ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

HOPF BIFURCATION IN A GENERALIZED GOODWIN MODEL WITH DELAY

M.Sc. THESIS

Eyşan ŞANS

Department of Mathematics Engineering

Mathematics Engineering Program

JUNE 2024

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

HOPF BIFURCATION IN A GENERALIZED GOODWIN MODEL WITH DELAY

M.Sc. THESIS

Eyşan ŞANS
(509201234)

Department of Mathematics Engineering

Mathematics Engineering Program

Thesis Advisor: Assoc. Prof. Dr. Cihangir ÖZEMİR

JUNE 2024

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

GECİKMELİ GENELLEŞTİRİLMİŞ GOODWIN MODELİNDE HOPF ÇATALLANMASI

YÜKSEK LİSANS TEZİ

Eyşan ŞANS
(509201234)

Matematik Mühendisliği Anabilim Dalı

Matematik Mühendisliği Programı

Tez Danışmanı: Doç. Dr. Cihangir ÖZEMİR

HAZİRAN 2024

Eyşan ŞANS, a M.Sc. student of ITU Graduate School student ID 509201234, successfully defended the thesis entitled "HOPF BIFURCATION IN A GENERALIZED GOODWIN MODEL WITH DELAY", which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor : **Assoc. Prof. Dr. Cihangir ÖZEMİR** ....
Istanbul Technical University

Jury Members : **Prof. Dr. Kamil ORUÇOĞLU** ....
Istanbul Technical University

Prof. Dr. Özgür MARTİN ....
Mimar Sinan Fine Arts University

Date of Submission : 24 May 2024
Date of Defense : 26 June 2024

To my family,

FOREWORD

I would like to express my sincere gratitude to my supervisor Assoc. Prof. Cihangir Özemir for his endless support, encouragement and guidance throughout my research. In addition to sharing his knowledge, he gave me moral support during the hardest times while writing this study. His invaluable support and trust have enabled me to complete this work. Without his valuable guidance, encouragement and contributions, I would not have dared to begin this work.

June 2024 Eyşan ŞANS

TABLE OF CONTENTS

Page

SYMBOLS

λ	**:** Wage share
β	**:** Rate of employment
θ	**:** Capacity utilization
v	**:** Capital coefficient
w	**:** Nominal wage rate
σ	**:** Technical capital productivity
γ	**:** Technical labour productivity

LIST OF TABLES

LIST OF FIGURES

Page

HOPF BIFURCATION IN A GENERALIZED GOODWIN MODEL WITH DELAY

SUMMARY

In the theory of dynamical systems, delay differential equations have an important place. While in a non-delayed dynamical system the rate of change of state variables depends instantaneously on the state variables, in delayed dynamical systems this functional dependence can be with a time delay. In real life problems, this may occur, for example, when the signals transmitted to the processor of a physical system that collects and evaluates signals from different points in space are transmitted with a time difference due to the path difference.

Methods and simulation tools are available in the literature for analysing the stability of a dynamic system formulated without delay, either locally at equilibrium points or globally. The "stable" and "unstable" conditions that we encounter in stability analysis can be target conditions according to the physical model under investigation. For example, in a dynamic system that approximately models the vibrations of a structural element vibrating under the effect of an earthquake, it is desired that the vibrations evolve to zero equilibrium point over time and that the zero equilibrium point is stable. In a mechanical system which is desired to generate energy with its vibrations, it will be the target condition that the vibrations are not damped. Stability analysis is performed to determine the parameter conditions that will give the stable and unstable conditions of the equilibrium points. However, if the dynamical system modelling the relevant physical system actually has a delayed time dynamics, the system may actually be unstable in a parameter set that is predicted as a stable equilibrium point by the non-delayed analysis. Therefore, the analysis of the relevant dynamics needs to be carried out in the formulation of the theory of delayed dynamical systems.

Goodwin's model is one of the well-known dynamical systems in macroeconomics which formulates the mechanism between the employment ratio and the wage share in a closed economy. The model is formulated under the assumptions of steady technical progress and steady growth in technical force. Only two factors of production are considered: labour and capital. Working class consume all their wages, whereas all profits are invested by the capital holders. A constant capital-output ratio is assumed, and the relation between the inflation rate and unemployment rate is determined by a linearized Phillips curve.

There is an argument in the literature that the functional dependence of the Phillips curve, which expresses the relationship between the inflation rate and the unemployment rate, depends on the time delay. There are only a few publications that consider this dependence with a delay and dynamically analyse modified versions of the Goodwin model.

The Goodwin model, which is essentially a mathematical economics analogue of the predator-prey system of population dynamics, despite its simplicity, explains to some extent the periodic behaviour of state variables observed at certain time intervals. Assumptions about the structure of the model can be relaxed to take into account more complex situations occurring in an economy. For this reason, the model has been modified with various evaluations in the existing literature. In this work, we consider a generalised, higher dimensional Goodwin model that takes into account capacity utilisation and capital coefficient variables in addition to the employment rate and wage share variables.

The emergence of a topologically non-equivalent phase portrait with the change of parameters in a dynamical system is called bifurcation. There are various types of bifurcations in the literature. Let a dynamical system with at least two state variables have a complex eigenvalue pair of the linearisation of an equilibrium point. If at a particular value of a bifurcation parameter chosen from the parameters of the system (i) the real part of the eigenvalue is zero, (ii) the derivative of the real part of the eigenvalue with respect to the bifurcation parameter is non-zero, (iii) and if the quantity referred to as the first Lyapunov coefficient is nonzero, the dynamical system undergoes a Hopf bifurcation with the variation of this particular value of the bifurcation parameter. Two conditions are encountered in phase diagrams. In one of them, the bifurcation parameter increases to its critical value and the equilibrium point is stable at the critical value. After the critical value, the equilibrium point is unstable and a stable limit cycle emerges. In the other case, as the bifurcation parameter increases to its critical value, the equilibrium point is stable; there is also an unstable limit cycle containing the equilibrium point. At the critical value, the limit cycle disappears and the equilibrium point is stable. At values greater than the critical value, the equilibrium point becomes unstable. As can be seen from this, Hopf bifurcation can occur when the linearisation of an equilibrium point has a pair of purely imaginary eigenvalues with variation of its parameters.

While the eigenvalue equation of linearisation around an equilibrium point in non-delayed dynamical systems is a polynomial equation, the eigenvalue equation of linearisation of delayed dynamical systems is an equation containing an exponential function and a polynomial. When investigating the existence of Hopf bifurcation in delayed dynamical systems, in principle, the validity of the other necessary conditions for this equation to have a purely imaginary root is also examined.

The main objective of this study is to perform stability analysis of a generalised delayed Goodwin system and to investigate the existence of Hopf bifurcation. With these analyses, the stability of an equilibrium point of economic variables is investigated.

This thesis consists of four Chapters.

In the first Chapter, the purpose of the thesis, the related literature research is presented and the research questions related to the analyses to be made are expressed and how the answers to these questions will be investigated are stated.

In the second Chapter, delay dynamical systems are discussed and information about local stability analysis of these systems is given. Also, general information about Hopf bifurcation is given. Then, an example from the literature on how Hopf bifurcation analysis is applied to a delay dynamical system is given. Afterwards, the Lyapunov exponent, which is a measure of how small changes in initial conditions in a dynamical system lead to differences between trajectories, is given in the case of delay-free dynamical systems and the results of its application to a known model are presented. Further, the predator-prey model of population dynamics is mentioned to show its relationship with the original Goodwin model, and then the assumptions of the Goodwin model are noted and its derivation is presented. In the Goodwin model, the employment variable corresponds to the prey and the share of labour to the predator. For this reason, the Goodwin model is also known as class conflict. The Phillips curve, which is one of the functional dependencies of the model, is given a brief overview in a separate sub-section. Finally, a generalised Goodwin model, which is the main model under study and whose delayed version is analysed, is presented for completeness.

The third Chapter presents the original analyses carried out in this thesis. The models considered are special sub-cases of the generalised Goodwin system with four dynamic variables. In the first subsection, in the case of variable speed technical progress and labour intensity, the subsystem of the main system with two unknown functions that do not depend on the other two variables is considered. Here, a time delay in the employment function corresponding to the Phillips curve is assumed. Two of the necessary conditions of Hopf bifurcation, having a purely imaginary eigenvalue and the condition that the derivative of the real part is different from zero, are studied and the critical delay value that can realise Hopf bifurcation is formulated. The parameter conditions that allow the system to be (i) stable at the non-zero equilibrium point for each value of the delay parameter, (ii) stable up to a critical value of the delay parameter and then unstable and undergo Hopf bifurcation are found separately. It is shown that the conditions for Hopf bifurcation are fulfilled for a certain family of parameters and the results are supported by figures obtained from the numerical solution. The second subsection contains a similar analysis. In the third subsection, the Lyapunov exponents for the system with the main four unknown state variables, in a case where the equilibrium point is unstable, are found with a Matlab code and two positive Lyapunov exponents are observed, potentially indicating hyperchaotic behaviour. In the last subsection, the type of Hopf bifurcation is determined by performing directional analysis for Hopf bifurcation. The last Chapter summarizes the results and highlights possible open problems.

With the analysis conducted in this thesis, we attempt to provide an approach to the original Goodwin model and its generalisations in the literature in terms of delayed analysis. When the employment-wage share cycles constructed with real data available in the literature for a particular country's economy are analysed, it is observed that the cycles are valid for a certain time interval, after which the cycle jumps to another equilibrium point of the phase plane. One explanation for this situation may be the change in the values of the parameters in the system of the relevant country that give the position of the equilibrium point. Another explanation for this situation, which can be obtained from the results of this thesis, is that the related equilibrium point becomes unstable at a certain point due to the realisation of the dynamics in the system with a delay and the drawn phase curve moves away from the equilibrium point and tends to realise another cycle in the phase space.

GECİKMELİ GENELLEŞTİRİLMİŞ GOODWIN MODELİNDE HOPF ÇATALLANMASI

ÖZET

Dinamik sistemler teorisinde gecikmeli diferansiyel denklemler önemli bir yer tutar. Gecikmeli olmayan bir dinamik sistemde durum değişkenlerinin değişim oranı durum değişkenlerine anlık olarak bağlı iken, gecikmeli dinamik sistemlerde bu fonksiyonel bağlılık bir zaman gecikmesi ile olabilmektedir. Gerçek yaşam problemlerinde bu durum, örneğin, fiziksel bir sistemin uzaydaki farklı noktalarından sinyal toplayıp değerlendirme yapan işlemcisine iletilen sinyallerin yol farkından dolayı zaman farkı ile iletilmesinden meydana gelebilmektedir.

Gecikmesiz olarak formüle edilmiş bir dinamik sistemin yerel olarak denge noktalarında veya global olarak stabilitesinin analizi için literatürde metotlar ve simülasyon yöntemleri mevcuttur. Stabilite analizinde karşımıza çıkan "kararlı" ve "kararsız" durumları, incelenen fiziksel modele göre hedeflenebilen durumlar olabilir. Örneğin, deprem etkisi altında titreşen bir yapı elemanının titreşimlerini yaklaşık olarak modelleyen bir dinamik sistemde titreşimlerin zamanla sıfır denge noktasına evrilmesi, sıfır denge noktasının kararlı olması istenir. Titreşimleriyle enerji üretmesi istenen bir mekanik sistemde ise titreşimlerin sönümlenmemesi hedef durum olacaktır. Stabilite analizi yapılarak denge noktalarının kararlı ve kararsız durumlarını verecek parametre koşulları belirlenir. Ancak, ilgili fiziksel sistemi modelleyen dinamik sistem gerçekte gecikmeli bir zaman dinamiğine sahip ise, sistemin bir denge noktasını kararlı olarak öngören bir parametre kümesinde sistem gerçekte kararsız olabilir. Bu nedenle, ilgili dinamiğin analizinin gecikmeli dinamik sistemler teorisinin formülasyonunda gerçekleştirilmesi gerekir.

Goodwin modeli, kapalı bir ekonomide istihdam oranı ile ücret payı arasındaki mekanizmayı formüle eden, makroekonomide iyi bilinen dinamik sistemlerden biridir. Model, istikrarlı teknik ilerleme ve teknik güçte istikrarlı büyüme varsayımları altında formüle edilmiştir. Sadece iki üretim faktörü dikkate alınmaktadır: emek ve sermaye. İşçi sınıfı tüm ücretlerini tüketirken, tüm karlar sermaye sahipleri tarafından yatırılmaktadır. Sabit bir sermaye-çıktı oranı varsayılmakta ve enflasyon oranı ile işsizlik oranı arasındaki ilişki doğrusallaştırılmış bir Phillips eğrisi ile belirlenmektedir.

Literatürde, enflasyon oranı ile işsizlik oranı arasındaki ilişkiyi ifade eden Phillips eğrisinin fonksiyonel bağımlılığının zaman gecikmeli bir bağlılık olduğuna ilişkin argüman bulunmaktadır. Bu bağlılığı gecikmeli olarak ele alıp Goodwin modelinin değiştirilmiş versiyonlarının dinamik analizini yapan az sayıda yayın mevcuttur.

Goodwin modelinin varsayımları şöyle sıralanabilir: (i) İstikrarlı teknik ilerleme söz konusudur. (ii) İşgücünde istikrarlı büyüme bulunmaktadır, (iii) Sadece iki üretim

faktörü dikkate alınmaktadır: emek ve sermaye. (iv) Tüm miktarların net ve gerçek olduğu varsayılmaktadır. (v) İşçi sınıfı tüm ücretlerini tüketirken, tüm karlar sermaye sahipleri tarafından yatırıma ayrılır. (vi) Sabit bir sermaye-çıktı oranı varsayılmaktadır. (vii) Tam istihdama yakın durumda reel ücret oranı yükselmektedir.

Temel olarak popülasyon dinamiğindeki av-avcı sisteminin matematiksel ekonomideki bir karşılığı olan Goodwin modeli, basitliğine rağmen, belirli zaman aralıklarında gözlemlenen durum değişkenlerinin periyodik davranışını bir dereceye kadar açıklamaktadır. Modelin yapısına ilişkin varsayımlar, bir ekonomide meydana gelen daha karmaşık durumları dikkate alacak şekilde gevşetilebilir. Bu nedenle, model mevcut literatürde çeşitli değerlendirmelerle modifiye edilmiştir. Bu çalışmada, istihdam oranı ve ücret payı değişkenlerine ek olarak kapasite kullanımı ve sermaye katsayısı değişkenlerini de dikkate alan genelleştirilmiş, daha yüksek boyutlu bir Goodwin modeli ele alınmaktadır.

Bir dinamik sistemde parametrelerin değişimiyle topolojik olarak eşdeğer olmayan bir faz portresinin ortaya çıkışı çatallanma olarak adlandırılır. Literatürde çeşitli çatallanma türleri mevcuttur. En az iki durum değişkenli bir dinamik sistemde, bir denge noktasının doğrusallaştırmasının kompleks bir özdeğer çifti bulunsun. Sistemin parametrelerinden seçilen bir çatallanma parametresinin özel bir değerinde (i) özdeğerin reel kısmı sıfır, (ii) özdeğerin reel kısmının çatallanma parametresine göre türevi sıfırdan farklı, (iii) birinci Lyapunov katsayısı olarak adlandırılan büyüklük sıfırdan farklı ise, dinamik sistem çatallanma parametresinin bu özel değerinin değişimi ile bir Hopf çatallanmasına uğrar. Faz diyagramlarında iki durumla karşılaşılır. Bunlardan birinde sistemde, çatallanma parametresi kritik değerine artarak ilerlerken ve kritik değerde denge noktası kararlıdır. Kritik değer sonrasında denge noktası kararsızdır ve kararlı bir limit çevrim orataya çıkar. Diğer durumda, çatallanma parametresi kritik değerine artarak ilerlerken denge noktası kararlıdır; ayrıca, denge noktasını içeren kararsız bir limit çevrim bulunur. Kritik değerde, limit çevrim kaybolur, denge noktası kararlıdır. Kritik değerden büyük değerlerde denge noktası kararsız hale gelir. Buradan da anlaşılabileceği gibi, Hopf çatallanması, bir denge noktasının doğrusallaştırmasının, parametrelerinin değişimi ile, bir pür imajiner özdeğer çiftine sahip olması durumunda gerçekleşebilir.

Gecikmesiz dinamik sistemlerde bir denge noktası civarında doğrusallaştırmanın özdeğer denklemi bir polinom denklemi iken, gecikmeli dinamik sistemlerin doğrusallaştırmasının özdeğer denklemi bir üstel fonksiyon ve polinom içeren bir denklemdir. Gecikmeli dinamik sistemlerde de Hopf çatallanmasının varlığı araştırılırken, ilke olarak yine, bu denklemin bir pür imajiner köke sahip olma durumu ve diğer gerekli koşulların geçerliliği incelenir.

Bu çalışmanın temel amacı, gecikmeli genelleştirilmiş Goodwin diferansiyel denklem sisteminin stabilite analizini yapmak ve Hopf çatallanma durumunun varlığını incelemektir. Bu analizler ile ekonomik değişkenlerin bir denge noktasının kararlılığı araştırılmıştır.

Bu tez çalışması dört bölümden oluşmaktadır.

Birinci bölümde tezin amacı, ilgili literatür araştırması sunulmuş ve yapılacak analizlerle ilgili araştırma soruları ifade edilerek bu soruların cevaplarının ne şekilde araştırılacağı ifade edilmiştir.

İkinci bölümde gecikmeli dinamik sistemlerden bahsedilmiş ve bu sistemlerin yerel stabilite analizi hakkında bilgi verilmiştir. Ayrıca, Hopf çatallanması hakkında genel bilgiye yer verilmiştir. Sonrasında, Hopf çatallanması analizinin gecikmeli bir dinamik sisteme nasıl uygulandığına dair literatürden bir örnek verilmiştir. Devamında, bir dinamik sistemde başlangıç koşullarındaki küçük değişimlerin yörüngeler arasında nasıl farklılığa yol açtığının bir ölçüsü olan Lyapunov üsteli ile ilgili gecikmesiz dinamik sistemler durumunda temel bilgiler verilmiştir ve bilinen bir model üzerindeki uygulamasının sonuçları sunulmuştur. Ardından, orijinal Goodwin modeli ile ilişkisini göstermek açısından popülasyon dinamiğinin av-avcı modelinden bahsedilmiş ve sonrasında Goodwin modelinin varsayımları not edilerek türetilişi yapılmıştır. Goodwin modelinde istihdam değişkeni av, emeğin payı da avcı ile karşılık bulur. Bu nedenle, Goodwin modeli sınıf çatışması olarak da tanınmaktadır. Modelin içerdiği fonksiyonel bağlılıklardan olan Phillips eğrisine ayrıca bir alt bölüm ayrılarak kısa bir genel bakış sunulmuştur. Son olarak, üzerinde çalışılan esas model olan ve gecikmeli versiyonu incelenen bir genelleştirilmiş Goodwin modeli ile ilgili yayında mevcut sonuçlar bütünlük açısından verilmiştir.

Üçüncü bölüm, tez çalışmasında yapılan özgün analizlerin sunulduğu bölümdür. İncelenen modeller, ele alınan dört dinamik değişkenli genelleştirilmiş Goodwin sisteminin özel alt durumlarıdır. Birinci alt bölümde, değişken hızlı teknik ilerleme ve iş yoğunluğu durumunda ana sistemin, diğer iki değişkene bağlı olmayan iki bilinmeyen fonksiyonlu alt sistemi ele alınmıştır. Burada, Phillips eğrisine karşılık gelen istihdam fonksiyonunda bir zaman gecikmesi varsayılmıştır. Hopf çatallanmasının gerek koşullarından ikisi, pür imajiner özdeğere sahip olma ve reel kısmın türevinin sıfırdan farklı olma koşulu çalışılarak Hopf çatallanmasını gerçekleyebilecek kritik gecikme değeri formüle edilmiştir. Sistemin sıfır olmayan denge noktasında (i) gecikme parametresinin her değeri için kararlı, (ii) gecikme parametresinin kritik bir değerine kadar kararlı, sonrasında kararsız olmasını ve Hopf çatallanmasına uğramasını sağlayan parametre koşulları ayrı ayrı bulunmuştur. Hopf çatallanmasına ilişkin koşulların belli bir parametre ailesi için sağlandığı gösterilmiş ve sonuçlar sayısal çözüm sonucu elde edilen görsellerle desteklenmiştir. İkinci alt bölüm benzer bir analiz içerir. Üçüncü alt bölümde, ana dört bilinmeyen durum değişkenli sistem için denge noktasının kararsız olduğu bir durumda, Lyapunov üstelleri bir Matlab kodu ile bulunmuş ve hiperkaotik davranışa işaret etme potansiyeli olan iki pozitif Lyapunov üsteli gözlenmiştir. Son alt bölümde, Hopf çatallanması için yön analizi yapılarak Hopf çatallanmasının tipi belirlenmiştir. Son bölümde ise sonuçlar özetlenmiş ve olası açık problemler üzerinde durulmuştur.

Bu tez çalışmasında yapılan analiz ile, literatürde mevcut orijinal Goodwin modeli ve genelleştirmelerine gecikmeli analiz açısından yaklaşım sağlanmaya çalışılmıştır. Belli bir ülke ekonomisi için, literatürde mevcut, gerçek verilerle oluşturulan istihdam-ücret payı döngüleri incelendiğinde, oluşan döngülerin belli bir zaman aralığı için geçerli olduğu, bu zaman aralığı sonrasında döngünün faz düzleminin başka bir denge noktasına sıçradığı gözlenmektedir. Bu durumun bir açıklaması, denge noktasının konumunu veren, ilgili ülkeye ilişkin sistemde mevcut parametrelerin değerlerinin değişmesi olabilir. Bu duruma ilişkin, bu tez çalışmasının sonuçlarından elde edilebilecek başka bir açıklamanın, sistemde dinamiğin gecikme ile gerçekleşmesi nedeniyle ilgili denge noktasının belli bir noktada kararsız hale gelmesi ve çizilen faz eğrisinin denge noktasından uzaklaşarak faz uzayında başka bir döngüyü gerçekleştirmeye yönelmesi şeklinde ifade edilebileceği düşünülmektedir.

1. INTRODUCTION

This thesis work is based on the study of delayed-state stability analysis and Hopf bifurcation in a generalized Goodwin model. We first state the main problems analysed in the thesis. After that we give a literature survey that led us to the problems stated. We conclude the introduction by providing the research questions of the work and mention the methodologies that will take us to the answers of these questions.

