta \Phi_+$ we obtain Φ'_+, whose explicit form is given as follows:

$$\begin{aligned}
\Phi'_+ &= -\frac{1}{8\sqrt{\Delta}} e^{i\theta_+} e^A \Big(v_1\, dv_1 + v_2\, dv_2 + (v_3 - i\lambda_1\lambda_2)\, dv_3 - (\lambda_1\lambda_2\lambda_3 + i\lambda_3 v_3)\, h^3 \\
&\quad - (\lambda_1^2\lambda_2^2\lambda_3^2 + i\,\lambda_1\lambda_2\lambda_3^2 v_3)\, dv_1 \wedge dv_2 \wedge dv_3 + (i\,\lambda_1\lambda_2\lambda_3 - \lambda_3 v_3)\, h^1 \wedge h^2 \wedge h^3 \\
&\quad - i\, v_1\, h^1 \wedge h^2 \wedge dv_1 - i\, v_2\, h^1 \wedge h^2 \wedge dv_2 - (\lambda_1\lambda_2 + i\, v_3)\, h^1 \wedge h^2 \wedge dv_3 \\
&\quad - \frac{1}{\Delta}(i\,\lambda_3^3 v_3^2 + \lambda_1\lambda_2\lambda_3^3 v_3 - i\,\lambda_3\Delta)\, dv_1 \wedge dv_2 \wedge h^3 \\
&\quad - \frac{1}{\Delta}(i\,\lambda_2^2\lambda_3 v_2 v_3 + \lambda_1\lambda_2^3\lambda_3 v_2)\, dv_1 \wedge h^3 \wedge dv_3 \\
&\quad - \frac{1}{\Delta}(i\,\lambda_1^2\lambda_3 v_1 v_3 + \lambda_1^3\lambda_2\lambda_3 v_1)\, h^3 \wedge dv_2 \wedge dv_3 \\
&\quad + \frac{1}{\Delta}(i\,\lambda_1\lambda_2\lambda_3^2 v_1 v_3 + \lambda_1^2\lambda_2^2\lambda_3^2 v_1)\, dv_1 \wedge dv_3 \wedge \mathscr{A}_3 \\
&\quad + \frac{1}{\Delta}(i\,\lambda_1\lambda_2\lambda_3^2 v_2 v_3 + \lambda_1^2\lambda_2^2\lambda_3^2 v_2)\, dv_2 \wedge dv_3 \wedge \mathscr{A}_3 \\
&\quad + \frac{1}{\Delta}(\lambda_1\lambda_2\lambda_3^2 v_1 v_3 - i\,\lambda_1^2\lambda_2^2\lambda_3^3 v_1 - i\,\lambda_2^2\lambda_3 v_1 v_2^2 - i\,\lambda_1^2\lambda_3 v_1^3)\, dv_1 \wedge h^3 \wedge \mathscr{A}_3 \\
&\quad + \frac{1}{\Delta}(i\,\lambda_1^2\lambda_2^2\lambda_3^3 v_2 - \lambda_1\lambda_2\lambda_3^3 v_2 v_3 - i\,\lambda_2^2\lambda_3 v_2^3 - \lambda_1^2\lambda_3 v_2 v_1^2)\, dv_2 \wedge h^3 \wedge \mathscr{A}_3 \\
&\quad - \frac{1}{\Delta}(\lambda_1\lambda_2^3\lambda_3 v_2^2 + \lambda_3\lambda_2\lambda_1^3 v_1^2 + i\,\lambda_2^2\lambda_3 v_2^2 v_3 + i\,\lambda_1^2\lambda_3 v_3 v_1^2)\, dv_3 \wedge h^3 \wedge \mathscr{A}_3 \\
&\quad + \frac{1}{\Delta}(i\,\lambda_1\lambda_2\lambda_3^3 v_3 - \lambda_3^3 v_3^2 + \lambda_3\Delta)\, h^1 \wedge h^2 \wedge h^3 \wedge dv_1 \wedge dv_2 \\
&\quad + \frac{1}{\Delta}(\lambda_2^2\lambda_3 v_2 v_3 - i\,\lambda_1\lambda_2^3 v_2 v_3)\, h^1 \wedge h^2 \wedge h^3 \wedge dv_1 \wedge dv_3 \\
&\quad + \frac{1}{\Delta}(i\,\lambda_1^3\lambda_2\lambda_3 v_1 - \lambda_1^2\lambda_3 v_1 v_3)\, h^1 \wedge h^2 \wedge h^3 \wedge dv_2 \wedge dv_3 \\
&\quad + \frac{1}{\Delta}(i\,\lambda_1^2\lambda_2^2\lambda_3^2 - \lambda_1\lambda_2\lambda_3^2 v_3)\, h^1 \wedge h^2 \wedge dv_1 \wedge dv_2 \wedge dv_3 \Big). \qquad (6.15)
\end{aligned}$$

