ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

ON THE SURFACES ON THE DEGENERATED HYPERCYLINDER $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$

M.Sc. THESIS

Ali GİNELİ

Department of Mathematics Engineering

Mathematics Engineering Programme

JANUARY 2025

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

$\mathscr{LC}^2 \times \mathbb{R}$ DEJENERE HİPERSİLİNDİRİNDEKİ YÜZEYLER ÜZERİNE

YÜKSEK LİSANS TEZİ

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Ali GİNELİ, a M.Sc. student of ITU Graduate School 509141210 successfully defended the thesis entitled "ON THE SURFACES ON THE DEGENERATED HYPERCYLINDER $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$", which he prepared after fulfilling the requirements specified in the associated legislation, before the jury whose signatures are below.

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Date of Defense : **27 January 2025**

To my family

FOREWORD

I started my master's degree at Istanbul Technical University in 2014-2015 spring term. Due to the intensity of my professional life, I took a break from my master's thesis during the thesis phase. I returned to write my thesis in the 2022-2023 academic year. At this stage, my dear advisor Assoc. Prof. Dr. Nurettin Cenk Turgay was very helpful both in refreshing my academic knowledge and throughout my thesis process. I am deeply grateful to him for his valuable contributions. I feel deeply indebted to my family, especially my mother, for their unwavering moral and financial support throughout my entire educational journey. I would also like to extend my heartfelt thanks to my spouse, who stood by me and provided constant support during my master's education. I would also like to thank all my teachers, friends and everyone who helped me in this process.

AI Asisted Copy Editing. During the preparation of this work, the author utilized ChatGPT for conducting grammatical checks.

January 2025

Ali GİNELİ
(M.Sc. Student)

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SYMBOLS

$\mathbb{E}_r^n$	: n- dimensional Semi-Euclidean Space with index r
$\mathscr{LC}^n$	: n- dimensional light cone of the Minkowski $n+1$-space
∇	: Levi- Civita Connection of the Submanifold
$\tilde{\nabla}$	: Levi- Civita Connection of the Semi-Euclidean Space
$\nabla^{\perp}$	: Normal Connection
A	: Shape Operator
h	: Second Fundemantel Form
H	: Mean Curvature Vector
K	: Gaussian Curvature Vector
R	: Riemannian Curvature Tensor
w_{ij}	: Connection Form

ON THE SURFACES ON THE DEGENERATED

HYPERCYLINDER $\mathscr{LC}^2 \times \mathbb{R}$

SUMMARY

In the 19th century, Gauss' definition of curvature and his studies on the local properties of surfaces laid the foundations of differential geometry. Riemann's work on manifolds integrated the concept of curvature with a general metric structure, providing a new perspective on modern geometry. In the 20th century, Einstein's general theory of relativity opened the way for studying the light-cone structures of these manifolds. In such studies, Minkowski space has been a fundamental model, and the Lorentz metric structure serves as an ideal setting for examining the geometry of light-cones.

In this context, the $\mathscr{LC}^2 \times \mathbb{R}$ manifold is defined as the Cartesian product of a light-cone and the real line, $\mathbb{R}$. The behavior of space-like surfaces in this environment is mathematically investigated.

This study examines the $\mathscr{LC}^2 \times \mathbb{R}$ manifold, which represents the light-cone geometry in Minkowski space. A space-like surface is defined by the positive norm of the tangent space at each point under the semi-Riemannian metric. The fundamental structures and properties used in the study are expressed as follows.

Let M be a surface in $\mathscr{LC}^2 \times \mathbb{R}$ with the position vector

$$x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v), x_4(u,v)).$$

Then light-like vector field ϕ normal to the surface is defined as

$$\phi = (x_1, x_2, x_3, 0), \quad \langle \phi, \phi \rangle = 0.$$

In the analysis that begins with the decomposition of the vector $\mathbf{C}_4 = (0,0,0,1)$ into its tangent and normal components, the decomposition is expressed as

$$C_4 = (C_4)^T + (C_4)^\perp.$$

The effect of the normal connection $\nabla^\perp$ and the definitions of various geometric quantities (α, A_ϕ) are explored. It is shown that the normal component

$$(C_4)^\perp = -\alpha\phi$$

is also light-like. Moreover, the vector $(C_4)^T$ is defined as a unit-length vector.