1.1 Purpose of Thesis

Our starting point is the following generalization of the Goodwin model, in the form

$$\dot{\beta} = [\varphi(\beta) + f(\lambda) - u(\lambda) - n]\beta, \tag{1.1a}$$

$$\dot{\lambda} = [\varphi(\beta) + \phi_1(\beta) - u(\lambda) - \phi_2(\theta)]\lambda, \tag{1.1b}$$

$$\dot{\theta} = [\psi(\lambda) + f(\lambda) - z(\lambda, v)]\theta, \tag{1.1c}$$

$$\dot{v} = [-f(\lambda) + z(\lambda, v)]v \tag{1.1d}$$

studied in [1]. In this system β denotes employment ratio, λ is for wage share, θ is the rate of capacity utilization, and v is the actual capital coefficient. To be concise, we express the functions appearing on the right hand sides in Chapter 3, where the analysis is presented.

First, based on this model, we consider a delayed sub-case of this model as

$$\dot{\beta}(t) = \Big[\beta_0 + \gamma_2\beta(t) - \delta_0\lambda(t)\Big]\beta(t), \tag{1.2a}$$

$$\dot{\lambda}(t) = \Big[\lambda_0 - \nu_2\lambda(t) + \gamma_2\beta(t) + \rho_1\beta(t-\tau)\Big]\lambda(t) \tag{1.2b}$$

where $\beta_0, \gamma_2, \delta_0, \lambda_0, \nu_2, \rho_1$ are constants. We determine the conditions on the parameters of the system so that the nonzero equilibrium point is stable or experiences a Hopf bifurcation, depending on the values of the delay $\tau \geq 0$. We also perform a direction analysis for the Hopf bifurcation.

Second, as a similar case, we perform the same analysis for the subsystem

$$\dot{\beta}(t) = \left[\tilde{\beta}_0 - \tilde{\delta}_0\lambda(t)\right]\beta(t), \tag{1.3a}$$

$$\dot{\lambda}(t) = \left[\tilde{\lambda}_0 - \mu_2 v_2\lambda(t) + \rho_1\beta(t-\tau)\right]\lambda(t), \tag{1.3b}$$

which is slightly different than **(1.2)**, including a new parameter and an extra equilibrium point condition when reduced from the main system **(1.1)**.

Finally, we approach the system **(1.1)** in the nondelayed form for a set of parameters at which the system is unstable at the equilibrium point and calculate the Lyapunov exponents of the system in this case.

1.2 Literature Review

There have been extended models in a large amount of researches since Goodwin introduced his simple but useful model about struggle between employment rate and wage shares based on Volterra-Lotka predator-prey model. Besides, some researchers prove the stability of the original model while others add new variable to make their model more reliable and realistic. In this regard, some of these studies will be briefly explained.

Goodwin's original work [2] shows the relation between the wage share and the employment ratio in a closed economy as

$$\dot{u} = \left[-(\alpha+\gamma)+\rho v\right]u, \tag{1.4a}$$

$$\dot{v} = \left[\frac{1}{\sigma}-(\alpha+\beta)-\frac{1}{\sigma}u\right]v. \tag{1.4b}$$

Desai and his co-workers amend Goodwin's original model by adding two extensions. They define a non-linear real wage equation and they change the assumption with regard to investment of all profits in [3] as

$$\frac{\dot{u}}{u} = -(\gamma'+\alpha)+\rho'(1-v)^{-\delta}, \tag{1.5a}$$

$$\frac{\dot{v}}{v} = \lambda\ln\left(\frac{\bar{u}-u}{1-\bar{u}}\right)-(\alpha+\beta). \tag{1.5b}$$

Another extension of Goodwin's model in which the authors aim at modifying the dynamics so that the outputs for the state variables remain within the unit square is presented in [4] as

$$\frac{\dot{u}}{u} = K_1 \mu^{\eta_1}(1-u)^{\eta_1}[-(\alpha' - \gamma') + \rho' v], \quad \textbf{(1.6a)}$$

$$\frac{\dot{v}}{v} = K_2 V^{\mu_2}(1-v)^{\eta_2}[\frac{1}{\sigma'} - (\alpha' + \beta') - \frac{1}{\sigma'} u]. \quad \textbf{(1.6b)}$$

Ref. [5] is an attempt in the same direction and they examine Goodwin's class struggle model by applying the planar Hamiltonian systems as a new approach to provide economically acceptable interval for solution and they show results of small perturbation. One of the main assumptions in Goodwin's model is that workers do not save. Author of [6] relax this condition and study an extension, considering the share of capital held by workers as an additional variable. Assuming variable capacity utilization and variable actual capital coefficient, [1] obtains a four-dimensional model that contains Goodwin as a special case. Based on Minsky's "Financial Instability Hypothesis" [7], in [8], Keen adds a banking sector to the Goodwin growth cycle. In this case, the stable limit cycles of the Goodwin model may no longer exist in the long run and instabilities may occur. The stability analysis of Keen model is performed in [9] in the form

$$\dot{\omega} = \omega[\Phi(\lambda) - \alpha], \quad \textbf{(1.7a)}$$

$$\dot{\lambda} = \lambda[\frac{K(1-\omega-r/u)}{v} - \alpha - \beta - \delta], \quad \textbf{(1.7b)}$$

$$\dot{u} = u[\frac{K(1-\omega-r/u)}{v} - r - \delta] - u^2[K(1-\omega-r/u) - (1-\omega)]. \quad \textbf{(1.7c)}$$

The authors of [10] take the second step to analyze the same dynamics with an inflation term and proceed further to construct a four-dimensional system considering speculative money flow to the economy. They obtain the system

$$\dot{\omega} = \omega[\Phi(\lambda) - \alpha - (1-\gamma)i(\omega)], \quad \textbf{(1.8a)}$$

$$\dot{\lambda} = \lambda[g(\pi) - \alpha - \beta], \quad \textbf{(1.8b)}$$

$$\dot{c} = r_L K(\pi) - r_f \pi - c[g(\pi) + i(\omega)] + (r_L - r_f) f, \quad \textbf{(1.8c)}$$

$$\dot{f} = \Psi(g(\pi) + i(\omega)) - f[g(\pi) + i(\omega)]. \quad \textbf{(1.8d)}$$

In [11], Goodwin's model is modified through a change in the assumption on the technical progress and also by introducing a memory variable which affects the behaviors of both the workers and the capitalists with a three dimensional system in the form

$$
\begin{aligned}
\dot{u} &= [-(\gamma+\beta)+(\rho_1+\rho_2 y)v]u, && \textbf{(1.9a)}\\
\dot{v} &= \frac{\varphi K_0(1-u)-(\beta+n+\eta)[K_0+\zeta(y)]-\zeta'(y)(\frac{u-u^*-y}{T})}{K_0+\zeta(y)}v, && \textbf{(1.9b)}\\
\dot{y} &= \frac{u-u^*-y}{T}. && \textbf{(1.9c)}
\end{aligned}
$$

The proposed model experiences a Hopf bifurcation at a critical value related to the memory effect. Within the context of delay dynamical systems we can mention [12], where the authors consider a finite time delay between investment orders and deliveries of finished capital goods and a delayed reaction of real wages to the unemployment levels. They obtain the system

$$
\begin{aligned}
\dot{v}_t &= \frac{1}{\sigma}\left(k^*-\varepsilon\frac{v_{t-\tau_1}-v^*}{v^*}\right)(1-u_{t-\tau_1})v_{t-\tau_1}e^{-g\tau_1}-gv_t, && \textbf{(1.10a)}\\
\dot{u}_t &= \left(\frac{\rho}{1-v_{t-\tau_2}}-(\alpha+\gamma)+\zeta\frac{\dot{v}_t}{v_t}\right)u_t. && \textbf{(1.10b)}
\end{aligned}
$$

In [13], three extensions of Goodwin's model are presented, and these extensions cover inflation, expected inflation and excess capacity. Frederick van der Ploeg [14] extends Goodwin's original model to allow savings decisions with the help of the direction of technological improvement as follows

$$
\begin{aligned}
\frac{\dot{\theta}}{\theta} &= -v_0+v_1\varepsilon-(1-v_2)\omega^*, && \textbf{(1.11a)}\\
\frac{\dot{\varepsilon}}{\varepsilon} &= (1+\mu)\Big[\{\frac{\sigma_1(1-\theta)+\sigma_2\theta}{\alpha^*}\}-\gamma-n\Big]-\omega^*. && \textbf{(1.11b)}
\end{aligned}
$$

In Ref. [15], a dynamic game is defined to understand behaviour of economic agents if they have rational or myopic behaviour under original content of Goodwin's model. Author of [16] discusses a new utility function for both capitalists and workers. Harvie [17] conducts an experiment with 10 OECD countries for Goodwin model in his work, and in [18], author takes a constant capital accumulation rate between (0,1] to improve Harvie's result with the following two dimensional system

$$
\begin{aligned}
\frac{\dot{\omega}}{\omega} &= \gamma+\rho\lambda-\alpha, && \textbf{(1.12a)}\\
\frac{\dot{\lambda}}{\lambda} &= \frac{k(1-\omega)}{v}-(\alpha+\beta+\delta). && \textbf{(1.12b)}
\end{aligned}
$$

The work [19], the original system is modified with the help of adding rate of capacity utilization and perform their new system to the US economy. Authors of [20] show

how government policy lag effect with three dimensional system as follows

$$\begin{aligned}\dot{u} &= \varepsilon\{H(1-v, i-\pi^e) - (1-c_k)(1-\delta)(1-v) + \mu(u^* - u)\}u \quad \textbf{(1.13a)}\\ &\equiv f_1(u, v, \pi^e; \mu),\\ \dot{v} &= (1-\gamma)\{F(u) + \pi^e - \alpha\}v \equiv f_2(u, v, \pi^e), \quad \textbf{(1.13b)}\\ \dot{\pi^e} &= \beta[\gamma\{F(u) + \pi^e - \alpha\} - \pi^e] = f_3(u, \pi^e). \quad \textbf{(1.13c)}\end{aligned}$$

Finally, we would like to mention again the the following generalization of the Goodwin model explicitly in the form

$$\begin{aligned}\dot{\beta} &= [\varphi(\beta) + f(\lambda) - u(\lambda) - n]\beta, \quad \textbf{(1.14a)}\\ \dot{\lambda} &= [\varphi(\beta) + \phi_1(\beta) - u(\lambda) - \phi_2(\theta)]\lambda, \quad \textbf{(1.14b)}\\ \dot{\theta} &= [\psi(\lambda) + f(\lambda) - z(\lambda, v)]\theta, \quad \textbf{(1.14c)}\\ \dot{v} &= [-f(\lambda) + z(\lambda, v)]v, \quad \textbf{(1.14d)}\end{aligned}$$

studied in [1]. In this system β denotes employment ratio, λ is for wage share, θ is the rate of capacity utilization, and v is the actual capital coefficient. The authors study the stability of this system in four different cases. First, they consider a "*simplified Goodwin-like case*" [1] in which the system **(1.14)** takes the form

$$\begin{aligned}\dot{\beta} &= [f(\lambda) - v_1 - n]\beta, \quad \textbf{(1.15a)}\\ \dot{\lambda} &= [\phi_1(\beta) - v_1]\lambda, \quad \textbf{(1.15b)}\\ \dot{v} &= [-f(\lambda) + z(\lambda, v)]v, \quad \textbf{(1.15c)}\\ \dot{\theta} &= \frac{1}{\sigma v}. \quad \textbf{(1.15d)}\end{aligned}$$

Second, in case of "*variable speed of technical progress and changes in work intensity*" [1] they study the subcase

$$\begin{aligned}\dot{\beta} &= [\varphi(\beta) + f(\lambda) - v_1 - v_2\lambda - n]\beta, \quad \textbf{(1.16a)}\\ \dot{\lambda} &= [\varphi(\beta) + \phi_1(\beta) - v_1 - v_2\lambda]\lambda, \quad \textbf{(1.16b)}\\ \dot{v} &= [-f(\lambda) + z(\lambda, v)]v, \quad \textbf{(1.16c)}\\ \theta &= \frac{1}{\sigma v}. \quad \textbf{(1.16d)}\end{aligned}$$

Then, another subcase

$$\dot{\beta} = [f(\lambda) - u(\lambda) - n]\beta, \quad \textbf{(1.17a)}$$
$$\dot{\lambda} = [\phi_1(\beta) - u(\lambda)]\lambda, \quad \textbf{(1.17b)}$$
$$\dot{v} = [-f(\lambda) + z(\lambda, v)]v, \quad \textbf{(1.17c)}$$
$$\dot{\theta} = [\psi(\lambda) + f(\lambda) - z(\lambda, v)]\theta, \quad \textbf{(1.17d)}$$

is studied under the title "*non-neutral technical progress*". Finally, they perform the stability analysis of the full system **(1.14)**. The systems **(1.16)**, **(1.17)** has been the main models analyzed in this thesis work in the context of delay dynamical systems. We have analyzed the full model **(1.14)** in the sense of Lyapunov exponents.

1.3 Hypothesis Research Questions

In the original Goodwin model, the Phillips curve reflects the relation between the inflation rate and unemployment. When we look at his original paper [21] page 297, we see that the curve obtained from data fit contains some loops. He states that "*A loop in this direction could result from a time lag in the adjustment of wage rates. If the rate of change of wage rates during each calendar year is related to unemployment lagged seven months, ... the loop disappears ..... points lie closely along a smooth curve*" [21]. This means that, the relation provided by the Phillips curve may contain a time delay. When we focus on the system **(1.16)**, this amounts to consider the function $\phi_1(\beta(t))$ with a time delay, i.e., in the form $\phi_1(\beta(t-\tau))$. Therefore, based on this system, we form the research questions of this thesis work as follows.

(RQ1) The system **(1.16)** is stable at the nonzero equilibrium point for a set of parameters. When considered in the delayed context mentioned above, are there ranges of parameters so that the system (i) is stable (ii) undergoes a Hopf bifurcation (iii) is unstable?

(RQ2) How are the results of the same analysis in (RQ1) if applied to **(1.17)**?

(RQ3) Authors of [1] study the full system **(1.14)** for stability of equilibrium points, but it seems they cannot find a range of parameters that satisfy the related stability conditions. In specific cases of the parameters, are there positive Lyapunov exponents for the system that might be a trace for chaotic of hyperchaotic behaviour?

(RQ4) For the analysis performed in answering (RQ1), is the bifurcation subcritical or supercritical?

The analyses that were performed in this thesis work for answering those research questions are as follows.

(A1) Considering a time delay in the function ϕ_1 of **(1.16)**, the necessary conditions for a Hopf bifurcation to occur are investigated; which are (i) having a pure imaninary eigenvalue (ii) transversality conditon. Then, for a specific case of the parameters, it is shown that these conditions are indeed satisfied. The analysis is presented in Section 3.1. Actually, having a nonzero first Lyapunov coeffiecient is the third condition for the Hopf bifurcation, we present the analysis separately in Section 3.4.

(A2) The analysis in (A1) for the case of **(1.17)** is presented in Section 3.2.

(A3) Using a Matlab package, at an unstable case, we calculate the Lyapunov coefficients and present the results in Section 3.3.

(A4) We evaluate the first Lyapunov coefficient for the analysis (A1). Although the subsystem we work on is a two state-variable Lotka-Volterra delayed type system and has been studied extensively in the literature, the calculations for the the case we considered has not been analysed before, to the best of our knowledge. The analysis, which is generally called as "Direction of Hopf Bifurcation" is presented in Section 3.4.

2. STABILITY OF DELAY DYNAMICAL SYSTEMS AND PRIMER ON GOODWIN MODEL

We start by introducing linear stability analysis of delay dynamical systems. We mention Hopf bifurcation, and present an example from literature how Hopf bifurcation is investigated in a delay-dynamical system. We mark the importance of Lyapunov exponents of a dynamical system and give a well-known example. Finally, we mention the original Goodwin model and for completeness, include the results available in the original work on the generalized Goodwin model, of which content has formed the systems studied in this thesis.

2.1 Nonlinear Delay Dynamical Systems and Stability

In this section, we first present linearisation of a delay dynamical system around an equilibrium point. After that we will provide some basic information about Hopf bifurcation in a dynamical system. We will conclude this part with an example which illustrates how the Hopf bifurcation analysis is performed in a nonlinear delay dynamical system.

2.1.1 Linearization nonlinear delay differential equations

A delay differential equation with a single constant delay can be represented in [22] as

$$\dot{X} = F(t, X(t), X(t-\tau)). \qquad \textbf{(2.1)}$$

A stable fixed point X^* means that its all trajectories at the near of the fixed point approach itself asymptotically when $t \to \infty$ and for this reason, all the time derivatives vanish identically. **(2.1)** is satisfied by $X(t) = X(t-\tau) = X^*$ at any point, and it satisfies the equation

$$f(X(t) = X(t-\tau) = X^*) = 0, \qquad X^* = (x_1^*, x_2^*, \ldots, x_n^*)^T. \qquad \textbf{(2.2)}$$

Around the equilibrium point X^*, stability of the equilibrium point is examined by perturbing it by infinitesimally displacing the solution with the help of time-dependent function $\delta X(t)$, and persisting over an interval of at least the values of the longest delay, τ_{max}, for the multiple delays. Denoting $X = X(t)$ and $X_\tau = X(t-\tau)$, we have

$$X = X^* + \delta X, \quad X_\tau = X^* + \delta X_\tau. \tag{2.3}$$

Then, we find

$$\dot{X} = \delta\dot{X} = f(X^* + \delta X, X^* + \delta X_\tau). \tag{2.4}$$

The infinitesimal displacements from the equilibrium point over the interval $(t_0 - \tau, t_0)$ are symbolised by δX. The equation **(2.4)** can be linearized around the equilibrium point by using the Taylor series expansion as follows

$$\begin{gathered}\delta\dot{X} = J_0\delta X + J_\tau\delta X_\tau,\\ (J_0)_{i,j} = \left(\frac{\partial F_i}{\partial x_j}\right)|_{x_j = x_j^*} \quad for \quad i,j = 1,2,...,n,\\ (J_\tau)_{i,j} = \left(\frac{\partial F_i}{\partial x_{\tau j}}\right)|_{x_{\tau j} = x_j^*} \quad for \quad i,j = 1,2,...,n,\end{gathered} \tag{2.5}$$

where J_0 and J_τ are the Jacobian matrices with respect to X and X_τ respectively and evaluated at $X = X_\tau = X^*$. Assume that $\delta X(t)$ is the solution as exponential functions of time along with the exponents given by the eigenvalue of the corresponding Jacobian matrix,

$$\delta X(t) = Ae^{\lambda t}. \tag{2.6}$$

In **(2.6)**, A is for a constant column matrix. If equation **(2.6)** is substituted into the equation **(2.5)** and collecting the coefficients of $e^{\lambda t}$, following matrix equation is obtained

$$\lambda A = (J_0 + e^{-\lambda\tau}J_\tau)A. \tag{2.7}$$

This equation obviously can be satisfied with nonzero displacement amplitudes A if

$$|J_0 + e^{-\lambda\tau}J_\tau - \lambda I| = 0, \tag{2.8}$$

where I is the identity matrix.

The characteristic equation **(2.8)** is different from the characteristic equation of a non-delayed systems due to the exponential term. It is known that if all roots of a characteristic equation are negative or have negative real parts, the equilibrium point is stable. Otherwise, it is called unstable. In addition to this, if one of the eigenvalue is zero, then the stability is undecidable to the linear order and a recourse should be taken to consider the neglected higher order terms in the Taylor expansion in **(2.4)**.

Despite the existence of an infinite number of roots for Equation **(2.8)**, it is often possible to determine analytically whether a given equilibrium point is stable or not.

The information in Subsection 2.1.1 was provided from Ref. [22].

2.1.2 Hopf bifurcation

Now, we focus on a dynamical system which is dependent on parameters. For the continuous time case, let us consider as in Ref. [23]

$$\dot{x} = f(x, \alpha), \tag{2.9}$$

where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^m$ represents the parameters. When we change parameters, the phase portrait changes as regards to parameters as well.

Definition *The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation. [23]*

Now, we will explain the idea of Hopf bifurcation with an example. Let us consider Andronov-Hopf bifurcation on the planar system as follows

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2) \end{aligned} \tag{2.10}$$

which is dependent to one parameter.

In polar coordinates $(\rho,\, \theta)$ it takes the form

$$\begin{aligned} \dot{\rho} &= \rho(\alpha - \rho^2), \\ \dot{\theta} &= 1, \end{aligned} \tag{2.11}$$

and can be integrated explicitly. Phase portraits of the system at the origin can be drawn easily because ρ and θ are independent in **(2.11)**.

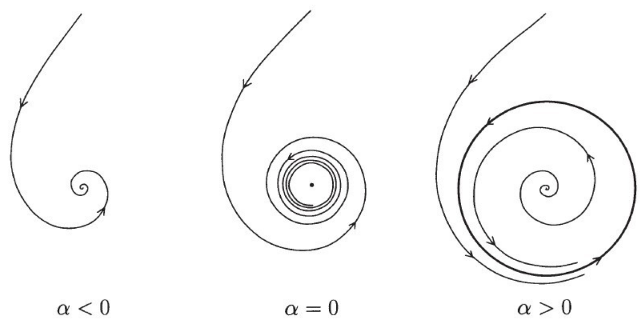


Figure 2.1 : Hopf bifurcation

Figure **(2.1)** was provided from Ref. [23].

According to the Figure **(2.1)**, when we choose any initial point, the equilibrium is stable focus as $\dot{\rho} < 0$ and $\rho(t) \to 0$ if $\alpha \leq 0$. Otherwise, when we investigate its behaviour when $\alpha > 0$, we have $\dot{\rho} > 0$ for $\sqrt{\alpha} > \rho > 0$. Therefore, the equilibrium becomes an unstable focus, and $\dot{\rho} < 0$ for $\rho > \sqrt{\alpha}$.

For any $\alpha > 0$ of radius $\rho_0 = \sqrt{\alpha}$ (at $\rho = \rho_0$ we have $\dot{\rho} = 0$) system **(2.11)** has a periodic orbit. Moreover, periodic orbit is stable since $\dot{\rho} > 0$ inside and $\dot{\rho} < 0$ outside the cycle. Therefore, $\alpha = 0$ is a bifurcation value. When α increases and crosses zero, there is a bifurcation in system **(2.10)** and it is called the Andronov-Hopf bifurcation. It causes small-amplitude periodic oscillations from the equilibrium state.