Lemma 6.1 [16,17] The transformed pure spinors Φ'_- and Φ'_+ can be written in the following form:

$$\Phi'_- = -\frac{i}{8} e^{A} e^{i\theta_-} e^{\frac{1}{2} z\wedge\bar{z}} \wedge \omega \qquad (6.16)$$

$$\Phi'_+ = -\frac{1}{8} e^{i\theta_+} e^{A} e^{-ij} \wedge z, \qquad (6.17)$$

where the complex 1-form $z = v + iw$, and the real and complex 2-forms j and ω are as given below

$$\begin{aligned}
z &= -\frac{1}{\sqrt{\Delta}}\Big((\lambda_1\lambda_2\lambda_3 + i\,\lambda_3 v_3)\, h^3 - v_1\, dv_1 - v_2\, dv_2 - (v_3 - i\,\lambda_1\lambda_2)\, dv_3 \Big) \\
j &= \frac{1}{\Delta}\Big(\Delta\, h^1\wedge h^2 + \lambda_1\lambda_2\lambda_3^2\, dv_1\wedge dv_2 + \lambda_1\lambda_2\lambda_3^2 v_1\, dv_1\wedge \mathscr{A}_3 \\
&\quad + \lambda_2^2\lambda_3 v_2\, dv_1\wedge h^3 - \lambda_1\lambda_2\lambda_3^2 v_2\, \mathscr{A}_3\wedge dv_2 \\
&\quad + \lambda_1^2\lambda_3 v_1\, h^3\wedge dv_2 - (\lambda_2^2\lambda_3 v_2^2 + \lambda_1^2\lambda_3 v_1^2)\, \mathscr{A}_3\wedge h^3 \Big) \\
\omega &= \frac{1}{\sqrt{\Delta}}\Big(\lambda_2\lambda_3\, h^1\wedge dv_1 + i\,\lambda_1\lambda_3\, h^1\wedge dv_2 + (v_1\lambda_1 + i\, v_2\lambda_2) h^1\wedge h^3 \\
&\quad + (i\, v_1\lambda_1 - v_2\lambda_2) h^2\wedge h^3 + i\,\lambda_2\lambda_3\, h^2\wedge dv_1 \\
&\quad - \lambda_1\lambda_3\, h^2\wedge dv_2 - (i\,\lambda_2\lambda_3 v_2 + \lambda_1\lambda_3 v_1)\, h^2\wedge \mathscr{A}_3 \Big).
\end{aligned}$$

Comparing (6.16),(6.17) with (4.183) one can see that they define an $SU(2)$ structure, as can be seen by taking $a = e^{i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ and $b = e^{-i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ in (4.183). Note that $|a|^2 = |b|^2 = e^{A}$, as needed. So, under NATD, a background with $SU(3)$ structure is transformed to a background with $SU(2)$ structure, as has been demonstrated many times in the literature previously, in particular in [16,51].

The results we present in (6.16-6.18) are in agreement with those obtained in [16] Whether these transformed pure spinors satisfy the supersymmetry equations was checked in [16] by direct computation. The results we obtained before make such a calculation redundant. Indeed, the pure spinors are obtained through the action of S_{NATD} and we have proved that this transformation maps solutions of the pure spinor equations to new solutions as we have discussed in section 5.2.2.

7. CONCLUSIONS

In this thesis, we studied manifolds of generalised G-structure relevant for compactification of type II superstring theory. A generalised G-structure is defined as a reduction of the structure group of the generalised frame bundle associated to the generalised tangent bundle $TM \oplus T^*M$. Preservation of $\mathcal{N} = 1$ supersymmetry requires that the structure group of the generalised tangent bundle $TM \oplus T^*M$ of the six dimensional internal manifold M is reduced from $SO(6,6)$ to $SU(3) \times SU(3)$. This is equivalent to the existence of two globally defined compatible pure spinors Φ_1 and Φ_2 of non-vanishing norm. In addition, these pure spinors should satisfy the pure spinor equations (5.1), (5.2). We have proved that these equations associated with preserved $\mathcal{N} = 1$ supersymmetry are covariant under $Pin(d,d)$ transformations.