This structure allows the derivation of the fundamental formulas for M, namely the connection ∇ and the second fundamental form h.

In the first main part of the study, Gaussian curvature of M is considered. The Gauss equation and properties of the second fundamental form are examined for the case when the Gaussian curvature is zero. It is concluded that a surface is flat if and only if it can be locally parametrized as

$$x(u,v) = (u + a(v))\eta(v) + uC_4,$$

where $\eta(v)$ is a light-like curve satisfying

$$\langle \eta, \eta \rangle = 0 \quad \text{and} \quad \langle \mathbf{C}_4, \eta \rangle = 0.$$

For such surfaces, the fundamental metric coefficients E, F, G are computed, and the Gaussian curvature is confirmed to be zero.

The structures of the tangent and normal bundles play a fundamental role in the geometric analysis of the surface. The tangent vector fields e_1, e_2 form an orthonormal basis for the tangent bundle, while the normal vector fields ϕ, ξ form a pseudo-orthonormal basis. Here, ϕ is light-like, and ξ is orthogonal to ϕ, satisfying $\langle \phi, \xi \rangle = -1$.

Using the normal connection $\nabla^{\perp}$, a relationship is defined as

$$\nabla^{\perp}_{C_4^T} \phi = \alpha \phi,$$

showing that the normal part of ϕ is light-like.

The differential properties of these bundles are analyzed using Gauss and Weingarten equations. The shape operators

$$A_{\phi} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_{\xi} = \begin{pmatrix} -e_1(\alpha) - \alpha^2 & -e_2(\alpha) \\ -e_2(\alpha) & \beta \end{pmatrix}$$

play an important role in the geometric characterization of the surface. These operators are used in connection with the second fundamental form. Moreover, the derivatives of the normal connection on ϕ and ξ are determined:

$$\nabla^{\perp}_{e_1} \phi = \alpha \phi, \quad \nabla^{\perp}_{e_1} \xi = -\alpha \xi, \quad \nabla^{\perp}_{e_2} \phi = \nabla^{\perp}_{e_2} \xi = 0.$$

The local curvature properties of space-like surfaces are explained in terms of geometric quantities such as the Gaussian curvature K and the mean curvature H. The eigenvalues λ_1, λ_2 of the shape operator represent the principal curvatures of the surface. The average of these eigenvalues gives the mean curvature H, and their product gives the Gaussian curvature K. Moreover, the classification of flat surfaces and surfaces are completed. Namely, it is proved that a surface in the $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$ is flat if and only if it is locally congruent to the surface parametrized by

$$x(u,v) = (u + a(v))\eta(v) + u(C_4).$$

Further surfaces with flat normal bundle are considered. It is obtained that a surface with flat normal bundle must necessarily be locally congruent to the surface parametrized by

$$x(u,v) = \gamma(u)\eta(v) + uC_4.$$

$\mathscr{LC}^2 \times \mathbb{R}$ DEJENERE HİPERSİLİNDİRİNDEKİ YÜZEYLER ÜZERİNE

ÖZET

19. yüzyılda Gauss'un eğrilik kavramını tanımlaması ve yüzeylerin lokal özellikleriyle ilgili çalışmalar yapmasıyla diferansiyel geometrinin temelleri atılmıştır. Riemann'ın manifoldlar üzerindeki çalışmaları, eğrilik kavramını genel bir metrik yapısı ile bütünleştirerek modern geometriye yeni bir bakış açısı kazandırmıştır. 20. yüzyılda Einstein'in genel görelilik teorisi ile bu manifoldların ışık-konisi yapıları üzerinden incelenmesinin yolu açılmıştır. Bu tür çalışmalarda Minkowski uzayı temel bir model olmuştur. Lorentz metrik yapısı, ışık-konisi geometrisi incelenmesi için ideal bir zemindir.