It is possible to determine Andronov-Hopf bifurcation via fixing any small neighborhood of the equilibrium and this kind of bifurcations are called local. Local bifurcations will be referred as bifurcations of equilibria or fixed points. It is not easy to detect some bifurcations by looking at small neighborhoods of equilibrium (fixed) points or cycles as well and these bifurcations are called global.

Theorem 1. *Consider a two dimensional system*

$$\frac{dx}{dt} = f(x,\alpha), \qquad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1, \tag{2.12}$$

where f is smooth and for all sufficiently small $|\alpha|$ the equilibrium $x = 0$ with eigenvalues

$$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm iw(\alpha), \tag{2.13}$$

where $\mu(0) = 0, w(0) = w_0 > 0$.

Suppose that the following conditions are satisfied:
(B.1) $l_1(0) \neq 0$, and l_1 is the first Lyapunov coefficient;
(B.2) $\mu'(0) \neq 0$.

In that case, there exists invertible coordinate and parameter changes and a time reparametrization such that **(2.12)** *transforms to*

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(||y||^4). \tag{2.14}$$

Finally, $O(||y||^4)$ terms can be dropped so that the following general result is obtained.

Theorem 2. *(Topological normal form for the Hopf bifurcation)*

Any two dimensional system with one parameter

$$\dot{x} = f(x, \alpha), \tag{2.15}$$

which has the equilibrium $x = 0$ *with* $\alpha = 0$*, with eigenvalues*

$$\lambda_{1,2}(0) = \pm i w_0, \quad w_0 > 0, \tag{2.16}$$

is locally topologically equivalent near the origin to one of the normal forms

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{2.17}$$

The information in Subsection 2.1.2 was provided from Ref. [23].

2.1.3 Stability of delay dynamical systems with an example

We note the results available in [24] to express clearly the Hopf bifurcation analysis of a delay dynamical system. The model analyzed in the work [24] is

$$\begin{aligned} \dot{x} &= (\frac{1}{b} - a)x(t) + z(t), \\ \dot{y} &= -by(t) + K[y(t) - y(t-\tau)], \\ \dot{z} &= -x(t) - cz(t). \end{aligned} \tag{2.18}$$

At the critical point $P_0(0,0,0)$, one finds the characteristic equation

$$\left[\lambda^2 + \left(c + a - \frac{1}{b}\right)\lambda + \left(1 + ac - \frac{c}{b}\right)\right] * \left[\lambda - (-b + K - Ke^{-\lambda\tau})\right] = 0. \tag{2.19}$$

Let $c + a - \frac{1}{b} = P$ and $1 + ac - \frac{c}{b} = Q$. Then, the two eigenvalues originating from the first factor in **(2.19)** have negative real parts if $P > 0$ and $Q > 0$. At this point, they study on the second part of the characteristic equation, which is the equation

$$\lambda - (-b + K - Ke^{-\lambda\tau}) = 0. \tag{2.20}$$

For $\lambda = i\omega$ one gets

$$i\omega + b - K + K(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.21}$$

Then, real and imaginary parts are separated as follows

$$\begin{cases} b - K + K\cos\omega\tau = 0, \\ \omega - K\sin\omega\tau = 0. \end{cases} \tag{2.22}$$

It is found that

$$\omega^2 = 2Kb - b^2. \quad \textbf{(2.23)}$$

It is easy to see that when $K \leq \frac{b}{2}$, **(2.23)** does not have any positive real solution and in the case $K > \frac{b}{2}$, **(2.23)** has the solution

$$\omega_+ = \sqrt{2Kb - b^2}. \quad \textbf{(2.24)}$$

It is found that

$$\tau_j = \frac{1}{\omega_+} \arccos \frac{K-b}{K} + \frac{2j\pi}{\omega_+}, \qquad j = 0, 1, 2... \quad \textbf{(2.25)}$$

After that, the transversality condition

$$\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}|_{\tau=\tau_j} > 0 \qquad (j = 0, 1, \ldots) \quad \textbf{(2.26)}$$

must be checked. When **(2.20)** is differentiated with respect to τ, it is easy to obtain that

$$\frac{d\lambda}{d\tau} + Ke^{-\lambda\tau}\left(-\frac{d\lambda}{d\tau}\tau - \lambda\right) = 0 \quad \textbf{(2.27)}$$

and hence

$$\frac{d\lambda}{d\tau} = \frac{-K\lambda e^{-\lambda\tau}}{1 + K\tau e^{-\lambda\tau}} \quad \textbf{(2.28)}$$

which gives

$$\begin{aligned} \mathrm{Re}\left\{\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_j} &= \mathrm{Re}\left\{\frac{iK\omega_+}{e^{i\omega_+\tau_j} - K\tau_j}\right\} \\ &= \frac{K\omega_+ \sin\omega_+\tau_j}{(\cos\omega_+\tau_j - K\tau_j)^2 + (\sin\omega_+\tau_j)^2} \\ &= \frac{\omega_+^2}{(\cos\omega_+\tau_j - K\tau_j)^2 + (\sin\omega_+\tau_j)^2} > 0. \end{aligned} \quad \textbf{(2.29)}$$

For $a = 15$, $b = 0.1$, $c = K = 1$, they calculate the first critical value of the delay parameter as $\tau_0 = 1.03473$ and present the following Figure **(2.2)**, Figure **(2.3)** and Figure **(2.4)** which support the existence of the Hopf bifurcation.

The information in Subsection 2.1.3 was provided from Ref. [24].

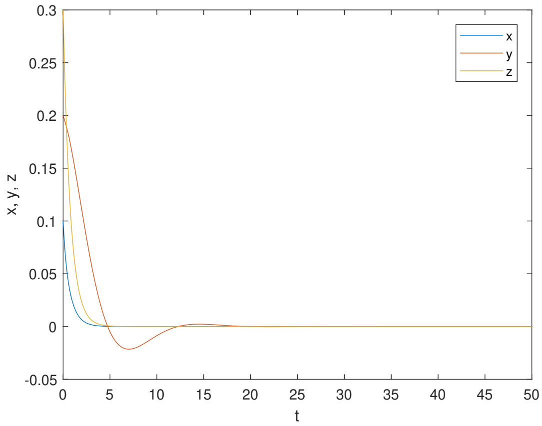


Figure 2.2 : $\tau = 0.8 < \tau_0 = 1.03473.$

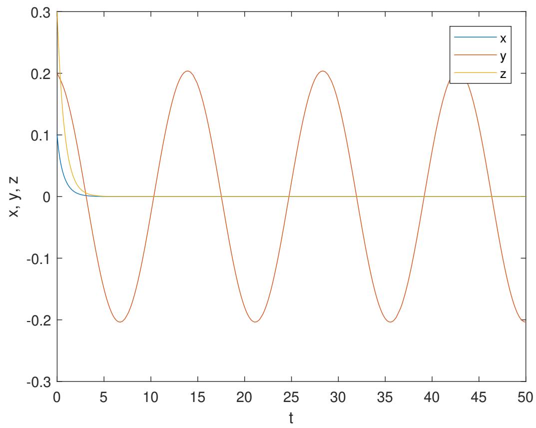


Figure 2.3 : $\tau = \tau_0 = 1.03473.$

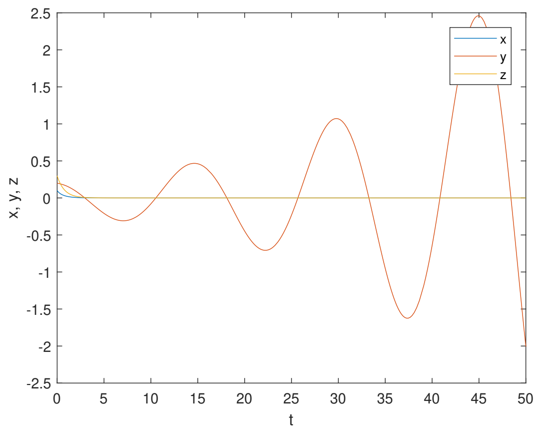


Figure 2.4 : $\tau = 1.1 > \tau_0 = 1.03473.$

2.2 Lyapunov Exponent

Lyapunov exponents are defined as the average exponential rate of divergence or convergence of neighbouring orbits. Ref. [25] suggests that Lyapunov exponent is a quantitative tool to measure chaotic behaviour of a dynamical system. Although these orbits are indistinguishable, their future behaviour may not be predictable with the exponential divergence.

Lyapunov exponents show how different directions are related to expanding and contracting nature in phase space. Suppose $x(t)$ is a point on the trajectory of a dynamical system and consider a point $x(t)+\delta(t)$ which is its neighbour. When we express $||\delta(t)|| \sim ||\delta_0||e^{\lambda t}$, the future distance between these neighbouring trajectories are characterized by the number λ, which is called the Lyapunov exponent.

Zero exponent indicates slowly changing magnitude of a principal axis tangent to the flow. Positive (negative) Lyapunov exponents correspond to axes which are on the average expanding (contracting). For dissipative system, there exists at least one negative exponent, and all exponents' sum are negative.

Exponential divergence is defined by a positive Lyapunov exponent. If a system has at least one positive Lyapunov exponent, there exists chaos which means our prediction for a dynamical system breaks down. Trajectories of a chaotic systems with aperiodic motion will rapidly diverges from each other and Lyapunov exponent of a dynamical systems shows speed of divergence of these trajectories. In addition, divergence of neighbouring trajectories indicates how a chaotic system is sensitive to its initial conditions.

Qualitative features of a systems of dynamics can be inferred from the signs of the Lyapunov exponents. An n-dimensional dynamical system has n Lyapunov exponents. There exists one positive Lyapunov exponent to create chaos for a one dimensional system. Besides, zero Lyapunov exponent causes a stable orbit and a negative Lyapunov exponent creates a periodic orbit. In a three dimensional continuous dissipative dynamical system one may have , a strange attactor with $(+,0,-)$, a two-torus with $(0,0,-)$, a limit cycle with $(0,-,-)$, and a fixed point with $(-,-,-)$.

There are three possible strange attractor types, and these are $(+,+,0,-)$, $(+,0,0,-)$, and $(+,0,-,-)$.

2.2.1 Lyapunov exponents of an *n*-dimensional dynamical system

Consider an n-dimensional dynamical system described in [22] by the system of first order coupled ordinary differential equation

$$\dot{X} = F(X), \tag{2.30}$$

where $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. Suppose that there are two trajectories in the n-dimensional phase space and their initial conditions X_0 and $X_0' = X_0 + \delta X_0$ are neighbouring each other. They are evolved over time and become the vectors $X(t)$ and $X'(t) = X(t) + \delta X(t)$, with the Euclidean norm

$$d(X_0,t) = ||\delta X(X_0,t)|| \equiv \sqrt{\delta x_1^2 + \delta x_2^2 + \ldots + \delta x_n^2}. \tag{2.31}$$

$d(X_0,t)$ is for the distance between two trajectories X_0 and X_0'. The evolution of δX with regard to time is written by linearizing **(2.30)** and we found

$$\delta \dot{X} = M(X(t)).\delta X, \tag{2.32}$$

where $M = \partial F/\partial X|_{X=X_0}$ is the Jacobian matrix of F. Besides, the mean rate of divergence of these two close trajectories calculated by

$$\lambda(X_0, \delta X) = \lim_{t\to\infty} \frac{1}{t} \log(\frac{d(X_0,t)}{d(X_0,0)}). \tag{2.33}$$

2.2.2 Lyapunov exponent with an example: Lorenz model

Lorenz studies special case to understand behaviour of trajectories. When he illustrates his result, he finds a butterfly structure in phase space. Although it is known that trajectories do not cross themselves, it seems that as if they cross themselves repeatedly. Therefore, his result shows us that there is something strange. Thus, Lorenz shows that predicting the future of a chaotic system is quite difficult.

In this case, Lorenz's three-dimensional dynamical system has three Lyapunov exponents for $\sigma = 16, R = 45,92$, and $b = 4$.

$$\begin{cases} \dot{x} = \sigma(Y - X), \\ \dot{y} = X(R - Z) - Y, \\ \dot{z} = XY - bZ. \end{cases} \tag{2.34}$$

According to system **(2.34)**, three Lyapunov exponents, shown in Figure **(2.5)**, calculated by using the LCE package for Mathematica as $\lambda_1 = 1.50085$, $\lambda_2 = 0.00327127$, and $\lambda_3 = -22.4872$.

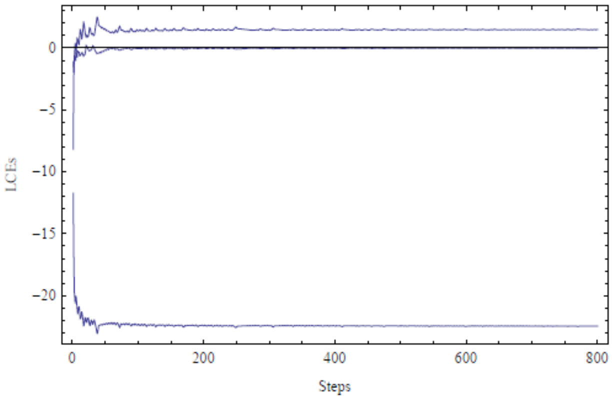


Figure 2.5 : Lyapunov exponents of the Lorenz dynamical system

2.3 Goodwin Model of Growth Cycle

In this section, we consider the predator-prey system which affects the Goodwin's work. After introducing predator-prey system, we will explain Goodwin's original model with its assumptions.

2.3.1 Predator-prey system

The predator-prey system shows how two species are related with each other. A. J. Lotka and V. Volterra worked on this model independently. According to [26], the model has the following assumptions:

(a) There are unlimited food for preys.

(b) Predators reach preys as a unique source of food.

(c) The rate of change of both populations is proportional to the size.

(d) Genetic adaptation and environment changes are not considered.

Under these assumptions the model is developed as follows

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta xy, \\ \frac{dy}{dt} &= -\gamma y + \delta xy, \end{aligned} \quad \textbf{(2.35)}$$

where x and y represent the prey and predator population respectively. β represents death rate of preys because of predator and α is the natural growth rate of preys without any predator in population. γ is the natural death rate of predator while there is not prey population. δ is the growth rate of predators due to preys.

When the equations in **(2.35)** are divided, one finds

$$\frac{dy}{dx} = -\frac{y}{x}\frac{\delta x - \gamma}{\beta y - \alpha}, \tag{2.36}$$

which can be rearranged to

$$\frac{\beta y - \alpha}{y}dy + \frac{\delta x - \gamma}{x}dx = 0. \tag{2.37}$$

Finally, one gets the equation

$$\delta x - \gamma \ln x + \beta y - \alpha \ln y = A, \tag{2.38}$$

with an arbitrary constant A.

2.3.2 Goodwin model

Richard M. Goodwin first introduced a nonlinear model to understand business cycle behaviour and in economics, and he adapted his model from Lotka-Volterra's predator-prey system.

Goodwin's original model is set up with the following assumptions which we note from [26]:

(a) Steady technical progress,
(b) Steady growth in the labour force,
(c) Only two factors of production, labour and capital; both homogeneous and non-specific,
(d) All quantities are real and net,
(e) All wages consumed, all profits saved and invested,
(f) A constant capital-output ratio,
(g) A real wage rate that rises in the neighbourhood of full employment.

In addition to the assumptions, symbols which are used in the dynamical system are listed in the following table.

Table 2.1 : Variables and their meanings of Goodwin's original model

Variable List	
q	Output
k	Capital
w	Wage
$a = a_0 e^{\alpha t}$	Labour productivity, α is the growth parameter
$s = q/k = 1/\sigma$	Capital productivity
$k/q = \sigma$	Capital-output ratio
$u = w/a$	Workers' share of product
$(1 - w/a)$	Capitalists' share of product
$(1 - w/a)q = \dot{k}$	Surplus = Profit =Savings = Investments
$\dot{k}/k = \dot{q}/q = (1 - w/a)/\sigma$	Profit rate
$n = n_0 e^{\beta t}$	Labour supply, β is the growth parameter
$l = q/a$	Employment
$v = l/n$	Employment rate

As explained in Ref. [26], logarithmic differentiation of the employment rate and the workers' share of product gives

$$\begin{aligned} \frac{\dot{v}}{v} &= \frac{\dot{l}}{l} - \frac{\dot{n}}{n} = \frac{\dot{q}}{q} - \alpha - \beta, \\ \frac{\dot{u}}{u} &= \frac{\dot{w}}{w} - \alpha, \end{aligned} \tag{2.39}$$

with $\frac{\dot{w}}{w} = \rho v - \gamma$ being the linearized Phillips curve. After that one obtains the Goodwin model as

$$\begin{aligned} \dot{v} &= \left[\frac{1-u}{\sigma} - (\alpha + \beta)\right] v, \\ \dot{u} &= [-(\gamma + \alpha) + \rho v]\, u. \end{aligned} \tag{2.40}$$

The origin $(0,0)$ is an equilibrium, and the nonzero equilibrium (v^*, u^*) is found as

$$\begin{aligned} v^* &= \frac{\gamma + \alpha}{\rho}, \\ u^* &= [1 - (\beta + \alpha)\sigma]. \end{aligned} \tag{2.41}$$

To provide economical meaning, Goodwin assumes that $u^* > 0$, and this means that $\frac{1}{\sigma} > (\alpha + \beta)$.

Using the linear approximation method at the two equilibria, the following Jacobian matrices are obtained:

$$J(0,0) = \begin{bmatrix} \frac{1}{\sigma} - (\alpha + \beta) & 0 \\ 0 & -(\gamma + \alpha) \end{bmatrix}, \qquad J(v^*, u^*) = \begin{bmatrix} 0 & -\frac{1}{\sigma} v^* \\ \rho u^* & 0 \end{bmatrix}. \tag{2.42}$$

For the origin, $J(0,0)$ is a diagonal matrix, and eigenvalues are real with opposite sign, so the origin is a saddle point. Besides, at (v^*, u^*), eigenvalues are purely imaginary. The stability is undetermined, and one proceeds exactly in the case of the predator-prey model as follows. The system **(2.40)** is considered as

$$\begin{aligned} \frac{dv}{dt} &= [s-(\alpha+\beta)-us]v, \\ \frac{du}{dt} &= [-(\gamma+\alpha)+\rho v]u. \end{aligned} \tag{2.43}$$

Dividing these equations one gets

$$\frac{du}{dv} = \frac{\rho v-\gamma-\alpha}{s-(\alpha+\beta)-us}\frac{u}{v}, \tag{2.44}$$

or equivalent,

$$[s-(\alpha+\beta)-us]vdu+[(\gamma+\alpha)-\rho v]udv = 0. \tag{2.45}$$

We organize to

$$\left[\frac{s-(\alpha+\beta)}{u}-s\right]du+\left[\frac{(\gamma+\alpha)}{v}-\rho\right]dv = 0. \tag{2.46}$$

and integrating **(2.46)**, one gets

$$[s-(\alpha+\beta)]\ln u - su + (\gamma+\alpha)\ln v - \rho v = A. \tag{2.47}$$

The numerical solution of the system **(2.40)** is illustrated by the parameters $\sigma = 3$, $\alpha = 0.001$, $\beta = 0.001$, $\gamma = 0.95$, $\rho = 1$ as in Figure **(2.6)**.

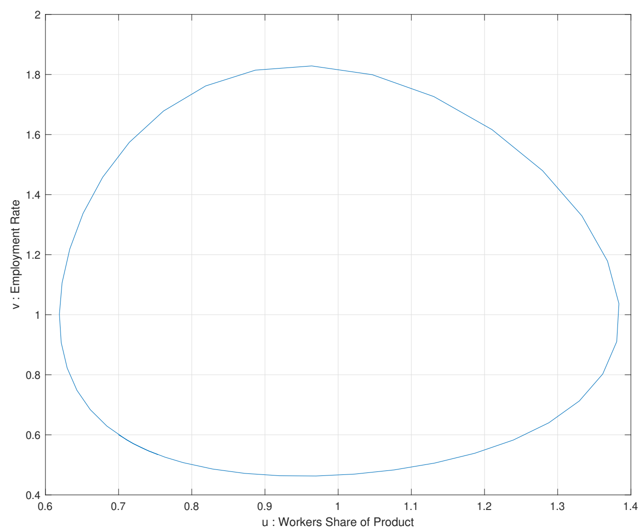


Figure 2.6 : Goodwin cycle

2.3.3 The Phillips curve

Ref. [26] said that Alban W. Phillips developed Phillips curve to show a statistical relationship between unemployment rate and money wage rate. Phillips curve indicates that there is nonlinear inverse relation between unemployment and money wage:

$$\frac{\dot{w}}{w} = f(U), \qquad f' < 0, \tag{2.48}$$

where U is the unemployment and the rate of change of the money wage denoted as $\dot{w}/w$.

Phillips used UK's data in 1861-1957 to illustrate his work in Figure **(2.7)**. To use this curve as a policy instrument, curve shows relationship between price level and unemployment instead of demonstrating relationship between money wage and unemployment under the assumption of choosing markets as perfectly competitive or monopolistic. In the Keynesian framework, according to Phillips curve, inflation exceeds real wages, so labour demand increases.

Figure **(2.8)** demonstrates relationship between unemployment rate and inflation in UK for 1861–1913 and 1948–1957 separately.

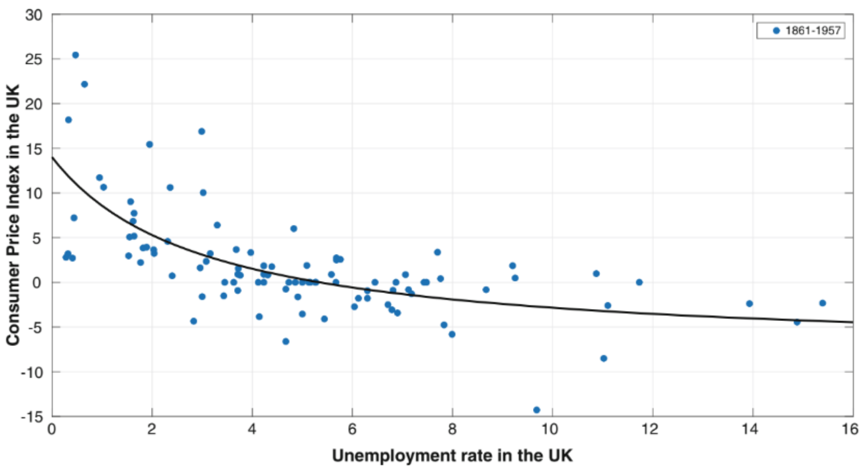


Figure 2.7 : Relation between unemployment rate and inflation in the United Kingdom, 1861- 1957.