Non-Abelian T-duality (NATD) is an extension of Abelian T-duality. It generates solutions to string backgrounds with non-Abelian isometries. One can obtain NATD for a given d dimensional Type II background with the isometry group G through the action of a coordinate-dependent $O(d,d)$ matrix T_{NATD}. Coordinate dependence is determined by the structure constants $C_{ij}^{\;k}$ of the Lie algebra of the isometry group G. It gives rise to geometric flux $f_{ij}^{\;k} = C_{ij}^{\;k}$. The metric, the B-field, and the dilaton field transform under T_{NATD}. The RR fields transform under S_{NATD} which is the corresponding $Pin(d,d)$ transformation obtained by the double covering homomorphism from $Pin(d,d)$ to $O(d,d)$. We examine the NATD transformation of pure spinor equations associated with preserved $\mathcal{N} = 1$ supersymmetry.

Similar studies have been carried out in the literature in analyzing $\mathcal{N} = 1$ supersymmetry under NATD. The advantage of our approach is that we exploit the fact that NATD can be regarded as an $O(d,d)/Pin(d,d)$ transformation. This has been understood only recently; it has been used in [20] to show that NATD is a solution generating transformation for the field equations for the metric, the B-field, and the p-form fields. The first step to proving the covariance of the pure spinor equations associated with preserved $\mathcal{N} = 1$ supersymmetry under $Pin(d,d)$ transformations was embedding these equations in the framework of Double Field Theory, so that the covariance under a general constant $Pin(d,d)$ transformation became manifest. In order to do this, we extended the exterior derivative operator to an $O(d,d)$ covariant differential operator. We also wrote the Hodge duality operator in a $Pin(d,d)$ covariant way. We discussed the case when the $Pin(d,d)$ transformation is coordinate dependent, and it showed that whether the transformed pure spinors satisfy the pure spinor equations or not is completely determined by the fluxes generated by the $Pin(d,d)$ transformation. The idea of preserving flux has been used earlier in the literature to study field equations relating to supergravity under NATD conditions in [20]. We demonstrated in section 5.2.2 that solutions under NATD were mapped to solutions because the geometric flux associated with the isometry group is the same as the flux generated by the NATD matrix.

In the last chapter of the thesis, we focused on a specific type of geometries that are known to be solutions of Type IIB supergravity. This geometry had $SU(2)$ isometry and admitted an $SU(3)$-structure. As we discussed in section 4.3.2.4, considering $SU(3)$ pure spinors associated to the $SU(3)$-structure as a special case of $SU(3) \times SU(3)$ pure spinors gives rise to calculating the $Pin(d,d)$ action on pure spinors. Since NATD is an $O(d,d)/Pin(d,d)$ transformation, we calculated the NATD transformation of these pure spinors along with the transformation of the B-field, metric, and dilaton. In the discussion above, it was pointed out that new pure spinors will automatically solve supersymmetry equations. Whether these transformed pure spinors satisfy the supersymmetry equations was checked in [16] by direct computation. Due to the results we obtained before, this calculation is no longer necessary. We also showed that $SU(3)$ pure spinors are transformed to pure spinors associated with an $SU(2)$ structure. We explicitly obtained the transformed $SU(3)$ pure spinors.

There is also an advantage to viewing NATD as a $Pin(d,d)$ transformation because it makes it easier to apply NATD to other backgrounds that are not covered by the ansatz described in section 6 with different isometry groups and supporting a generic $SU(3) \times SU(3)$-structure. The methods we employed in this thesis are particularly suitable for the analysis of the supersymmetry and structure group of the resulting backgrounds. Furthermore, the method used here would also be useful for analyzing backgrounds supporting a $G_2 \times G_2$-structure. By considering both the $SU(3)$ pure spinors and the G_2 pure spinors as special cases of the $G_2 \times G_2$ pure spinors, it is possible to calculate the NATD transformation of these pure spinors. We intend to address these concerns in future work.

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