Bu bağlamda, $\mathscr{LC}^2 \times \mathbb{R}$ manifoldu, bir ışık-konisinin $\mathbb{R}$ doğrusu ile kartezyen çarpımı olarak tanımlanmıştır. Uzay-benzeri yüzeylerin bu ortamda nasıl davrandığı matematiksel olarak incelenmiştir.

Bu çalışmada, $\mathscr{LC}^2 \times \mathbb{R}$ manifoldu, Minkowski uzayındaki ışık-konisi geometrisini temsil eder. Uzay-benzeri bir yüzey, her bir noktadaki tanjant uzayın yarı-Riemann metriği altında pozitif norm taşımasıyla tanımlanır. Çalışmada kullanılan temel yapılar ve özellikler aşağıdaki şekildeki gibi ifade edilmiştir.

$\mathscr{LC}^2 \times \mathbb{R}$ manifoldunun içinde bir M yüzeyinin pozisyon vektörü

$$x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v), x_4(u,v))$$

şeklinde ifade edilmiştir.

Daha sonra ışık-benzeri normal vektör alanı ϕ,

$$\phi = (x_1, x_2, x_3, 0), \quad \langle \phi, \phi \rangle = 0$$

şeklinde tanımlanmıştır.

Çalışmada, bu vektör $\mathbf{C}_4 = (0,0,0,1)$'ün tanjant ve normal bileşenlerine ayrılmasıyla başlayan analizde,

$$C_4 = (C_4)^T + (C_4)^{\perp}$$

şeklindeki ayrışma ele alınmıştır. Normal bağlantının $\nabla^{\perp}$ etkisi ve çeşitli geometrik miktarların α, A_ϕ tanımları incelenmiş, sonuçta normal bileşen olan

$$(C_4)^{\perp} = -\alpha\phi$$

ifadesi de ışık-benzeri olduğu gösterilmiştir. Ayrıca,

$$(C_4)^T$$

birim uzunlukta bir vektör olarak tanımlanmıştır.

Bu yapı, M'nin temel formülleri olan bağlantı ∇ ve ikinci temel form h ifadelerinin türetilmesine olanak sağlamıştır.

Uzay-benzeri yüzeylerin lokal eğrilik özellikleri, Gauss eğriliği K ve ortalama eğrilik H gibi geometrik niceliklerle açıklanır. Şekil operatörünün özdeğerleri (λ_1, λ_2), yüzeyin temel eğriliklerini ifade eder. Bu özdeğerlerin ortalaması ortalama eğrilik H'yi, çarpımı ise Gauss eğriliği K'yi verir.

Çalışmanın ilk ana kısmında, M'nin Gaussian eğriliği ele alınmıştır. Gaussian eğriliğin sıfır olduğu durum için Gauss denklemi ve ikinci temel formun özellikleri incelenmiştir. Düz yüzeylerin sınıflandırılmasıyla ilgili sonuç olarak, bir yüzey düz ise, o yüzey yerel olarak

$$x(u,v) = (u + a(v))\eta(v) + uC_4$$

şeklinde parametrik olarak ifade edilebilir. Burada $\eta(v)$, ışık-benzeri bir eğri olup,

$$\langle \eta, \eta \rangle = 0 \quad \text{ve} \quad \langle \mathbf{C}_4, \eta \rangle = 0$$

koşullarını sağlamaktadır. Bu yüzeyler için M'nin temel metrik katsayıları olan E, F, G hesaplanmış ve Gaussian eğriliğin sıfır olduğu doğrulanmıştır.

Tanjant ve normal demetlerin yapıları, yüzeyin geometrik analizinde temel bir rol oynamaktadır. Tanjant demet e_1, e_2 ortonormal bir baz oluşturur ve tanjant uzayı temsil eder. Normal demet ϕ, ξ pseudo-ortonormal bir bazdır. Burada ϕ, ışık-benzeri ξ ise ϕ ye dik bir vektördür. Yani $\langle \phi, \xi \rangle = -1$ koşulunu sağlar.