Figure **(2.7)** was provided from Ref. [26].

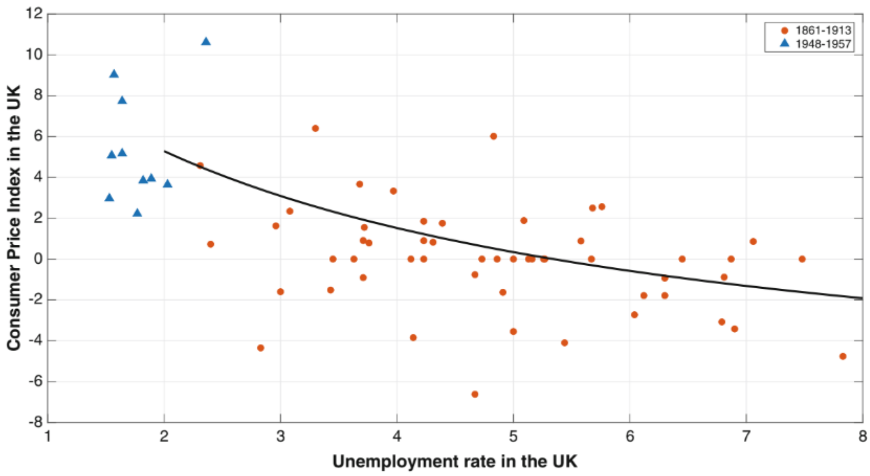


Figure 2.8 : Relation between unemployment rate and inflation in the United Kingdom, 1861–1913 (orange dots) and 1948–1957 (blue diamonds).

Figure **(2.8)** was provided from Ref. [26].

Phillips curve has been criticized from different aspects. Although there is a relationship between unemployment and inflation in the short run, it is not possible to be generalized for the long run because workers and employers make an agreement on wage with considering inflation to arrange price increases at rates near expected inflation. Besides, characteristics of economy determines the natural level of employment and rise in inflation causes rise in employment for a short time.

The information in Subsection 2.3 was provided from Ref. [26].

2.4 Generalizations of the Goodwin Model

Authors of [1] analyze the employment-distribution effect as the cycle generation factor. They focus on four modifications and it is assumed that modifications make system more realistic. The modifications are

- Production is determined according to the demand surplus,
- When employment increases, work intensity may decrease,
- Non-neutral technical progress and variability of technical progress may be allowed,
- Money wage and price setting behaviour determine the real wage rate.

Upon the modifications, the general model which is constructed by four non-linear differential equations, the state variables being the rate of employment, wage share, rate of capacity utilization, actual capital coefficient respectively, is given as

$$\dot{\beta} = [\varphi(\beta)+f(\lambda)-u(\lambda)-n]\beta, \quad \textbf{(2.49a)}$$

$$\dot{\lambda} = [\varphi(\beta)+\phi_1(\beta)-u(\lambda)-\phi_2(\theta)]\lambda, \quad \textbf{(2.49b)}$$

$$\dot{\theta} = [\psi(\lambda)+f(\lambda)-z(\lambda,v)]\theta, \quad \textbf{(2.49c)}$$

$$\dot{v} = [-f(\lambda)+z(\lambda,v)]v. \quad \textbf{(2.49d)}$$

The functions appearing in the system are

$$\varphi(\beta) = -\gamma_1+\gamma_2\beta, \quad \textbf{(2.50a)}$$

$$f(\lambda) = \delta[g(1-\lambda)-s_w], \quad \textbf{(2.50b)}$$

$$u(\lambda) = \mu_1+\mu_2\nu_1+\mu_2\nu_2\lambda, \quad \textbf{(2.50c)}$$

$$\phi_1(\beta) = -\rho_0+\rho_1\beta, \quad \textbf{(2.50d)}$$

$$\phi_2(\theta) = b_2(1-a_3)\theta/(1-a_3b_3), \quad \textbf{(2.50e)}$$

$$\psi(\lambda) = \nu_1-\mu_1-\mu_2\nu_1+\nu_2(1-\mu_2)\lambda, \quad \textbf{(2.50f)}$$

$$z(\lambda,v) = c(1-\lambda)/v, \quad \textbf{(2.50g)}$$

with the constants

$$\rho_0 = \frac{a_1(1-b_3)-b_1(1-a_3)}{1-a_3b_3}, \quad \textbf{(2.51a)}$$

$$\rho_1 = \frac{a_2(1-b_3)}{1-a_3b_3}. \quad \textbf{(2.51b)}$$

The employment rate which affects the work intensity is described by $\varphi(\beta)$. $f(\lambda)$ represents the rate of actual production. $u(\lambda)$ is the rate of technical capital productivity. $\phi_1(\beta)$ and $\phi_2(\theta)$ are for the real wage which depends on employment and capacity utilization respectively. $\psi(\lambda)$ is for the rate of technical capital coefficient. $z(\lambda,v)$ is the rate of the capital stock.

The following parameter restrictions are applied.

Table 2.2 : General parameter restrictions.

General parameter restrictions	
$\mu_1 \geq 0, \quad 1 \geq \mu_2 \geq 0$	$b_1 \geq 0, \quad b_2 \geq 0, \quad 1 \geq b_3 \geq 0$
$\nu_1 \geq 0, \quad \nu_2 > 0$	$1 > c > 0$
$n \geq 0$	$1 > s_\pi > s_w > 0$
$\gamma_1 \geq 0, \quad \gamma_2 \geq 0$	$\delta > 0$
$a_1 > 0, \quad a_2 > 0, \quad 1 \geq a_3 \geq 0$	$1 > a_3 b_3$
$g := c - (s_\pi - s_w) > 0$	

The information in Subsection 2.4 was provided from Ref. [1].

Remark 1. *The content of Chapter 2 aims at providing brief information about the tools used in this thesis work from the dynamical systems theory and mentions the results of the article [1] of which main model is used as a basis for the problems studied in this work. The information available in this Chapter is collected from the references provided and therefore is not to be understood as an original content submitted by the author of this thesis in fulfillment of this degree.*

3. ANALYSIS OF THE MODIFIED DELAYED GOODWIN MODEL

This Chapter contains the original work performed in this thesis study. As we explained in the Introduction, the main model

$$\dot{\beta} = [\varphi(\beta) + f(\lambda) - u(\lambda) - n]\beta, \quad \textbf{(3.1a)}$$

$$\dot{\lambda} = [\varphi(\beta) + \phi_1(\beta) - u(\lambda) - \phi_2(\theta)]\lambda, \quad \textbf{(3.1b)}$$

$$\dot{\theta} = [\psi(\lambda) + f(\lambda) - z(\lambda, v)]\theta, \quad \textbf{(3.1c)}$$

$$\dot{v} = [-f(\lambda) + z(\lambda, v)]v \quad \textbf{(3.1d)}$$

is studied in [1] in various subcases from the point of stability. We consider three of these cases: (i) variable speed of technical progress and changes in work intensity, (ii) non-neutral technical progress, and (iii) the full model.

Section 3.1 is devoted to case (i) with a delay term and occurrence of a Hopf bifurcation is studied at the nonzero equilibrium point in a two-dimensional subsystem of **(3.1)**. After that case, (ii) is studied in the delayed context in Section 3.2. For the full model in case (iii), we evaluate Lyapunov exponents in the non-delayed four-dimensional model in the unstable region of parameters. Finally, we present the calculations for determining the direction and the stability of Hopf bifurcation in Section 3.4 for the case (i).

3.1 Variable Speed of Technical Progress and Changes in Work Intensity

According to the analysis in [1], in this case, the system **(3.1)** reduces to

$$\dot{\beta} = [\varphi(\beta) + f(\lambda) - u(\lambda) - n]\beta, \quad \textbf{(3.2a)}$$

$$\dot{\lambda} = [\varphi(\beta) + \phi_1(\beta) - u(\lambda)]\lambda, \quad \textbf{(3.2b)}$$

$$\dot{v} = [-f(\lambda) + z(\lambda, v)]v, \quad \textbf{(3.2c)}$$

$$\theta = \frac{1}{\sigma v} \quad \textbf{(3.2d)}$$

where σ is a constant. For this case, the parameters are chosen to satisfy $\mu_1 = 0$, $\mu_2 = 1$, $v_1 > 0$, $v_2 > 0$. As the first two equations **(3.2a)**, **(3.2b)** form a decoupled system, we will perform our analysis on the subsystem formed by these two equations. The functions appearing in the system are given as

$$\varphi(\beta) = -\gamma_1 + \gamma_2\beta, \tag{3.3a}$$

$$f(\lambda) = \delta[g(1-\lambda) - s_w], \tag{3.3b}$$

$$u(\lambda) = \nu_1 + \nu_2\lambda, \tag{3.3c}$$

$$\phi_1(\beta) = -\rho_0 + \rho_1\beta \tag{3.3d}$$

with the constants

$$\rho_0 = \frac{a_1(1-b_3) - b_1(1-a_3)}{1-a_3b_3}, \tag{3.4a}$$

$$\rho_1 = \frac{a_2(1-b_3)}{1-a_3b_3}. \tag{3.4b}$$

In the subsystem **(3.2a)**, **(3.2b)**

$$\dot{\beta} = [\varphi(\beta) + f(\lambda) - u(\lambda) - n]\beta, \tag{3.5a}$$

$$\dot{\lambda} = [\varphi(\beta) + \phi_1(\beta) - u(\lambda)]\lambda, \tag{3.5b}$$

the function $\phi_1(\beta)$ originates from the relation between the inflation rate and employment rate. There are supporting arguments in literature that this relation actually includes a time-delay, therefore, we consider this term with a time delay and consider the delayed system

$$\dot{\beta}(t) = \Big[\varphi\big(\beta(t)\big) + f\big(\lambda(t)\big) - u\big(\lambda(t)\big) - n\Big]\beta(t), \tag{3.6a}$$

$$\dot{\lambda}(t) = \Big[\varphi\big(\beta(t)\big) + \phi_1\big(\beta(t-\tau)\big) - u\big(\lambda(t)\big)\Big]\lambda(t). \tag{3.6b}$$

It takes the form

$$\dot{\beta}(t) = \Big[\beta_0 + \gamma_2\beta(t) - \delta_0\lambda(t)\Big]\beta(t), \tag{3.7a}$$

$$\dot{\lambda}(t) = \Big[\lambda_0 - \nu_2\lambda(t) + \gamma_2\beta(t) + \rho_1\beta(t-\tau)\Big]\lambda(t). \tag{3.7b}$$

where

$$\beta_0 = (g - s_w)\delta - \gamma_1 - \nu_1 - n, \tag{3.8}$$

$$\lambda_0 = -\gamma_1 - \rho_0 - \nu_1, \tag{3.9}$$

$$\delta_0 = \nu_2 + g\delta. \tag{3.10}$$

Actually, the system **(3.7)** is a delayed logistic type Lotka-Volterra system. Although there is a vast range of publications which deal with delayed versions of the Lotka-Volterra systems, we did not find one that exactly matches **(3.7)**. Therefore, to the best of our knowledge, this system has not been considered in the literature before.

We shall be dealing with the equilibrium

$$\beta_e = \frac{g\delta(\gamma_1+\nu_1+\nu_2+\rho_0)-\nu_2(n+s_w\delta-\rho_0)}{\rho_1\nu_2+g\delta(\rho_1+\gamma_2)}, \tag{3.11}$$

$$\lambda_e = \frac{\gamma_2(g\delta+\rho_0-n-s_w\delta)-\rho_1(n+\gamma_1+s_w\delta+\nu_1-g\delta)}{\rho_1\nu_2+g\delta(\rho_1+\gamma_2)}. \tag{3.12}$$

The related Jacobi matrices at the equilibrium (β_e,λ_e) are

$$J_0 = \begin{bmatrix}\gamma_2\beta_e & -\delta_0\beta_e \\ \gamma_2\lambda_e & -\nu_2\lambda_e\end{bmatrix}, \qquad J_\tau = \begin{bmatrix}0 & 0 \\ \rho_1\lambda_e & 0\end{bmatrix}. \tag{3.13}$$

The characteristic equation for an eigenvalue of the linearization of the system **(3.7)**

$$|J_0+e^{-x\tau}J_\tau-xI| = \begin{vmatrix}\gamma_2\beta_e-x & -\delta_0\beta_e \\ \gamma_2\lambda_e+e^{-x\tau}\rho_1\lambda_e & -\nu_2\lambda_e-x\end{vmatrix} = 0 \tag{3.14}$$

gives that

$$P(x) = x^2+p_0x+r_0+q_0e^{-x\tau} = 0 \tag{3.15}$$

with

$$p_0 = \nu_2\lambda_e-\gamma_2\beta_e, \tag{3.16a}$$

$$r_0 = g\delta\gamma_2\beta_e\lambda_e, \tag{3.16b}$$

$$q_0 = \delta_0\rho_1\beta_e\lambda_e. \tag{3.16c}$$

When $\tau=0$, **(3.15)** takes the form

$$P_0(x) = x^2+p_0x+r_0+q_0 = 0. \tag{3.17}$$

Lemma 1. *Since* $r_0+q_0>0$, *if*

$$p_0>0 \tag{3.18}$$

all roots of the characteristic equation **(3.17)** *have negative real parts and thus the equilibrium point is stable at* $\tau=0$.

Let $x = i\omega$ be a root of **(3.15)**, $\omega > 0$. Then we have

$$p_0\omega = \; = \; q_0 \sin(\omega\tau), \tag{3.19a}$$

$$\omega^2 - r_0 \; = \; q_0 \cos(\omega\tau). \tag{3.19b}$$

Squaring and adding up both sides we get

$$\omega^4 + (p_0^2 - 2r_0)\omega^2 + r_0^2 - q_0^2 = 0. \tag{3.20}$$

Putting $\omega^2 = z > 0$, we obtain

$$h(z) = z^2 + (p_0^2 - 2r_0)z + r_0^2 - q_0^2 = 0. \tag{3.21}$$

In order to have a pure imaginary eigenvalue $x = i\omega$, this equation must have at least one positive solution $\tilde{z}$. Defining the discriminant

$$\Delta = (p_0^2 - 2r_0)^2 - 4(r_0^2 - q_0^2) \tag{3.22}$$

we have the following results.

(H1) $\Delta < 0$: No real $\tilde{z}$.

(H2) $\Delta = 0$, $2r_0 - p_0^2 \leq 0$: No positive $\tilde{z}$.

(H3) $\Delta = 0$, $2r_0 - p_0^2 > 0$: One positive $\tilde{z}$.

(H4) $r_0^2 - q_0^2 < 0$: One positive $\tilde{z}$.

(H5) $\Delta > 0$, $r_0^2 - q_0^2 > 0$, $2r_0 - p_0^2 < 0$: No positive $\tilde{z}$.

(H6) $\Delta > 0$, $r_0^2 - q_0^2 > 0$, $2r_0 - p_0^2 > 0$: 2 positive $\tilde{z}$.

Therefore, we have the following result.

Lemma 2. *(A) In cases (H1), (H2), (H5),* **(3.21)** *has no root* $\tilde{z} > 0$. *(B) In cases (H3), (H4) and (H6),* **(3.21)** *has a positive root* $\tilde{z}$.

Suppose there exists a positive solution to **(3.21)**. Without loss of generality, we can assume that there are two positive roots $\tilde{z}_1$, $\tilde{z}_2$, hence there are two values of $\omega > 0$ as

$$\omega_1 = \sqrt{\tilde{z}_1}, \qquad \omega_2 = \sqrt{\tilde{z}_2}, \tag{3.23}$$

We evaluate the critical values of τ so that **(3.15)** has the pure imaginary root $i\omega$ as

$$\tau_k^j = \frac{1}{\omega_k}\left[\cos^{-1}\left(\frac{\omega_k^2 - r_0}{q_0}\right) + 2j\pi\right] \tag{3.24}$$

for $k = 1,2$ and $j = 0,1,2,\ldots$. Let us define

$$\tau_0 = \tau_{k_0}^0 = \min(\tau_1^0, \tau_2^0), \qquad \omega_0 = \omega_{k_0}, \qquad z_0 = \omega_0^2. \tag{3.25}$$

Now we note the following Lemma from [27].

Lemma 3. *Consider the exponential polynomial equation*

$$\begin{aligned} P(x, e^{-x\tau_1}, \ldots, e^{-x\tau_m}) = {} & x^n + p_1^{(0)} x^{n-1} + \ldots + p_{n-1}^{(0)} x + p_n^{(0)} \\ & + [p_1^{(1)} x^{n-1} + \ldots + p_{n-1}^{(1)} x + p_n^{(1)}] e^{-x\tau_1} \\ & + \ldots \\ & + [p_1^{(m)} x^{n-1} + \ldots + p_{n-1}^{(m)} x + p_n^{(m)}] e^{-x\tau_m} = 0, \end{aligned} \tag{3.26}$$

where $\tau_i \geq 0$ $(i = 1,2,\ldots,m)$ *and* $p_j^{(i)}$ $(i = 0,1,2,\ldots,m; j = 1,2,\ldots,n)$ *are constants. As* $(\tau_1, \tau_2, \ldots, \tau_m)$ *vary, the sum of the order of the zeros of* $P(x, e^{-x\tau_1}, \ldots, e^{-x\tau_m})$ *on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

According to this Lemma, we have the following conclusion.

Corollary 1. *Suppose the condition* **(3.18)** *is satisfied.*

(i) If one of the conditions listed in Lemma 2(A) is true, then, the roots of **(3.15)** *are with negative real parts for all* $\tau \geq 0$.

(ii) If one of the conditions listed in Lemma 2(B) is true, then, the roots of **(3.15)** *are with negative real parts for all* $\tau \in [0, \tau_0)$.

Lemma 4. *Suppose that one of the conditions in Lemma 2(B) holds and that* $h'(z_0) \neq 0$. *Then,* $i\omega_0$ *is a simple (i.e., not multiple) pure imaginary root of the equation* **(3.15)** *when* $\tau = \tau_0$. *Additionally, the transversality condition*

$$\frac{d(\mathrm{Re}(x(\tau)))}{d\tau}\Big|_{\tau=\tau_0} \neq 0 \tag{3.27}$$

is satisfied and the sign of $d(\mathrm{Re}(x(\tau)))/d\tau|_{\tau=\tau_0}$ *is consistent with that of* $h'(z_0)$.

Proof. First let us show that $i\omega_0$ is a simple pure imaginary root of the equation **(3.15)** when $\tau = \tau_0$ if $h'(z_0) \neq 0$. Assume that $x = i\omega_0$ is a multiple root of **(3.15)**. Then, we must have $\left.\frac{dP(x)}{dx}\right|_{x=i\omega_0} = 0$. Keep in mind that we have the eigenvalue $x = i\omega_0$ when $\tau = \tau_0$. From the equalities

$$P(i\omega_0) = (i\omega_0)^2 + p_0 i\omega_0 + r_0 + q_0 e^{-i\omega_0\tau_0} = 0, \tag{3.28}$$

$$P'(i\omega_0) = 2i\omega_0 + p_0 - \tau_0 q_0 e^{-i\omega_0\tau_0} = 0 \tag{3.29}$$

we eliminate the exponential term and get

$$p_0 - \tau_0\omega_0^2 + r_0\tau_0 + i\omega_0(2 + p_0\tau_0) = 0. \tag{3.30}$$

The real and imaginary parts of this equality must vanish, which together require $2\omega_0^2 + p_0^2 - 2r_0 = 0$, which means $h'(\omega_0^2) = h'(z_0) = 0$. This is a contradiction to the assumption, therefore $i\omega_0$ is a simple root. From this part of the Lemma, we shall make use of the result

$$\left.\frac{dP}{dx}\right|_{x=i\omega_0} \neq 0. \tag{3.31}$$

In order to show the validity of the transversality condition **(3.27)**, we evaluate the derivative of **(3.15)** and obtain

$$\left(2x + p_0 - \tau q_0 e^{-x\tau}\right)\frac{dx}{d\tau} = xq_0 e^{-x\tau} \tag{3.32}$$

which actually is

$$\frac{dP}{dx}\frac{dx}{d\tau} = xq_0 e^{-x\tau}. \tag{3.33}$$

Equation **(3.31)** allows us to proceed as

$$\left.\frac{dx}{d\tau}\right|_{\tau=\tau_0, x=i\omega_0} = \frac{i\omega_0 q_0 e^{-i\omega_0\tau_0}}{\left.\frac{dP}{dx}\right|_{x=i\omega_0}}. \tag{3.34}$$

and the denominator is nonzero within some vicinity of $x = i\omega_0$ by **(3.31)**. We obtain

$$\left.\frac{dx}{d\tau}\right|_{\tau=\tau_0, x=i\omega_0} = \frac{i\omega_0 q_0}{(2i\omega_0 + p_0)e^{i\omega_0\tau_0} - \tau q_0}. \tag{3.35}$$

Evaluating this expression at $x = i\omega_0$ we obtain

$$\begin{aligned}\left.\frac{d(\mathrm{Re}(x(\tau)))}{d\tau}\right|_{\tau=\tau_0} &= \mathrm{Re}\left(\left.\frac{dx}{d\tau}\right|_{\tau=\tau_0, x=i\omega_0}\right) \\ &= \mathrm{Re}\left(\frac{i\omega_0 q_0}{(2i\omega_0 + p_0)e^{i\omega_0\tau_0} - \tau q_0}\right) \\ &= \frac{\omega_0^2(2\omega_0^2 + p_0^2 - 2r_0)}{D} = \frac{\omega_0^2}{D}h'(\omega_0^2)\end{aligned} \tag{3.36}$$

with

$$D = [p_0 \cos(\omega_0 \tau_0) - 2\omega_0 \sin(\omega_0 \tau_0) - q_0 \tau_0]^2 + [p_0 \sin(\omega_0 \tau_0) + 2\omega_0 \cos(\omega_0 \tau_0)]^2. \tag{3.37}$$