Normal bağlantı $\nabla^\perp$ kullanılarak,

$$\nabla^\perp_{C_4^T} \phi = \alpha \phi$$

şeklinde bir ilişki tanımlanmıştır. Burada ϕ'nin normal kısmının ışık-benzeri olduğu gösterilmiştir.
Ayrıca bu demetlerin diferansiyel özellikleri Gauss ve Weingarten denklemleri kullanılarak incelenmiştir.

$$A_\phi = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_\xi = \begin{pmatrix} -e_1(\alpha) - \alpha^2 & -e_2(\alpha) \\ -e_2(\alpha) & \beta \end{pmatrix}$$

şeklindeki şekil operatörleri, yüzeyin geometrik karakterizasyonunda önemli bir role sahiptir. Bu operatörler, yüzeyin ikinci temel formu ile bağlantılı olarak kullanılmıştır. Ayrıca, normal bağlantının ϕ ve ξ vektörleri üzerindeki türevleri belirlenmiştir.

$$\nabla^\perp_{e_1} \phi = \alpha \phi, \quad \nabla^\perp_{e_1} \xi = -\alpha \xi, \quad \nabla^\perp_{e_2} \phi = \nabla^\perp_{e_2} \xi = 0.$$

Çalışmanın ikinci kısmında, normal demeti düz olan, yani normal eğriliği $K^{\perp} = 0$ olan yüzeyler incelenmiştir. İlk olarak bu koşulun

$$e_2(\alpha) = 0$$

olmasına denk olduğu gösterilmiştir. M'nin pozisyon vektörü,

$$x(u,v) = \gamma(u)\eta(v) + uC_4$$

formunda elde edilmiştir. Burada

$$\gamma(u) = e^{\int \alpha(u)\,du}$$

ve $\eta(v)$, yine ışık-benzeri bir eğridir.

Çalışmanın temel sonuçlarından biri, $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$ ortamındaki bir yüzey ancak ve ancak

$$x(u,v) = (u + a(v))\eta(v) + u(C_4)$$

şeklinde tanımlanan yüzeyle yerel olarak eşdeğerse düz olabilir.

Çalışmanın temel sonuçlarından bir diğeri ise, $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$ ortamındaki bir yüzeyin normal demeti düz ise, bu yüzey

$$x(u,v) = \gamma(u)\eta(v) + uC_4$$

şeklinde ifade edilebilir.

Geometrik analizde düz yüzeyler ve düz normal demetli yüzeyler, önemli bir sınıflandırma problemine işaret etmektedir.

Bu çalışmada, $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$ ortamında uzay-benzeri yüzeylerin diferansiyel geometrisi detaylı bir şekilde incelenmiştir. Elde edilen sonuçlar, hem düz yüzeylerin hem de düz normal demetli yüzeylerin yapısal özelliklerini ortaya koymaktadır. Bu tür yüzeylerin diferansiyel geometrisi, Lorentz manifoldlarında ışık-benzeri yüzeylerin incelenmesi için temel bir çerçeve sunmaktadır.

1. INTRODUCTION

Let M be a submanifold of a semi-Euclidean space $\mathbb{E}^n_r$. Position vector x of M is a global vector field which has been studied by many geometers so far in order to understand extrinsic properties of M. Therefore, it has wide-ranging applications across mathematics, engineering, and the natural sciences, [1]. In differential geometry, the study of surfaces within various geometric frameworks provides fundamental insights into both theoretical and applied mathematics. Euclidean space, denoted as $\mathbb{E}^n$, serves as the classical setting for understanding surface properties through well-established tools like the metric tensor and curvature. Euclidean space offers a familiar, flat setting where surface geometry can be understood in terms of intrinsic properties like Gaussian curvature and extrinsic measures such as mean curvature. However, when extended to Minkowski space, $\mathbb{E}^n_1$, space-like surfaces those with exclusively space-like tangents present unique challenges and characteristics.

The study of spacelike surfaces in Minkowski space, $\mathbb{E}^n_1$, has become a pivotal area of interest in differential geometry, with notable intersections in mathematical physics. Defined by having tangent vectors that are exclusively space-like, these surfaces demonstrate distinct geometric and causal properties due to the indefinite metric structure of Minkowski space. These surfaces have been studied by many geometers so far. For example, authors in [2], have characterized the curvature conditions and stability aspects of space-like surfaces, offering insights into their classification and behavior under different geometrical constraints.