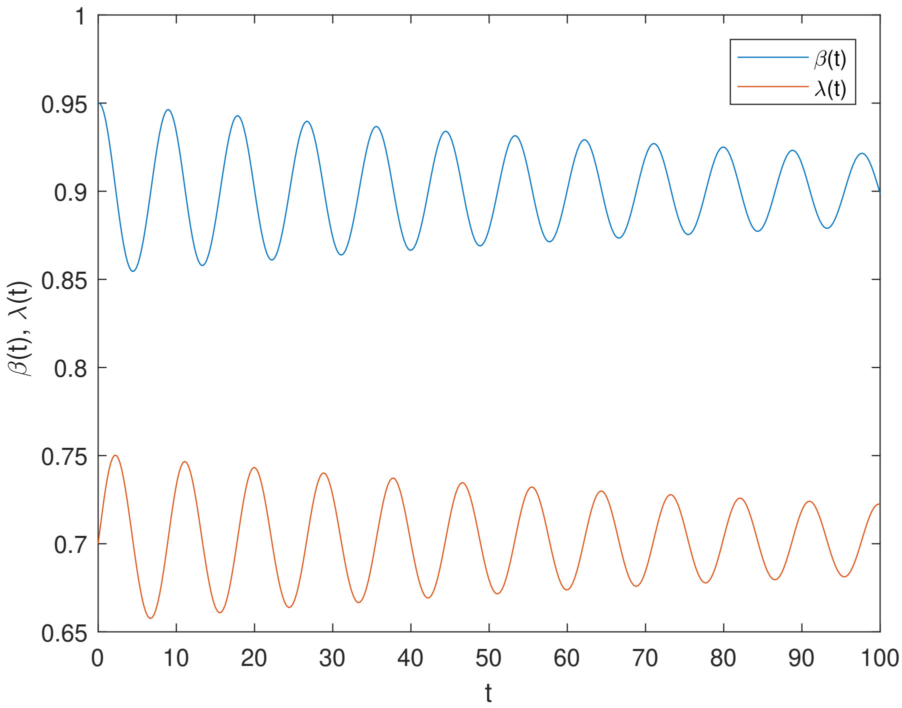
Obivously, $D \neq 0$ by **(3.31)**. □

Theorem 3. *Let $E = E(\beta_e, \lambda_e)$ be a positive equilibrium point of* **(3.7)** *in the unit box for which the condition* **(3.18)** *in Lemma 1 is satisfied and let τ_0 be defined as before.*

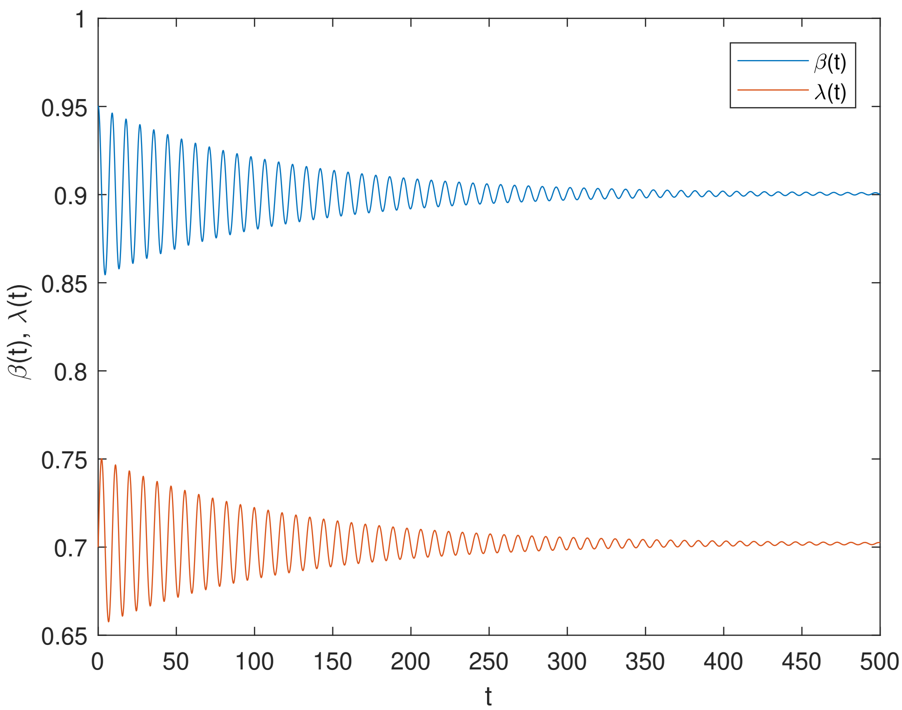
- *If one of the conditions listed in Lemma 2(A) is true, then, E is asymptotically stable for all $\tau \geq 0$.*
- *If one of the conditions listed in Lemma 2(B) is true and if $h'(z_0) \neq 0$, then E is an asymptotically stable equilibrium point when $\tau \in [0, \tau_0)$. E is unstable for $\tau \in (\tau_0, \tau_1)$, where τ_1 is the first value of τ obtained from* **(3.24)** *after τ_0. The system* **(3.7)** *undergoes a Hopf bifurcation at E when $\tau = \tau_0$.*

The result is illustrated in Figure **(3.1)**, Figure **(3.2)** and Figure **(3.3)** by using the parameter combinations as $\nu_1 = 0.02$, $\nu_2 = 0.04$, $\gamma_1 = 0.01$, $\gamma_2 = 0.012$, $\delta = 4.2$, $c = 0.38$, $n = 0.01$, $s_\pi = 0.24$, $s_w = 0.04$, $a_1 = 0.9$, $a_2 = 1$, $a_3 = 0.99$, $b_1 = 1.9$, $b_2 = 0$, $b_3 = 0.6$.

We find $\beta_e = 0.900484$, $\lambda_e = 0.702017$, $\tau_0 = 0.0348488$.

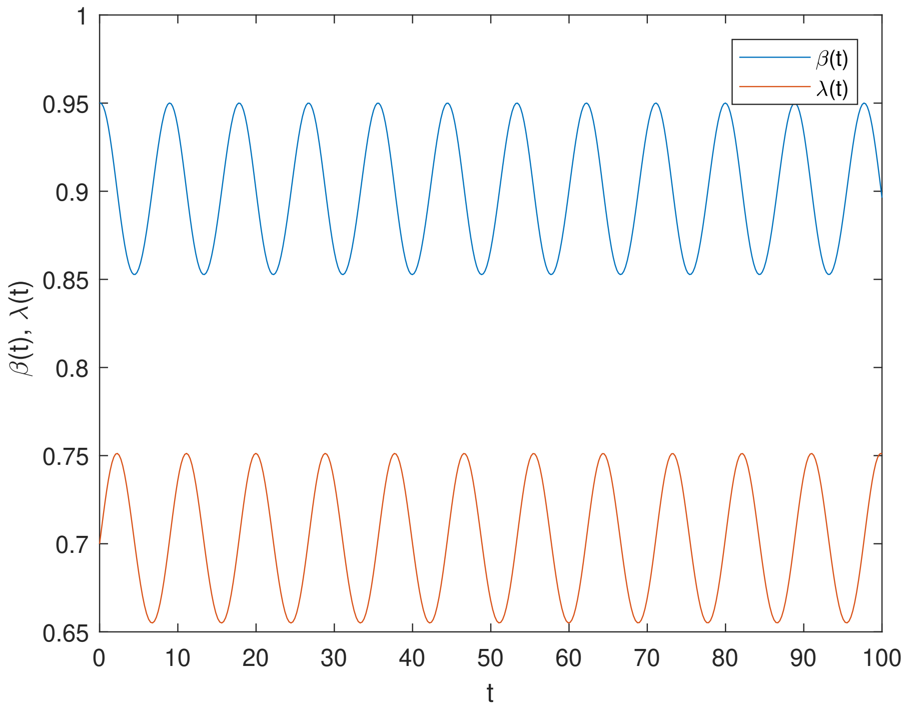


(a)

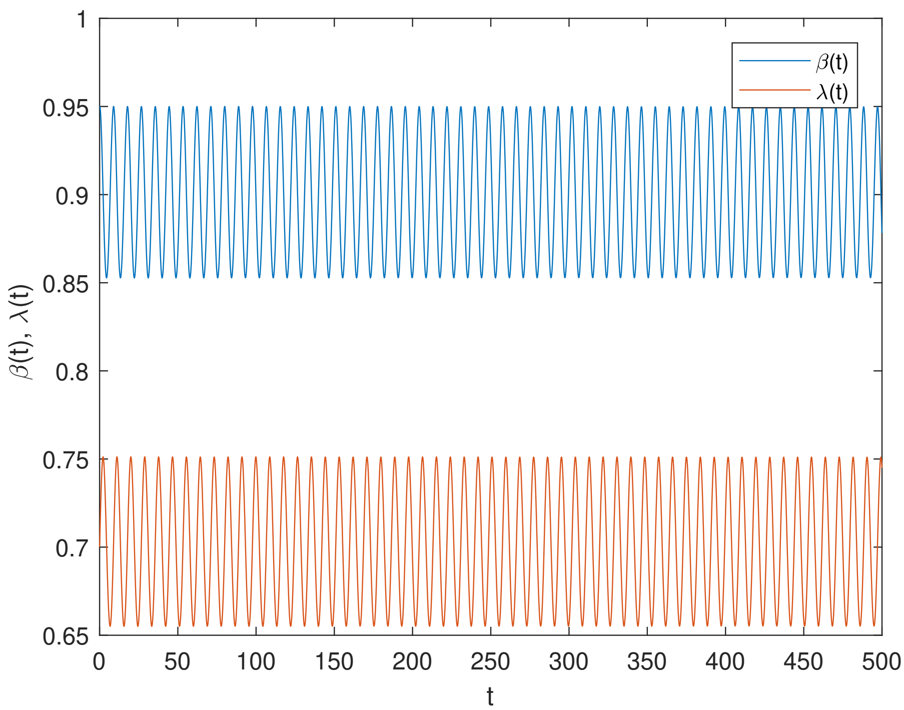


(b)

Figure 3.1 : $\tau = 0 < \tau_0 = 0.0348488$.

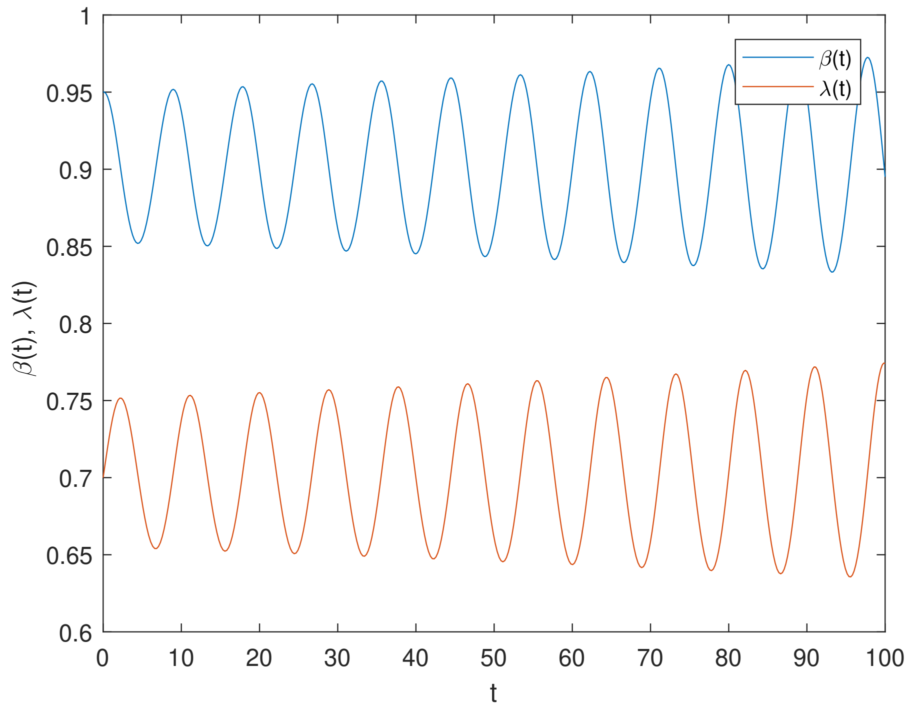


(a)

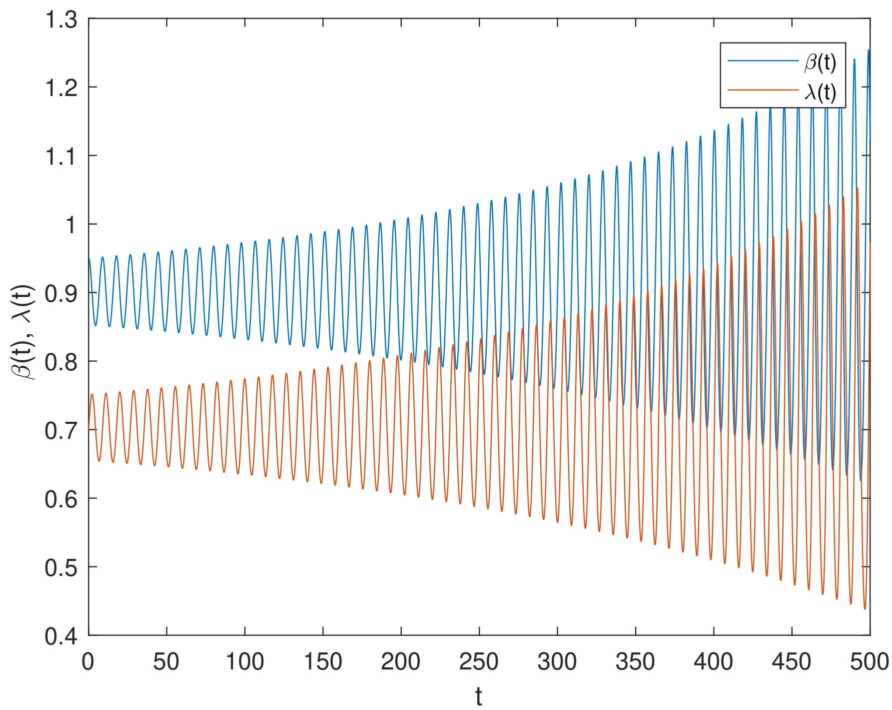


(b)

Figure 3.2 : $\tau = \tau_0 = 0.0348488$.



(a)



(b)

Figure 3.3 : $\tau = 0.05 > \tau_0 = 0.0348488$.

3.2 Non-Neutral Technical Progress

Allowing non-neutral technical progress and neglecting changes in work intensity and influence of capacity utilization, authors of [1] obtain the system

$$\dot{\beta} = [f(\lambda) - u(\lambda) - n]\beta, \quad \textbf{(3.38a)}$$

$$\dot{\lambda} = [\phi_1(\beta) - u(\lambda)]\lambda, \quad \textbf{(3.38b)}$$

$$\dot{v} = [-f(\lambda) + z(\lambda, v)]v, \quad \textbf{(3.38c)}$$

$$\dot{\theta} = [\psi(\lambda) + f(\lambda) - z(\lambda, v)]\theta, \quad \textbf{(3.38d)}$$

in which

$$\mu_1 \geq 0, \quad 0 < \mu_2 < 1, \quad \nu_1 \geq 0, \quad \nu_2 \geq 0 \quad \textbf{(3.39)}$$

and

$$u(\lambda) = \mu_1 + \mu_2\nu_1 + \mu_2\nu_2\lambda, \quad \textbf{(3.40a)}$$

$$\psi(\lambda) = \nu_1 - \mu_1 - \mu_2\nu_1(1-\mu_2)\lambda. \quad \textbf{(3.40b)}$$

As before, we are going to consider the subsystem formed by the equations **(3.38a)**, **(3.38b)**. However, first observe that as stated in [1], for the equilibrium point $(\tilde{\beta}_e, \tilde{\lambda}_e, \tilde{v}_e, \tilde{\theta}_e)$ of this system, through **(3.38c)**, **(3.38d)** one gets $\psi(\tilde{\lambda}_e) = 0$. As in [1], name this value of $\tilde{\lambda}_e$ as $\tilde{\lambda}_e^*$, which is

$$\tilde{\lambda}_e^* = \frac{\mu_1 - \nu_1(1-\mu_2)}{\nu_2(1-\mu_2)}. \quad \textbf{(3.41)}$$

Further, from **(3.38a)** and **(3.38b)**, one obtains

$$\tilde{\lambda}_e = \frac{\delta(g - s_w) - n - \mu_1 - \mu_2\nu_1}{\delta g + \mu_2\nu_2}, \quad \textbf{(3.42)}$$

$$\tilde{\beta}_e = \frac{a_1(1-b_3) - b_1(1-a_3) + (1-a_3b_3)(\mu_1 + \mu_2\nu_1 + \mu_2\nu_2\lambda_e)}{a_2(1-b_3)}. \quad \textbf{(3.43)}$$

Clearly, $\tilde{\lambda}_e^*$ and $\tilde{\lambda}_e$ must be equal. Assuming the parameters happen to occur to satisfy $\tilde{\lambda}_e^* = \tilde{\lambda}_e$, we consider the separate dynamics between $\beta(t)$ and $\lambda(t)$ in **(3.38a)** and **(3.38b)** with a delay term same as the previous subsection with the same motivation. Hence we consider the model

$$\dot{\beta}(t) = \Big[f\big(\lambda(t)\big) - u\big(\lambda(t)\big) - n\Big]\beta(t), \quad \textbf{(3.44a)}$$

$$\dot{\lambda}(t) = \Big[\phi_1\big(\beta(t-\tau)\big) - u\big(\lambda(t)\big)\Big]\lambda(t). \quad \textbf{(3.44b)}$$

Clearly, **(3.44)** is a special case of **(3.6)** in which $\varphi(\beta) = -\gamma_1 + \gamma_2\beta \equiv 0$, up to a change in the constants included in $u(\lambda)$. Therefore, the bifurcation analysis performed in the previous subsection with $\gamma_1 = \gamma_2 = 0$ will follow by quite similar lines with a slight change in the coefficients. We shall briefly provide the main results for this special case.

For completeness, let us remember that

$$f(\lambda) = \delta[g(1-\lambda) - s_w], \tag{3.45a}$$

$$\phi_1(\beta) = -\rho_0 + \rho_1\beta \tag{3.45b}$$

with the constants

$$\rho_0 = \frac{a_1(1-b_3) - b_1(1-a_3)}{1-a_3b_3}, \tag{3.46a}$$

$$\rho_1 = \frac{a_2(1-b_3)}{1-a_3b_3}. \tag{3.46b}$$

It takes the form

$$\dot{\beta}(t) = \left[\tilde{\beta}_0 - \tilde{\delta}_0\lambda(t)\right]\beta(t), \tag{3.47a}$$

$$\dot{\lambda}(t) = \left[\tilde{\lambda}_0 - \mu_2\nu_2\lambda(t) + \rho_1\beta(t-\tau)\right]\lambda(t). \tag{3.47b}$$

where

$$\tilde{\beta}_0 = (g - s_w)\delta - \mu_1 - \mu_2\nu_1 - n, \tag{3.48}$$

$$\tilde{\lambda}_0 = -\rho_0 - \mu_1 - \mu_2\nu_1, \tag{3.49}$$

$$\tilde{\delta}_0 = g\delta + \mu_2\nu_2. \tag{3.50}$$

In addition to **(3.43)**, **(3.42)**, we can express the nonzero equilibrium ponits as

$$\tilde{\lambda}_e = \frac{\tilde{\beta}_0}{\tilde{\delta}_0}, \qquad \tilde{\beta}_e = \frac{\mu_2\nu_2\tilde{\beta}_0 - \tilde{\lambda}_0\tilde{\delta}_0}{\rho_1\tilde{\delta}_0}. \tag{3.51}$$

The related Jacobi matrices at the equilibrium $\tilde{E}(\tilde{\beta}_e, \tilde{\lambda}_e)$ are

$$J_0 = \begin{bmatrix} 0 & -\tilde{\delta}_0\tilde{\beta}_e \\ 0 & -\mu_2\nu_2\tilde{\lambda}_e \end{bmatrix}, \qquad J_\tau = \begin{bmatrix} 0 & 0 \\ \rho_1\tilde{\lambda}_e & 0. \end{bmatrix} \tag{3.52}$$

The characteristic equation for an eigenvalue of the linearization of the system **(3.47)**

$$|J_0 + e^{-x\tau}J_\tau - xI| = \begin{vmatrix} -x & -\tilde{\delta}_0\tilde{\beta}_e \\ e^{-x\tau}\rho_1\tilde{\lambda}_e & -\mu_2\nu_2\tilde{\lambda}_e - x \end{vmatrix} = 0 \tag{3.53}$$

gives that

$$P(x) = x^2 + \tilde{p}_0 x + \tilde{q}_0 e^{-x\tau} = 0 \quad \textbf{(3.54)}$$

with

$$\tilde{p}_0 = \mu_2 \nu_2 \tilde{\lambda}_e, \quad \textbf{(3.55a)}$$

$$\tilde{q}_0 = \tilde{\delta}_0 \rho_1 \tilde{\beta}_e \tilde{\lambda}_e. \quad \textbf{(3.55b)}$$

When $\tau = 0$, **(3.54)** takes the form

$$P_0(x) = x^2 + \tilde{p}_0 x + \tilde{q}_0 = 0 \quad \textbf{(3.56)}$$

Lemma 5. *Since $\tilde{p}_0 > 0$ and $\tilde{q}_0 > 0$, all roots of the characteristic equation* **(3.17)** *have negative real parts and thus the equilibrium point is stable at $\tau = 0$.*

Let $x = i\omega$ be a root of **(3.54)**, $\omega > 0$. Then we have

$$\tilde{p}_0 \omega = \; = \tilde{q}_0 \sin(\omega\tau), \quad \textbf{(3.57a)}$$

$$\omega^2 = \tilde{q}_0 \cos(\omega\tau). \quad \textbf{(3.57b)}$$

Squaring and adding up both sites we get

$$\omega^4 + \tilde{p}_0^2 \omega^2 - \tilde{q}_0^2 = 0. \quad \textbf{(3.58)}$$

Putting $\omega^2 = z > 0$, we obtain

$$h(z) = z^2 + \tilde{p}_0^2 z - \tilde{q}_0^2 = 0. \quad \textbf{(3.59)}$$

In order to have a pure imaginary eigenvalue $x = i\omega$, this equation must have at least one positive solution $\hat{z}$. Since $h(0) = -\tilde{q}_0^2 < 0$, **(3.59)** has at least one positive solution $\hat{z}$. Without loss of generality, we can assume that there are two positive roots $\hat{z}_1$, $\hat{z}_2$, hence there are two values of $\tilde{\omega} > 0$ as

$$\tilde{\omega}_1 = \sqrt{\hat{z}_1}, \qquad \tilde{\omega}_2 = \sqrt{\hat{z}_2}, \quad \textbf{(3.60)}$$

We evaluate the critical values of τ so that **(3.54)** has the pure imaginary root $i\tilde{\omega}$ as

$$\tilde{\tau}_k^j = \frac{1}{\tilde{\omega}_k}\left[\cos^{-1}\left(\frac{\tilde{\omega}_k^2}{\tilde{q}_0}\right) + 2j\pi\right] \quad \textbf{(3.61)}$$

for $k = 1, 2$ and $j = 0, 1, 2, \ldots$. Let us define

$$\tilde{\tau}_0 = \tilde{\tau}_{k_0}^0 = \min(\tilde{\tau}_1^0, \tilde{\tau}_2^0), \qquad \tilde{\omega}_0 = \tilde{\omega}_{k_0}, \qquad \hat{z}_0 = \tilde{\omega}_0^2. \quad \textbf{(3.62)}$$

Following this, we have the following conclusion.

Corollary 2. *All the roots of* **(3.54)** *are with negative real parts for all* $\tau \in [0, \tilde{\tau}_0)$.

Lemma 6. $i\tilde{\omega}_0$ *is a simple (i.e., not multiple) pure imaginary root of the equation* **(3.15)** *when* $\tau = \tilde{\tau}_0$. *Additionally, we have*

$$\frac{d(\mathrm{Re}(x(\tau)))}{d\tau}\Big|_{\tau=\tilde{\tau}_0} > 0 \tag{3.63}$$

and therefore the transversality condition is satisfied.

Proof. The proof follows when we take $\tilde{r}_0 = 0$ in the proof of Lemma 4. We repeat the calculations for the convenience of the reader. First let us show that $i\tilde{\omega}_0$ is a simple pure imaginary root of the equation **(3.54)** when $\tau = \tilde{\tau}_0$. Assume that $x = i\tilde{\omega}_0$ is a multiple root of **(3.15)**. Then, we must have $\frac{dP(x)}{dx}\Big|_{x=i\tilde{\omega}_0} = 0$. From the equalities

$$P(i\tilde{\omega}_0) = (i\tilde{\omega}_0)^2 + \tilde{p}_0 i\tilde{\omega}_0 + \tilde{q}_0 e^{-i\tilde{\omega}_0\tilde{\tau}_0} = 0, \tag{3.64}$$

$$P'(i\tilde{\omega}_0) = 2i\tilde{\omega}_0 + \tilde{p}_0 - \tilde{\tau}_0\tilde{q}_0 e^{-i\tilde{\omega}_0\tilde{\tau}_0} = 0 \tag{3.65}$$

we eliminate the exponential term and get

$$\tilde{p}_0 - \tilde{\tau}_0\tilde{\omega}_0^2 + i\tilde{\omega}_0(2 + \tilde{p}_0\tilde{\tau}_0) = 0. \tag{3.66}$$

The imaginary part cannot vanish as $\tilde{p}_0 > 0$, $\tilde{\tau}_0 > 0$. Therefore the assumption is false and

$$\frac{dP}{dx}\Big|_{x=i\tilde{\omega}_0} \neq 0. \tag{3.67}$$

Further, let us note that if we put the conditions $\tilde{p}_0 - \tilde{\tau}_0\tilde{\omega}_0^2 = 0$ and $2 + \tilde{p}_0\tilde{\tau}_0 = 0$, they together require $2\tilde{\omega}_0^2 + \tilde{p}_0^2 = 0$, which means $h'(\tilde{\omega}_0^2) = h'(\hat{z}_0) = 0$. In order to prove the second part, we evaluate the derivative of **(3.54)** and obtain

$$\left(2x + \tilde{p}_0 - \tau\tilde{q}_0 e^{-x\tau}\right)\frac{dx}{d\tau} = x\tilde{q}_0 e^{-x\tau} \tag{3.68}$$

which actually is

$$\frac{dP}{dx}\frac{dx}{d\tau} = x\tilde{q}_0 e^{-x\tau}. \tag{3.69}$$

Equation **(3.67)** allows us to proceed as

$$\frac{dx}{d\tau}\Big|_{\tau=\tilde{\tau}_0, x=i\tilde{\omega}_0} = \frac{i\tilde{\omega}_0\tilde{q}_0 e^{-i\tilde{\omega}_0\tilde{\tau}_0}}{\frac{dP}{dx}\Big|_{x=i\tilde{\omega}_0}} \tag{3.70}$$

and the denominator is nonzero within some vicinity of $x = i\tilde{\omega}_0$ by **(3.67)**. We obtain

$$\left.\frac{dx}{d\tau}\right|_{\tau=\tilde{\tau}_0, x=i\omega_0} = \frac{i\tilde{\omega}_0\tilde{q}_0}{(2i\tilde{\omega}_0+\tilde{p}_0)e^{i\tilde{\omega}_0\tilde{\tau}_0}-\tilde{\tau}_0\tilde{q}_0}. \tag{3.71}$$

Evaluating this expression at $x = i\omega_0$ we obtain

$$\begin{aligned}\left.\frac{d(\mathrm{Re}(x(\tau)))}{d\tau}\right|_{\tau=\tilde{\tau}_0} &= \mathrm{Re}\left(\left.\frac{dx}{d\tau}\right|_{\tau=\tilde{\tau}_0, x=i\tilde{\omega}_0}\right)\\ &= \mathrm{Re}\left(\frac{i\tilde{\omega}_0\tilde{q}_0}{(2i\tilde{\omega}_0+\tilde{p}_0)e^{i\tilde{\omega}_0\tilde{\tau}_0}-\tilde{\tau}_0\tilde{q}_0}\right)\\ &= \frac{\tilde{\omega}_0^2(2\tilde{\omega}_0^2+\tilde{p}_0^2)}{\tilde{D}} = \frac{\tilde{\omega}_0^2 h'(\tilde{\omega}_0^2)}{\tilde{D}}\end{aligned} \tag{3.72}$$

with

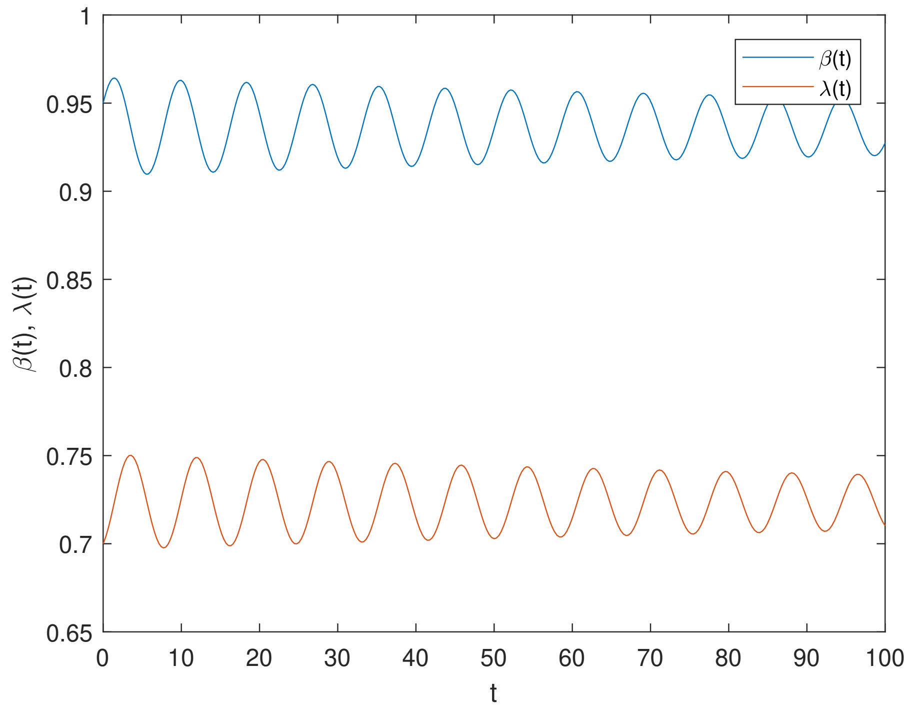
$$\tilde{D} = [\tilde{p}_0\cos(\tilde{\omega}_0\tilde{\tau}_0) - 2\tilde{\omega}_0\sin(\tilde{\omega}_0\tilde{\tau}_0) - \tilde{q}_0\tilde{\tau}_0]^2 + [\tilde{p}_0\sin(\tilde{\omega}_0\tilde{\tau}_0) + 2\tilde{\omega}_0\cos(\tilde{\omega}_0\tilde{\tau}_0)]^2. \tag{3.73}$$

Obviously, $\tilde{D} \neq 0$ by **(3.67)**. □

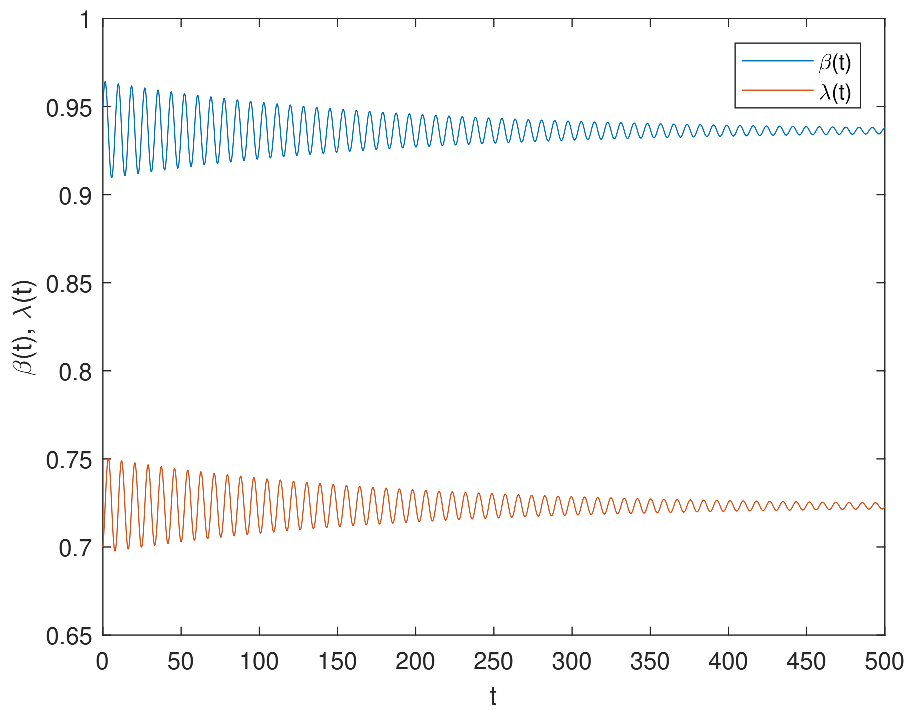
Theorem 4. *Let $\tilde{E} = \tilde{E}(\tilde{\beta}_e, \tilde{\lambda}_e)$ be a positive equilibrium point of* **(3.44)** *and let $\tilde{\tau}_0$ be defined as before. $\tilde{E}$ is an asymptotically stable equilibrium point when $\tau \in [0, \tilde{\tau}_0)$. $\tilde{E}$ is unstable for $\tau \in (\tilde{\tau}_0, \tilde{\tau}_1)$, where $\tilde{\tau}_1$ is the first value of τ obtained from* **(3.61)** *after $\tilde{\tau}_0$. The system* **(3.47)** *undergoes a Hopf bifurcation at $\tilde{E}$ when $\tau = \tilde{\tau}_0$.*

The result is illustrated Figure **(3.4)**, Figure **(3.5)** and Figure **(3.6)** by using the parameter combination $\mu_1 = 0.018$, $\mu_2 = 0.5$, $\nu_1 = 0.015$, $\nu_2 = 0.03$, $\gamma_1 = 0$, $\gamma_2 = 0$, $\delta = 4$, $c = 0.4$, $n = 0.01$, $s_\pi = 0.24$, $s_w = 0.04$, $a_1 = 0.9$, $a_2 = 1$, $a_3 = 1$, $b_1 = 1.9$, $b_2 = 0$, $b_3 = 0.6$.

We find $\tilde{\beta}_e = 0.936626$, $\tilde{\lambda}_e = 0.741718$, $\tilde{\tau}_0 = 0.0196509$.

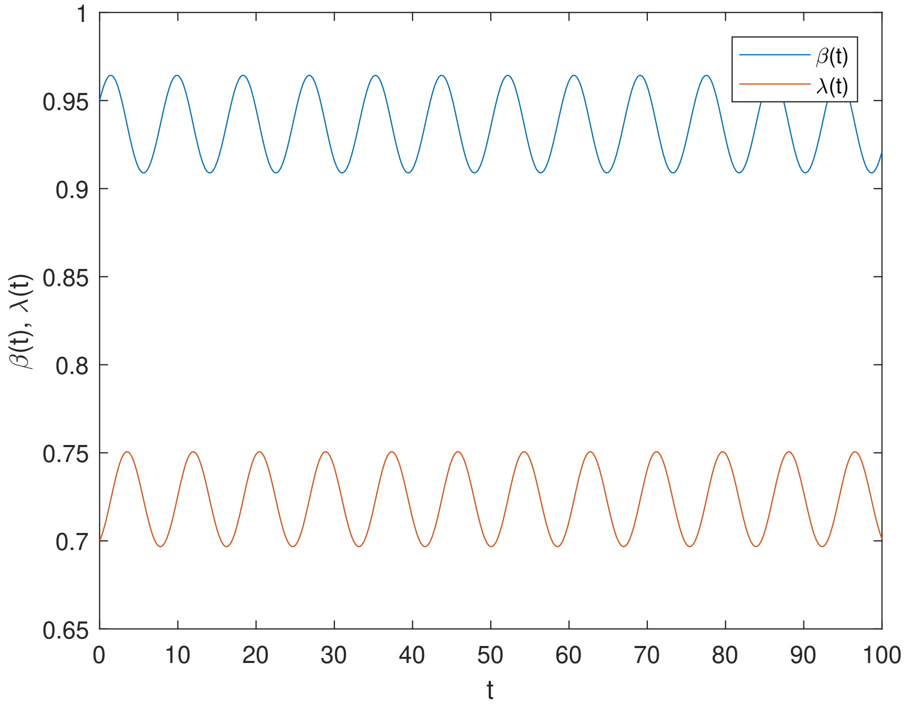


(a)

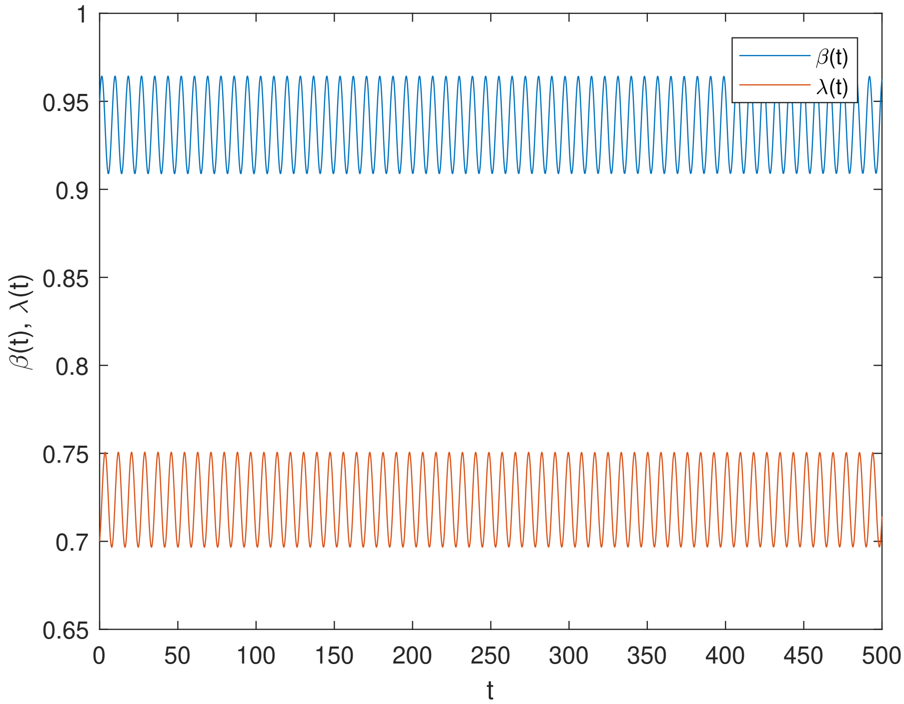


(b)

Figure 3.4 : $\tau = 0 < \tilde{\tau}_0 = 0.0196509$.

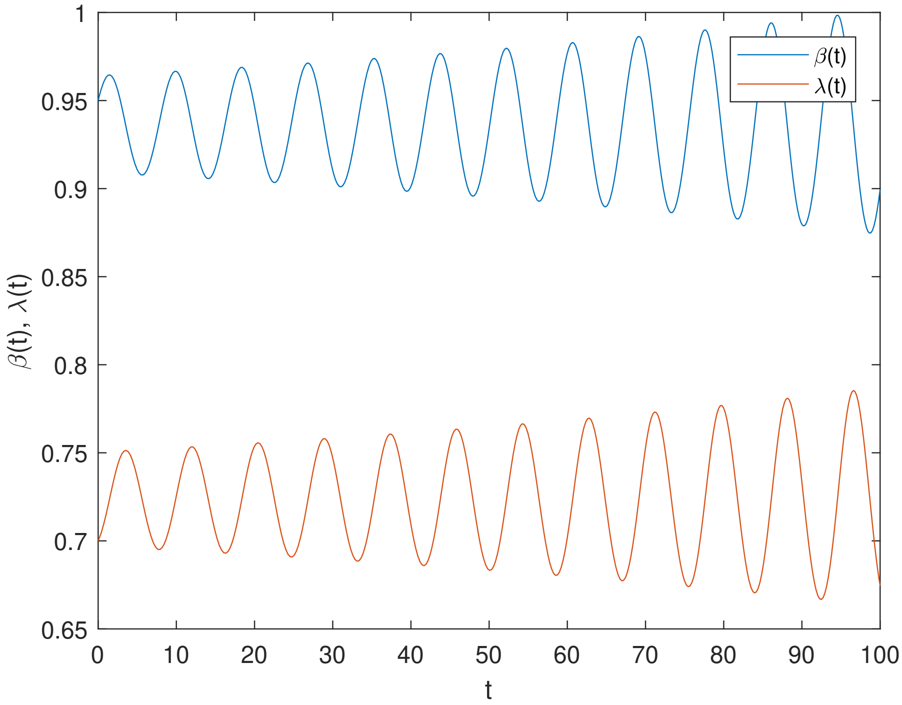


(a)

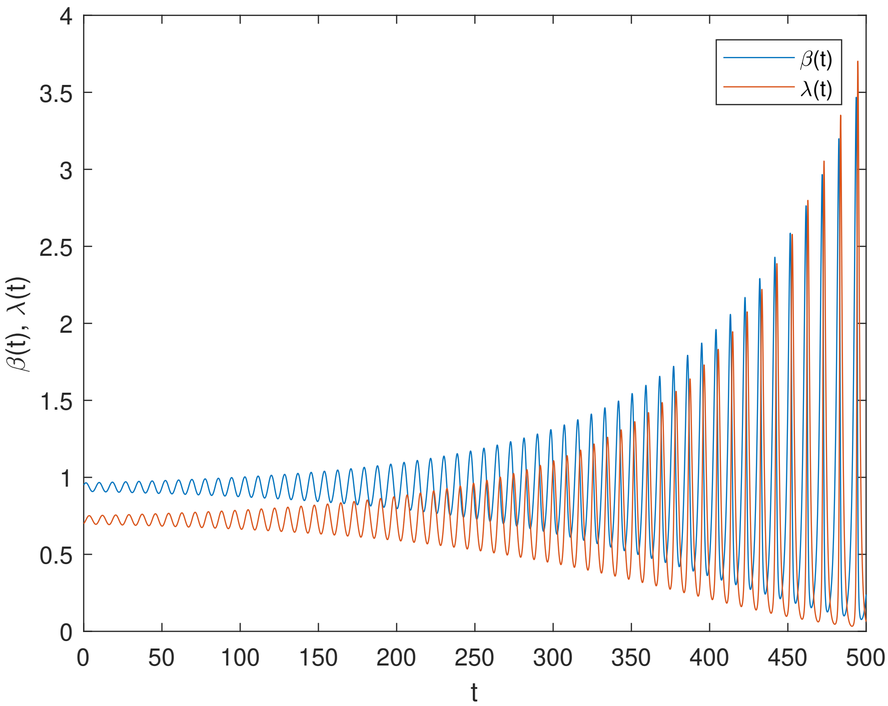


(b)

Figure 3.5 : $\tau = \tilde{\tau}_0 = 0.0196509.$



(a)



(b)

Figure 3.6 : $\tau = 0.05 > \tilde{\tau}_0 = 0.0196509$.

3.3 The General Model Reconsidered for Lyapunov Exponent

Now we consider the full system **(3.1)** of [1], in which the equilibrium points are calculated as

$$\begin{aligned}
\lambda_e &= [\mu_1 - \nu_1(1-\mu_2)]/[\nu_2(1-\mu_2)], \\
\beta_e &= \{\gamma_1 + n + \mu_1/(1-\mu_2) - \delta[(g(1-\lambda_e) - s_w]\}/\gamma_2, \\
v_e &= [c(1-\lambda_e)]/\{\delta[g(1-\lambda_e) - s_w]\}, \\
\theta_e &= \Big[[\varphi(\beta_e) + \phi_1(\beta_e) - u(\lambda_e)]/[b_2(1-a_3)]\Big](1-a_3b_3).
\end{aligned} \tag{3.74}$$

Around this equilibrium point, they find the Jacobian matrix

$$J = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix} \tag{3.75}$$

where

$$\begin{aligned}
a_{11} &= \varphi'(\beta_e)\beta_e = \gamma_2\beta_e > 0, \\
a_{12} &= [f'(\lambda_e) - u'(\lambda_e)]\beta_e = -(\delta g + \mu_2\nu_2)\beta_e < 0, \\
a_{21} &= [\varphi'(\beta_e) + \phi_1'(\beta_e)]\lambda_e = [\gamma_2 + a_2(1-b_3)/(1-a_3b_3)]\lambda_e > 0, \\
a_{22} &= -u'(\lambda_e)\lambda_e = -\mu_2\nu_2\lambda_e < 0, \\
a_{23} &= -\phi_2'(\theta_e)\lambda_e = -b_2(1-a_3)\lambda_e/(1-a_3b_3) < 0, \\
a_{32} &= [\psi'(\lambda_e) + f'(\lambda_e) - \frac{\partial z}{\partial \lambda_e}]\theta_e = [\nu_2(1-\mu_2) - \delta g + c/v_e]\theta_e, \\
a_{34} &= -\frac{\partial z}{\partial v_e}\theta_e = [c(1-\lambda_e)/v_e^2]\theta_e > 0, \\
a_{42} &= [-f'(\lambda_e) + \frac{\partial z}{\partial \lambda_e}]v_e = (\delta g - c/v_e)v_e, \\
a_{44} &= \frac{\partial z}{\partial v_e}v_e = -c(1-\lambda_e)/v_e < 0.
\end{aligned} \tag{3.76}$$

According to above matrix, they find characteristic equation

$$s^4 + c_1s^3 + c_2s^2 + c_3s + c_4 = 0 \tag{3.77}$$

where

$$
\begin{aligned}
c_1 &= -(a_{11}+a_{22}+a_{44}), \\
c_2 &= a_{11}a_{22}+(a_{11}+a_{22})a_{44}-a_{12}a_{21}-a_{23}a_{32}, \\
c_3 &= (a_{12}a_{21}-a_{11}a_{22}+a_{23}a_{32})a_{44}+(a_{11}a_{32}-a_{34}a_{42})a_{23}, \\
c_4 &= a_{11}a_{23}(a_{34}a_{42}-a_{32}a_{44}).
\end{aligned} \tag{3.78}
$$

Let us note that these results are available in [1].

For the set of parameters

$$
\begin{array}{llll}
\mu_1 = 0.012 & \nu_1 = 0.015 & \gamma_1 = 0.09 & s_\pi = 0.24 \\
\mu_2 = 0.60 & \nu_2 = 0.02 & \gamma_2 = 0.10 & s_w = 0.04 \\
a_1 = 0.865 & a_2 = 1 & a_3 = 0.90 & \delta = 4 \\
b_1 = 0.190 & b_2 = 0.25 & b_3 = 0.50 & c = 0.40 \\
n = 0.01 & & &
\end{array} \tag{3.79}
$$

the equilibrium points are calculated as $\beta_e = 0.90$, $\lambda_e = 0.75$, $\theta_e = 0.80$ and $v_e = 2.50$. There are four non-zero eigenvalues $0.0425805 \mp 0.730579i$, -0.0441941, and 0.0000331856. Therefore, the equilibrium is unstable.

By using a Matlab package which uses Wolf's Algorithm [28], we evaluate the Lyapunov exponents and find them as 0.102509, 0.102763, 0.002579 and -0.059178. The following Figure **(3.7)** is the output of this evaluation. We conclude that, since there are two positive Lyapunov exponents, the system may exhibit (hyper)chaotic behavior with this set of parameters.

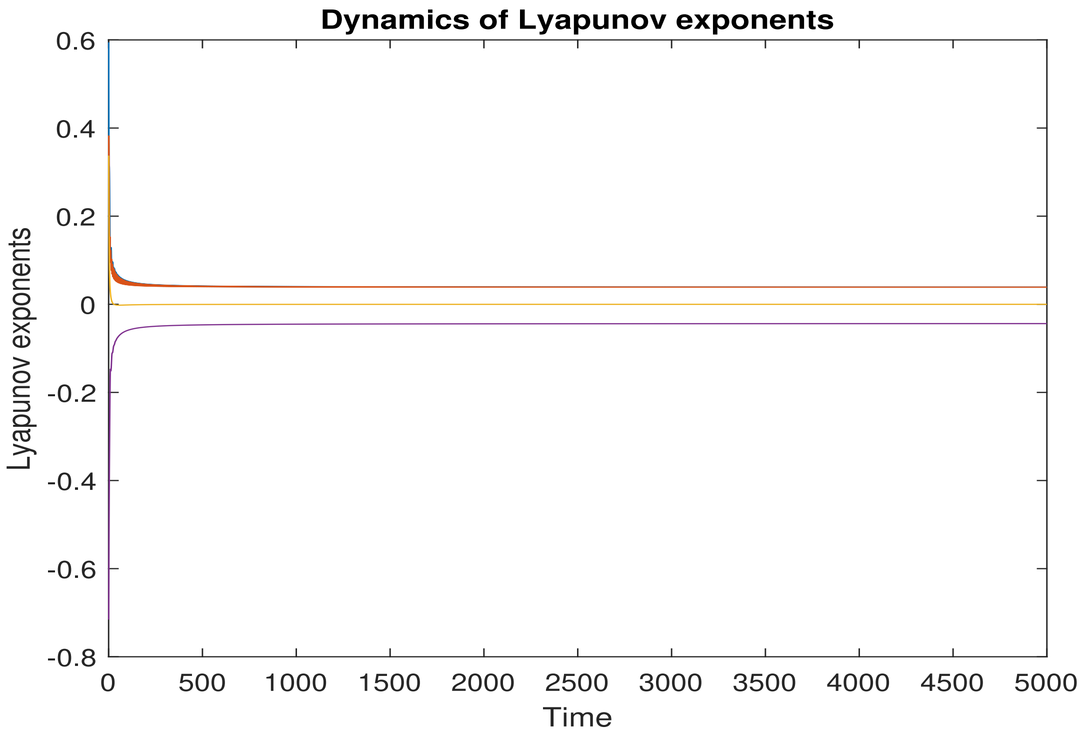


Figure 3.7 : Dynamics of Lyapunov exponents

3.4 Direction of Hopf Bifurcation

In Section 3.1, we considered the subsystem

$$\dot{\beta}(t) = \Big[\beta_0 + \gamma_2\beta(t) - \delta_0\lambda(t)\Big]\beta(t), \quad \textbf{(3.80a)}$$
$$\dot{\lambda}(t) = \Big[\lambda_0 - \nu_2\lambda(t) + \gamma_2\beta(t) + \rho_1\beta(t-\tau)\Big]\lambda(t) \quad \textbf{(3.80b)}$$

and investigated the conditions on the parameters so that the first two of the conditions (i) pure imaginary eigenvalues (ii) tranversality for a Hopf bifurcation to occur are satisfied. In this subsection, assuming these two conditions are satisfied, we shall evaluate the first Lyapunov coefficient.

First we make the change of variable $t = \tau\tilde{t}$. As $\dfrac{d}{dt} = \dfrac{1}{\tau}\dfrac{d}{d\tilde{t}}$, we obtain

$$\frac{d}{d\tilde{t}}\beta(\tau\tilde{t}) = \tau\Big[\beta_0 + \gamma_2\beta(\tau\tilde{t}) - \delta_0\lambda(\tau\tilde{t})\Big]\beta(\tau\tilde{t}), \quad \textbf{(3.81a)}$$
$$\frac{d}{d\tilde{t}}\lambda(\tau\tilde{t}) = \tau\Big[\lambda_0 - \nu_2\lambda(\tau\tilde{t}) + \gamma_2\beta(\tau\tilde{t}) + \rho_1\beta(\tau(\tilde{t}-1))\Big]\lambda(\tau\tilde{t}). \quad \textbf{(3.81b)}$$

Let us name $\tilde{\beta}(\tilde{t}) = \beta(\tau\tilde{t})$ and $\tilde{\lambda}(\tilde{t}) = \lambda(\tau\tilde{t})$ and get

$$\frac{d}{d\tilde{t}}\tilde{\beta}(\tilde{t}) = \tau\Big[\beta_0 + \gamma_2\tilde{\beta}(\tilde{t}) - \delta_0\tilde{\lambda}(\tilde{t})\Big]\tilde{\beta}(\tilde{t}), \quad \textbf{(3.82a)}$$

$$\frac{d}{d\tilde{t}}\tilde{\lambda}(\tilde{t}) = \tau\Big[\lambda_0 - v_2\tilde{\lambda}(\tilde{t}) + \gamma_2\tilde{\beta}(\tilde{t}) + \rho_1\tilde{\beta}(\tilde{t}-1)\Big]\tilde{\lambda}(\tilde{t}). \quad \textbf{(3.82b)}$$

This normalizes the delay τ to 1, giving a factor of τ on the right hand side. We drop tildes and hence fix

$$\dot{\beta}(t) = \tau\Big[\beta_0 + \gamma_2\beta(t) - \delta_0\lambda(t)\Big]\beta(t), \quad \textbf{(3.83a)}$$

$$\dot{\lambda}(t) = \tau\Big[\lambda_0 - v_2\lambda(t) + \gamma_2\beta(t) + \rho_1\beta(t-1)\Big]\lambda(t). \quad \textbf{(3.83b)}$$

We translate the equilibrium point $E(\beta_e, \lambda_e)$ by

$$u_1(t) = \beta(t) - \beta_e, \qquad u_2(t) = \lambda(t) - \lambda_e. \quad \textbf{(3.84)}$$

This gives us

$$\dot{u}_1(t) = \tau\Big[\gamma_2\beta_e u_1(t) - \delta_0\beta_e u_2(t) + \gamma_2 u_1^2(t) - \delta_0 u_1(t)u_2(t)\Big], \quad \textbf{(3.85a)}$$

$$\begin{aligned}\dot{u}_2(t) = \tau\Big[&\gamma_2\lambda_e u_1(t) - v_2\lambda_e u_2(t) + \rho_1\lambda_e u_1(t-1)\\ &+\gamma_2 u_1(t)u_2(t) - v_2 u_2^2(t) + \rho_1 u_1(t-1)u_2(t)\Big]. \end{aligned} \quad \textbf{(3.85b)}$$

Based on this form, we proceed to determine the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions according to the normal form theory by [29]. Regarding this center manifold analysis, we also would like to mention Ref. [30]. In the literature, the closest, but not identical model to **(3.7)** is studied in [31]. Therefore, following the calculations in [31] and also in [32] and [33], we continue as follows.

For clarity, we would like to present this system in the matrix form as

$$\begin{aligned}\begin{bmatrix}\dot{u}_1(t)\\ \dot{u}_2(t)\end{bmatrix} &= \tau\begin{bmatrix}\gamma_2\beta_e & -\delta_0\beta_e\\ \gamma_2\lambda_e & -v_2\lambda_e\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} + \tau\begin{bmatrix}0 & 0\\ \rho_1\lambda_e & 0\end{bmatrix}\begin{bmatrix}u_1(t-1)\\ u_2(t-1)\end{bmatrix}\\ &+ \tau\begin{bmatrix}\gamma_2 u_1^2(t) - \delta_0 u_1(t)u_2(t)\\ \gamma_2 u_1(t)u_2(t) - v_2 u_2^2(t) + \rho_1 u_1(t-1)u_2(t)\end{bmatrix}\end{aligned} \quad \textbf{(3.86)}$$

Let us define

$$u_t(\theta) = \begin{bmatrix}u_{1t}(\theta)\\ u_{2t}(\theta)\end{bmatrix} := u(t+\theta) = \begin{bmatrix}u_1(t+\theta)\\ u_2(t+\theta)\end{bmatrix}, \quad u_t : [-1, 0] \to \mathbb{R}^2. \quad \textbf{(3.87)}$$

Then, **(3.88)** becomes

$$\begin{bmatrix}\dot{u}_1(t)\\ \dot{u}_2(t)\end{bmatrix} = \tau\begin{bmatrix}\gamma_2\beta_e & -\delta_0\beta_e\\ \gamma_2\lambda_e & -\nu_2\lambda_e\end{bmatrix}\begin{bmatrix}u_{1t}(0)\\ u_{2t}(0)\end{bmatrix} + \tau\begin{bmatrix}0 & 0\\ \rho_1\lambda_e & 0\end{bmatrix}\begin{bmatrix}u_{1t}(-1)\\ u_{2t}(-1)\end{bmatrix} + \tau\begin{bmatrix}\gamma_2 u_{1t}^2(0) - \delta_0 u_{1t}(0)u_{2t}(0)\\ \gamma_2 u_{1t}(0)u_{2t}(0) - \nu_2 u_{2t}^2(0) + \rho_1 u_{1t}(-1)u_{2t}(0)\end{bmatrix} \quad \textbf{(3.88)}$$

For notational convenience, we set

$$(a,b) = \begin{bmatrix} a\\ b\end{bmatrix}_{2\times 1}, \qquad (a,b)^T = \begin{bmatrix} a & b\end{bmatrix}_{1\times 2}. \quad \textbf{(3.89)}$$

For $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta)) \in C([-1,0],\mathbb{R}^2)$, define the operator L_μ by

$$L_\mu\phi = \tau\begin{bmatrix}\gamma_2\beta_e & -\delta_0\beta_e\\ \gamma_2\lambda_e & -\nu_2\lambda_e\end{bmatrix}\begin{bmatrix}\phi_1(0)\\ \phi_2(0)\end{bmatrix} + \tau\begin{bmatrix}0 & 0\\ \rho_1\lambda_e & 0\end{bmatrix}\begin{bmatrix}\phi_1(-1)\\ \phi_2(-1)\end{bmatrix} \quad \textbf{(3.90)}$$

and

$$f(\phi,\mu) = \tau\begin{bmatrix}f_{11}\\ f_{12}\end{bmatrix} = \tau\begin{bmatrix}\gamma_2\phi_1^2(0) - \delta_0\phi_1(0)\phi_2(0)\\ \gamma_2\phi_1(0)\phi_2(0) - \nu_2\phi_2^2(0) + \rho_1\phi_1(-1)\phi_2(0)\end{bmatrix} \quad \textbf{(3.91)}$$

with $\tau = \tau_k + \mu$. See that, by virtue of these, we can denote the terms appearing in **(3.86)** as

$$L_\mu\begin{bmatrix}u_{1t}(\theta)\\ u_{2t}(\theta)\end{bmatrix} = \tau\begin{bmatrix}\gamma_2\beta_e & -\delta_0\beta_e\\ \gamma_2\lambda_e & -\nu_2\lambda_e\end{bmatrix}\begin{bmatrix}u_{1t}(0)\\ u_{2t}(0)\end{bmatrix} + \tau\begin{bmatrix}0 & 0\\ \rho_1\lambda_e & 0\end{bmatrix}\begin{bmatrix}u_{1t}(-1)\\ u_{2t}(-1)\end{bmatrix} \quad \textbf{(3.92)}$$

and

$$f(u_t,\mu) = \tau\begin{bmatrix}\gamma_2 u_{1t}^2(0) - \delta_0 u_{1t}(0)u_{2t}(0)\\ \gamma_2 u_{1t}(0)u_{2t}(0) - \nu_2 u_{2t}^2(0) + \rho_1 u_{1t}(-1)u_{2t}(0)\end{bmatrix} \quad \textbf{(3.93)}$$

Therefore, we have

$$\dot{u}(t) = L_\mu(u_t) + f(u_t,\mu). \quad \textbf{(3.94)}$$

Now we wish to write this system as a functional differential equation on $C([-1,0],\mathbb{R}^2)$ in the form

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t \quad \textbf{(3.95)}$$

which is more suitable as it includes a single unknown vector u_t compared to **(3.94)** which includes both $u(t)$ and u_t. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,\mu)$ in $\theta \in [-1,0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) \quad \textbf{(3.96)}$$

for $\phi \in C[-1,0]$. $\eta(\theta,\mu)$ can be chosen as

$$\eta(\theta,\mu) = \tau \begin{bmatrix} \gamma_2\beta_e & -\delta_0\beta_e \\ \gamma_2\lambda_e & -\nu_2\lambda_e \end{bmatrix} \delta(\theta) + \tau \begin{bmatrix} 0 & 0 \\ \rho_1\lambda_e & 0 \end{bmatrix} \delta(\theta+1) \qquad \textbf{(3.97)}$$

where $\delta(\theta)$ is the Dirac distribution.

For $\phi(\theta) \in C^1([-1,0],R^2)$, define

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s) = L_\mu(\phi), & \theta = 0, \end{cases} \qquad \textbf{(3.98)}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ f(\phi,\mu), & \theta = 0, \end{cases} \qquad \textbf{(3.99)}$$

Since $\dfrac{du_t}{d\theta} = \dfrac{du_t}{dt}$, **(3.86)** is equivalent to **(3.95)**. Indeed, for $\theta \in [-1,0)$, **(3.95)** is the trivial identity $\dfrac{du_t}{d\theta} = \dfrac{du_t}{dt}$ and for $\theta = 0$ it gives **(3.86)**.

For $\psi \in C^1([0,1],(R^2)^*)$, define the adjoint $A^*(0)$ of $A(0)$ as

$$A^*(0)\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0, \end{cases} \qquad \textbf{(3.100)}$$

where T denotes transpose and a bilinear inner product

$$\langle \psi,\phi \rangle = \overline{\psi}(0)\cdot\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \qquad \textbf{(3.101)}$$

where $\eta(\theta) = \eta(\theta,0)$ and a bar denotes complex conjugate. Then $A(0)$ and $A^*(0)$ are adjoint operators. In addition, we know that $\pm i\omega\tau_k$ are eigenvalues of A(0). Thus, they are also eigenvalues of A^*. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to $i\omega\tau_k$ and $q^*(s)$ be the eigenvector of $A^*(0)$ corresponding to $-i\omega\tau_k$; that is,

$$A(0)q(\theta) = i\omega\tau_k q(\theta), \qquad A^*(0)q^*(s) = -i\omega\tau_k q^*(s). \qquad \textbf{(3.102)}$$

We must find $q(\theta)$ that satisfy

$$\begin{cases} \dfrac{dq(\theta)}{d\theta} = i\omega\tau_k q(\theta) & \theta \in [-1,0), \\ L_{\mu=0}\big(q(\theta)\big) = i\omega\tau_k q(0), & \theta = 0. \end{cases} \qquad \textbf{(3.103)}$$

and $q^*(s)$ satisfying identities required by **(3.100)**. Then it is not difficult to show that

$$q(\theta) = \big(1,\alpha\big)e^{i\omega\tau_k\theta}, \qquad \alpha = \frac{\gamma_2\beta_e - i\omega}{\delta_0\beta_e} \qquad \textbf{(3.104)}$$

and

$$q^*(s) = B\big(\alpha^*, 1\big)e^{i\omega\tau_k s}, \qquad \alpha^* = \frac{i\omega - \nu_2\lambda_e}{\delta_0\beta_e}. \quad \textbf{(3.105)}$$

The evaluation of the inner product yields

$$\begin{aligned}
\langle q^*(s), q(\theta)\rangle &= \overline{q^*}(0)^T q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{q^*}(\xi-\theta)^T d\eta(\theta,0)q(\xi)d\xi \\
&= \overline{B}(\overline{\alpha}^*, 1)^T(1,\alpha) \\
&- \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{B}(\overline{\alpha}^*, 1)^T e^{-i\omega\tau_k(\xi-\theta)} d\eta(\theta,0)(1,\alpha)e^{i\omega\tau_k\xi}d\xi \\
&= \overline{B}(\overline{\alpha}^* + \alpha) - \overline{B}\int_{\theta=-1}^{0} \theta e^{i\omega\tau_k\theta}(\overline{\alpha}^*, 1)^T d\eta(\theta,0)(1,\alpha) \\
&= \overline{B}(\overline{\alpha}^* + \alpha) \\
&- \overline{B}\int_{\theta=-1}^{0} \theta e^{i\omega\tau_k\theta}(\overline{\alpha}^*, 1)^T \left\{ \tau_k \begin{bmatrix} \gamma_2\beta_e & -\delta_0\beta_e \\ \gamma_2\lambda_e & -\nu_2\lambda_e \end{bmatrix} \delta(\theta) \right. \\
&\qquad \left. + \tau_k \begin{bmatrix} 0 & 0 \\ \rho_1\lambda_e & 0 \end{bmatrix} \delta(\theta+1) \right\}(1,\alpha)d\theta \\
&= \overline{B}(\overline{\alpha}^* + \alpha) - \overline{B}.(-1).e^{i\omega\tau_k(-1)}(\overline{\alpha}^*, 1)^T \tau_k \begin{bmatrix} 0 & 0 \\ \rho_1\lambda_e & 0 \end{bmatrix}(1,\alpha) \\
&= \overline{B}(\overline{\alpha}^* + \alpha) + \tau_k\overline{B}e^{-i\omega\tau_k}\rho_1\lambda_e \\
&= \overline{B}\Big(\overline{\alpha}^* + \alpha + \tau_k e^{-i\omega\tau_k}\rho_1\lambda_e\Big) \qquad \textbf{(3.106)}
\end{aligned}$$

after which we choose

$$\overline{B} = \frac{1}{\overline{\alpha}^* + \alpha + \tau_k e^{-i\omega\tau_k}\rho_1\lambda_e} \quad \textbf{(3.107)}$$

to satisfy

$$\langle q^*(s), q(\theta)\rangle = 1. \quad \textbf{(3.108)}$$

We first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of **(3.95)** when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t\rangle, \qquad W(t,\theta) = u_t - zq - \bar{z}\bar{q} = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \quad \textbf{(3.109)}$$

On the center manifold C_0 we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \quad \textbf{(3.110)}$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + ..., \quad \textbf{(3.111)}$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Note that W is real if u_t is real. We consider only real solutions. For solution $u_t \in C_0$ of **(3.95)**, since $\mu = 0$,

$$\begin{aligned}
\dot{z} &= \langle q^*, \dot{u}_t \rangle \\
&= \langle q^*, A(0)u_t + R(0)u_t \rangle \\
&= \langle q^*, A(0)u_t \rangle + \langle q^*, R(0)u_t \rangle \\
&= \langle A^*(0)q^*, u_t \rangle + \langle q^*, f(u_t, 0) \rangle \\
&= \langle -i\omega\tau_k q^*, u_t \rangle + \langle q^*, f\big(W(z,\bar{z},\theta) + 2\text{Re}\{z(t)q(\theta)\}, 0\big) \rangle \\
&= \langle -i\omega\tau_k q^*, u_t \rangle + \langle q^*, f\big(W(z,\bar{z},\theta) + 2\text{Re}\{z(t)q(\theta)\}, 0\big) \rangle \\
&= i\omega\tau_k \langle q^*, u_t \rangle + \overline{q}^*(0)^T f_0(z,\bar{z}) \\
&= i\omega\tau_k z(t) + g(z,\bar{z}) \qquad \textbf{(3.112)}
\end{aligned}$$

with $g(z,\bar{z}) = \overline{q}^*(0)^T f_0(z,\bar{z})$ and

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + ... \qquad \textbf{(3.113)}$$

Let us note that

$$u_t(u_{1t}(\theta), u_{2t}(\theta)) = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta), \qquad \textbf{(3.114)}$$

and $q(\theta) = (1,\alpha)e^{i\omega\tau_k\theta}$, and, explicitly,

$$\begin{aligned}
u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \dots \\
u_{2t}(0) &= z\alpha + \bar{z}\bar{\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \dots \qquad \textbf{(3.115)} \\
u_{1t}(-1) &= ze^{-i\omega\tau_k} + \bar{z}e^{i\omega\tau_k} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \dots \\
u_{2t}(-1) &= z\alpha e^{-i\omega\tau_k} + \bar{z}\bar{\alpha}e^{i\omega\tau_k} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \dots
\end{aligned}$$

$$\begin{aligned}
g(z,\bar{z}) &= \overline{q}^*(0)^T f_0(z,\bar{z}) = \overline{B}\tau_k(\overline{\alpha}^*, 1)^T \begin{bmatrix} f_{11}^0 \\ f_{12}^0 \end{bmatrix} \\
&= \overline{B}\tau_k(\overline{\alpha}^*, 1)^T \begin{bmatrix} \gamma_2 u_{1t}^2(0) - \delta_0 u_{1t}(0)u_{2t}(0) \\ \gamma_2 u_{1t}(0)u_{2t}(0) - \nu_2 u_{2t}^2(0) + \rho_1 u_{1t}(-1)u_{2t}(0) \end{bmatrix} \textbf{(3.116)}
\end{aligned}$$

Substituting **(3.113)** into the left side of **(3.116)** and comparing the coefficients, we obtain

$$
\begin{aligned}
g_{20} &= 2\overline{B}\tau_k(\alpha\gamma_2+\overline{\alpha}^*\gamma_2-\alpha\bar{\alpha}^*\delta_0-\alpha^2\nu_2+\alpha\rho_1 e^{-i\omega\tau_k}),\\
g_{11} &= \overline{B}\tau_k(\alpha\gamma_2+\overline{\alpha}\gamma_2+2\overline{\alpha}^*\gamma_2-\alpha\overline{\alpha}^*\delta_0-\overline{\alpha}\,\overline{\alpha}^*\delta_0-2\alpha\overline{\alpha}\nu_2+e^{i\omega\tau_k}\alpha\rho_1+e^{-i\omega\tau_k}\overline{\alpha}\rho_1),\\
g_{02} &= 2\overline{B}\tau_k(\overline{\alpha}\gamma_2+\overline{\alpha}^*\gamma_2-\overline{\alpha}\,\overline{\alpha}^*\delta_0-\overline{\alpha}^2\nu_2+e^{i\omega\tau_k}\overline{\alpha}\rho_1),\\
g_{21} &= \overline{B}\tau_k\Big(2\alpha\gamma_2W_{11}^{(1)}(0)+4\overline{\alpha}^*\gamma_2W_{11}^{(1)}(0)+\overline{\alpha}\gamma_2W_{20}^{(1)}(0)+2\overline{\alpha}^*\gamma_2W_{20}^{(1)}(0)\\
&+ 2\gamma_2W_{11}^{(2)}(0)+\gamma_2W_{20}^{(2)}(0)-2\alpha\overline{\alpha}^*\delta_0W_{11}^{(1)}(0)-\overline{\alpha}\,\overline{\alpha}^*\delta_0W_{20}^{(1)}(0)-2\overline{\alpha}^*\delta_0W_{11}^{(2)}(0)\\
&- \overline{\alpha}^*\delta_0W_{20}^{(2)}(0)-4\alpha\nu_2W_{11}^{(2)}(0)-2\overline{\alpha}\nu_2W_{20}^{(2)}(0)+2\alpha\rho_1W_{11}^{(1)}(-1)\\
&+ \overline{\alpha}\rho_1W_{20}^{(1)}(-1)+2\rho_1e^{-i\omega\tau_k}W_{11}^{(2)}(0)+\rho_1e^{i\omega\tau_k}W_{20}^{(2)}(0)\Big) \qquad \textbf{(3.117)}
\end{aligned}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From **(3.95)** and **(3.109)**, we have

$$
\begin{aligned}
\dot{W} &= \dot{u}_t-\dot{z}q-\dot{\bar{z}}\bar{q}\\
&= A(0)u_t+R(0)u_t-(i\omega\tau_kz+g)q-(-i\omega\tau_k\bar{z}+\bar{g})\bar{q}\\
&= A(0)u_t+R(0)u_t-zA(0)q-\bar{z}A(0)\bar{q}-gq-\bar{g}\bar{q}\\
&= A(0)(u_t-zq-\bar{z}\bar{q})+R(0)u_t-2\mathrm{Re}\{gq\}\\
&= A(0)W+R(0)u_t-2\mathrm{Re}\{gq\}\\
&= \begin{cases} A(0)W-2\mathrm{Re}\{\bar{q}^*(0)\cdot f_0q(\theta)\}, & \theta\in[-1,0),\\ A(0)W-2\mathrm{Re}\{\bar{q}^*(0)\cdot f_0q(0)\}+f_0, & \theta=0,\end{cases}\\
&\overset{def}{=} A(0)W+H(z,\bar{z},\theta), \qquad \textbf{(3.118)}
\end{aligned}
$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}\theta\frac{\bar{z}^2}{2}\cdots \qquad \textbf{(3.119)}$$

Using $\dot{W}=W_z\dot{z}+W_{\bar{z}}\dot{\bar{z}}$ and Eqs. **(3.118)**, **(3.119)**, we obtain

$$(A(0)-2i\omega\tau_k)W_{20}(\theta)=-H_{20}(\theta), \qquad \textbf{(3.120)}$$

$$A(0)W_{11}(\theta)=-H_{11}(\theta), \qquad \textbf{(3.121)}$$

$$(A(0)+2i\omega\tau_k)W_{02}(\theta)=-H_{02}(\theta). \qquad \textbf{(3.122)}$$

From **(3.118)**, we know that for $\theta\in[-1,0)$,

$$H(z,\bar{z},\theta)=-\bar{q}^*(0)^Tf_0q(\theta)-q^*(0)^T\bar{f}_0\bar{q}(\theta)=-g(z,\bar{z})q(\theta)-\bar{g}(z,\bar{z})\bar{q}(\theta). \qquad \textbf{(3.123)}$$

Comparing the coefficients with **(3.119)** gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta),$$
$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), \qquad \textbf{(3.124)}$$

We get

$$\dot{W}_{20}(\theta) = 2i\omega\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \qquad \textbf{(3.125)}$$

Note that $q(\theta) = q(0)e^{i\omega\tau_k\theta}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\omega\tau_k}q(0)e^{i\omega\tau_k\theta} + \frac{i\bar{g}_{02}}{3\omega\tau_k}\bar{q}(0)e^{-i\omega\tau_k\theta} + E_1e^{2i\omega\tau_k\theta} \qquad \textbf{(3.126)}$$

where $E_1 = (E_{11}, E_{12})$ is a constant vector. Similarly, from **(3.120)** and **(3.124)**

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \qquad \textbf{(3.127)}$$

and

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega\tau_k}q(0)e^{i\omega\tau_k\theta} + \frac{i\bar{g}_{11}}{\omega\tau_k}\bar{q}(0)e^{-i\omega\tau_k\theta} + E_2. \qquad \textbf{(3.128)}$$

with a constant vector $E_2 = (E_{21}, E_{22})$. In what follows we shall seek appropriate E_1 and E_2 in **(3.126)** and **(3.128)**, respectively. It follows from the definition of A and **(3.120)** that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega\tau_k W_{20}(0) - H_{20}(0), \qquad \textbf{(3.129)}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \qquad \textbf{(3.130)}$$

where $\eta(\theta) = \eta(\theta, 0)$. From **(3.118)**, we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_k \begin{pmatrix} \gamma_2 - \alpha\delta_0 \\ \gamma_2\alpha - \alpha^2\nu_2 + \rho_1\alpha e^{-i\omega\tau_k} \end{pmatrix} \qquad \textbf{(3.131)}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_k \begin{pmatrix} 2\gamma_2 - \alpha\delta_0 - \overline{\alpha}\delta_0 \\ 2\gamma_2\mathrm{Re}\{\alpha\} - 2\nu_2|\alpha|^2 + 2\rho_1\mathrm{Re}\{\alpha e^{i\omega\tau_k}\} \end{pmatrix}. \qquad \textbf{(3.132)}$$

Substituting **(3.126)** and **(3.131)** into **(3.129)**, we obtain

$$\left(2i\omega\tau_k I - \textstyle\int_{-1}^{0} e^{2i\omega\tau\theta}d\eta(\theta)\right)E_1 = 2\tau_k \begin{pmatrix} \gamma_2 - \alpha\delta_0 \\ \gamma_2\alpha - \alpha^2\nu_2 + \rho_1\alpha e^{-i\omega\tau_k} \end{pmatrix} \qquad \textbf{(3.133)}$$

that is

$$\begin{pmatrix} 2i\omega - \gamma_2\beta_e & \delta_0\lambda_e \\ -\gamma_2\lambda_e - \rho_1\lambda_e e^{-2i\omega\tau_k} & 2i\omega + \nu_2\lambda_e \end{pmatrix} E_1 = \begin{pmatrix} 2\gamma_2 - 2\alpha\delta_0 \\ 2\gamma_2\alpha - 2\alpha^2\nu_2 + 2\rho_1\alpha e^{-i\omega\tau_k} \end{pmatrix} \qquad \textbf{(3.134)}$$

Similarly, substituting **(3.128)** and **(3.132)** into **(3.130)**, we get

$$\int_{-1}^{0} d\eta(\theta)E_2 = -\tau_k \begin{pmatrix} 2\gamma_2 - \alpha\delta_0 - \overline{\alpha}\delta_0 \\ 2\gamma_2\mathrm{Re}\{\alpha\} - 2\nu_2|\alpha|^2 + 2\rho_1\mathrm{Re}\{\alpha e^{i\omega\tau}\} \end{pmatrix}. \qquad \textbf{(3.135)}$$

which means

$$\begin{pmatrix} \gamma_2\beta_e & -\delta_0\lambda_e \\ (\gamma_2+\rho_1)\lambda_e & -\nu_2\lambda_e \end{pmatrix} E_2 = -\begin{pmatrix} 2\gamma_2 - \alpha\delta_0 - \overline{\alpha}\delta_0 \\ 2\gamma_2\mathrm{Re}\{\alpha\} - 2\nu_2|\alpha|^2 + 2\rho_1\mathrm{Re}\{\alpha e^{i\omega\tau}\} \end{pmatrix}. \qquad \textbf{(3.136)}$$

It follows from **(3.126)**, **(3.128) (3.135)**, and **(3.136)** that g_{21} can be expressed. Thus, we can compute the following values:

$$c_1(0) = \frac{i}{2\omega\tau_k}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \qquad \textbf{(3.137)}$$

$$\bar{\mu}_2 = -\frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}(\lambda'(\tau_k))}, \qquad \textbf{(3.138)}$$

$$\beta_2 = 2\mathrm{Re}\big(c_1(0)\big), \qquad \textbf{(3.139)}$$

$$T_2 = -\frac{\mathrm{Im}\big(c_1(0)\big) + \mu_2\mathrm{Im}\big(\lambda'(\tau_k)\big)}{\omega}, \qquad \textbf{(3.140)}$$

which determine the quantities of bifurcating periodic solutions at the critical value τ_k, i.e. $\bar{\mu}_2$ determines the directions of the Hopf bifurcation: if $\bar{\mu}_2 > 0$ ($\bar{\mu}_2 < 0$) then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_k$ ($\tau < \tau_k$); β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions in the center manifold are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

We take the parameters as $\nu_1 = 0.02$, $\nu_2 = 0.04$, $\gamma_1 = 0.01$, $\gamma_2 = 0.012$, $\delta = 4.2$, $\sigma = 0.55$, $c = 0.38$, $n = 0.01$, $s_\pi = 0.24$, $s_w = 0.04$, $a_1 = 0.9$, $a_2 = 1$, $a_3 = 0.99$, $b_1 = 1.9$, $b_2 = 0$, $b_3 = 0.6$.

We find that $c_1(0) = 0.00132164 - 0.0136561i$, hence $\bar{\mu}_2 < 0$, therefore, the bifurcation is subcritical. The bifurcating periodic solution exists when τ crosses τ_0 to the left. Since $\beta_2 > 0$, the bifurcating periodic solution is unstable.

4. CONCLUSIONS AND RECOMMENDATIONS

We conclude by outlining the achievements of the thesis and provide possible future work.

4.1 Conclusions

As a counterpart of predator-prey dynamics in mathematical economics, despite its simplicity, to some extent, the Goodwin model explains the periodic behavior of the state variables which are observed within certain time intervals. The assumptions on the construction of the model can be relaxed so as to take into account more complicated situations occurring in an economy. Therefore, the model has been modified in the available literature through various considerations. In this work we considered a generalized, higher dimensional Goodwin model which takes into account variable capacity utilization and capital coefficient in addition to the variables employment ratio and wage share.

Taking into account a delay effect in the Phillips curve, we show that, while the equilibrium point of the generalized system is stable in a parameter domain in the non-delay case, the delayed model may experience a Hopf bifurcation and periodic oscillations may arise. In specific cases, the equations for the wage share and the employment rate decouples from the full system. For these cases, we analytically determine the critical value of the delay parameter that will drive the stable equilibrium point to an unstable one through a Hopf bifurcation. We also observe two positive Lyapunov exponents in the non-delayed four-dimensional model in the unstable case, which is referred to as hyperchaotical behavior in literature. In one of the analysis for the Hopf bifurcation, we performed the direction analysis.

4.2 Future Suggestions

With the analysis conducted in this thesis, we attempt to provide an approach to the original Goodwin model and its generalisations in the literature in terms of delayed analysis. As a point which deserves further consideration we believe, we would like to note that, when the employment-wage share cycles constructed with real data available in the literature for a particular country's economy are analysed, it is observed that the cycles are valid for a certain time interval, after which the cycle jumps to another equilibrium point of the phase plane. One explanation for this situation may be the change in the values of the parameters in the system of the relevant country that give the position of the equilibrium point. Another explanation for this situation, which can be obtained from the results of this thesis, is that the related equilibrium point becomes unstable at a certain point due to the realisation of the dynamics in the system with a delay and the drawn phase curve moves away from the equilibrium point and tends to realise another cycle in the phase space.

Analysis of delay dynamical systems has a vast literature. However, we observed that the Goodwin model and its modifications have not been studied thoroughly. That was the main motivation in setting the content of the thesis the way we did above. We would like to remark that, other modifications of the Goodwin model and higher dimensional extensions await for being analyzed in the delayed context.

REFERENCES

[1] **Glombowski, J. and Krüger, M.** (1987). Generalizations of Goodwin's growth cycle model, *Economic Evolution and Structural Adjustment: Proceedings of Invited Sessions on Economic Evolution and Structural Change Held at the 5th International Conference on Mathematical Modelling at the University of California, Berkeley, California, USA July 29–31, 1985*, Springer, pp.260–290.

[2] **Goodwin, R.M.**, (1967). A Growth Cycle: Socialism, Capitalism and Economic Growth, **C.H. Feinstein**, editor, Essays in Economic Dynamics, Palgrave Macmillan UK, London, pp.165–170.

[3] **Desai, M.**, **Henry, B.**, **Mosley, A. and Pemberton, M.** (2006). A clarification of the Goodwin model of the growth cycle, *Journal of Economic Dynamics and Control*, *30*(12), 2661–2670.

[4] **Harvie, D.**, **Kelmanson, M.A. and Knapp, D.G.** (2007). A Dynamical Model of Business-Cycle Asymmetries: Extending Goodwin, *Economic Issues*, *12*(1).

[5] **Gaudenzi, M.**, **Madotto, M. and Zanolin, F.** (2013). On the generalizations of the Goodwin model.

[6] **Sordi, S.** (2001). Growth cycles when workers save, *Central European Journal of Operations Research*, *9*.

[7] **Minsky, H.P.** (1982). Can "it" happen again? A reprise, *Challenge*, *25*(3), 5–13.

[8] **Keen, S.** (1995). Finance and economic breakdown: Modeling Minsky's "financial instability hypothesis", *Journal of Post Keynesian Economics*, *17*(4), 607–635.

[9] **Grasselli, M.R. and Costa Lima, B.** (2012). An analysis of the Keen model for credit expansion, asset price bubbles and financial fragility, *Mathematics and Financial Economics*, *6*, 191–210.

[10] **Grasselli, M.R. and Nguyen Huu, A.** (2015). Inflation and speculation in a dynamic macroeconomic model, *Journal of Risk and Financial Management*, *8*(3), 285–310.

[11] **Tebaldi, C. and Colacchio, G.** (2007). Chaotic behavior in a modified Goodwin's growth cycle model, *Proceedings of the 25th International Conference of the System Dynamics Society and 50th Anniversary Celebration. System Dynamics Society*.

[12] **Sportelli, M. and De Cesare, L.** (2022). A Goodwin type cyclical growth model with two-time delays, *Structural Change and Economic Dynamics*, *61*, 95–102.

[13] **Desai, M.** (1973). Growth cycles and inflation in a model of the class struggle, *Journal of economic theory*, *6*(6), 527–545.

[14] **Van der Ploeg, F.** (1984). Implications of workers' savings for economic growth and the class struggle, *Nonlinear Models of Fluctuating Growth: An International Symposium Siena, Italy, March 24–27, 1983*, Springer, pp.1–13.

[15] **Balducci, R.**, **Candela, G. and Ricci, G.** (1984). A generalization of R. Goodwin model with rational behavior of economic agents, *Nonlinear Models of Fluctuating Growth: An International Symposium Siena, Italy, March 24–27, 1983*, Springer, pp.47–66.

[16] **Ambrosi, G.M.** (2015). The Goodwin growth cycle model as solution to a variational problem, *Cambridge Journal of Economics*, *39*(6), 1645–1658.

[17] **Harvie, D.** (2000). Testing Goodwin: growth cycles in ten OECD countries, *Cambridge Journal of Economics*, *24*(3), 349–376.

[18] **Grasselli, M.R. and Maheshwari, A.** (2018). Testing a Goodwin model with general capital accumulation rate, *Metroeconomica*, *69*(3), 619–643.

[19] **Araujo, R.A. and Moreira, H.N.** (2021). Testing a Goodwin's model with capacity utilization to the US economy, *Nonlinearities in Economics: An Interdisciplinary Approach to Economic Dynamics, Growth and Cycles*, 295–313.

[20] **Yoshida, H. and Asada, T.** (2007). Dynamic analysis of policy lag in a Keynes–Goodwin model: stability, instability, cycles and chaos, *Journal of Economic Behavior & Organization*, *62*(3), 441–469.

[21] **Phillips, A.W.** (1958). The relation between unemployment and the rate of change of money wage rates in the United Kingdom, 1861-1957, *Economica*, *25*(100), 283–299.

[22] **Lakshmanan, M. and Senthilkumar, D.V.** (2011). *Dynamics of Nonlinear Time-Delay Systems*, Springer Science & Business Media.

[23] **Kuznetsov, Y.A.** (1998). *Elements of Applied Bifurcation Theory*, Springer.

[24] **Gao, Q. and Ma, J.** (2009). Chaos and Hopf bifurcation of a finance system, *Nonlinear Dynamics*, *58*, 209–216.

[25] **Singh, J.P. and Roy, B.** (2016). The nature of Lyapunov exponents is (+,+,-,-). Is it a hyperchaotic system?, *Chaos, Solitons & Fractals*, *92*, 73–85.

[26] **Orlando, G. and Sportelli, M.** (2021). Growth and cycles as a struggle: Lotka–Volterra, Goodwin and Phillips, *Nonlinearities in Economics: An Interdisciplinary Approach to Economic Dynamics, Growth and Cycles*, 191–208.

[27] **Ruan, S. and Wei, J.** (2003). On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dynamics of Continuous Discrete and Impulsive Systems Series A*, *10*, 863–874.

[28] **Wolf, A.**, **Swift, J.B.**, **Swinney, H.L. and Vastano, J.A.** (1985). Determining Lyapunov exponents from a time series, *Physica D: Nonlinear Phenomena*, *16*(3), 285–317.

[29] **Hassard, B.D.**, **Kazarinoff, N.D. and Wan, Y.H.** (1981). *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge.

[30] **Balachandran, B.**, **Kalmár-Nagy, T. and Gilsinn, D.E.** (2009). *Delay Differential Equations*, Springer, New York.

[31] **Song, Y. and Wei, J.** (2005). Local Hopf bifurcation and global periodic solutions in a delayed predator–prey system, *Journal of Mathematical Analysis and Applications*, *301*(1), 1–21.

[32] **Çelik, C. and Merdan, H.** (2013). Hopf bifurcation analysis of a system of coupled delayed-differential equations, *Applied Mathematics and Computation*, *219*(12), 6605–6617.

[33] **Manjunath, S. and Raina, G.** (2014). Stability and bifurcation analysis of a Lotka-Volterra time delayed system, *The 26th Chinese Control and Decision Conference (2014 CCDC)*, IEEE, pp.2056–2063.

CURRICULUM VITAE

Name Surname : Eyşan Şans

EDUCATION :

- **B.Sc.** : 2016, Çankaya University, Faculty of Economics and Administrative Sciences, Department of Economics
- **B.Sc.** : 2016, Çankaya University, Faculty of Art and Sciences, Department of Mathematics and Computer Science

PUBLICATIONS, PRESENTATIONS AND PATENTS ON THE THESIS:

- **Şans, E.** , Akdemir, M., and Özemir, C. (2024). Hopf Bifurcation in a Generalized Goodwin Model with Delay. *3rd International Graduate Research Symposium (IGRS'24)*, May 8-10, 2024 İstanbul, Türkiye.