When M lies in one of the semi-Riemannian space forms $\mathbb{H}^{n-1}_{r-1}$ and $\mathbb{S}^{n-1}_r$, then x becomes a parallel normal vector field. This is also true if M is a subset of the light-cone $\mathbb{LC}^{n-1}$. This case was studied in recent papers [3,4]. Research on space-like surfaces in Minkowski space has been advanced significantly, with studies like [3] and [4] examining their curvature constraints, causal structures, and stability properties.

In [3], a specific smooth function was considered on any space-like surface within the light cone of four-dimensional Lorentz-Minkowski space. It was demonstrated how this function encodes both the extrinsic and intrinsic geometry of such a surface. By assuming the existence of a local maximum, it can be determined when the space-like surface must be entirely umbilical, leading to a Liebmann-type result. Two notable families of examples of space-like surfaces within a light cone were explicitly constructed. The main result of this paper is as follows:

Theorem 1.1. *[3] Let $x : \mathbb{S}^2 \to \mathbb{E}^4_1$ be a space-like immersion which factors through $\mathbb{LC}^3$. Then, for every $u \in \mathbb{E}^4_1$ which satisfies $\langle u,u\rangle = -1, u_0 < 0$, we have,*

$$\lambda_1 \leq \frac{2}{min\langle x,u\rangle^2},$$

and the equality holds for some u if and only if the surface is the totally umbilical round sphere $\mathbb{S}^2(u,r)$, $\quad r = \langle x,u\rangle \in \mathbb{R}^+$, where λ_1 is the least positive eigenvalue of the Laplace operator of $\mathbb{S}^2$.

In [4], codimension two space-like submanifolds contained in the light cone of the Lorentz-Minkowski spacetime were studied through an approach that allows their extrinsic and intrinsic geometries to be computed in terms of a single function u. As a first application of this approach, the totally umbilical submanifolds were classified. For compact, codimension two spacelike submanifolds within the light cone, it was shown that they are conformally diffeomorphic to the round sphere and were described by an explicit embedding expressed in terms of u. In the final part of the paper, the case where the submanifold was weakly trapped was considered, and in particular, some non-existence results for weakly trapped submanifolds within the light cone were derived. The primary result of this paper can be summarized as follows:

Theorem 1.2. *[4] Let $x : (\Omega, y) \to \mathbb{LC}^{n+1} \subset \mathbb{E}^{n+2}_1$ be a codimension two totally umbilical spacelike submanifold contained in $\mathbb{LC}^{n+1}$. Then there exists $a \in \mathbb{L}^{n+2}$, $a \neq 0$ and $\langle a,a\rangle = c \in \{-1,0,1\}$, and there exists $\tau \in \mathbb{R}$, $\tau > 0$, such that*

$$x(\Sigma) \subset \Sigma(a,\tau).$$

On the other hand, flat surfaces, surfaces with flat normal bundle and umbilical surfaces in Euclid and Minkowski spaces has been studied in many papers. Note that the

Gaussian curvature is the product of the principal curvatures at a given point on the surface. A surface is said to be flat if its Gaussian curvature vanishes. A surface embedded in $\mathbb{E}^3$ is flat if the surface is a developable surface (a surface that can be unfolded into a plane without distortion). Examples include: planes, cylinders and cones. These surfaces are also examples of developable surfaces, meaning their normal bundles are flat.

An umbilical surface is a surface where every point is an umbilic point. Umbilic points are points where the principal curvatures (the maximum and minimum curvatures at a point) are equal, meaning the surface is curved the same way in all directions. At every point on an umbilical surface, the two principal curvatures k_1 and k_2 are equal. Hence, the surface bends uniformly in all directions at each point. The shape operator is proportional to the identity matrix at every point. This means that the normal curvature is the same in all directions. The most important example of an umbilical surface is the sphere. For a sphere of radius r, the curvature at every point is $\frac{1}{r}$ and the surface curves the same way in all directions. A plane can also be considered an umbilical surface since the curvature is zero in all directions. An umbilical surface has the same curvature in all directions at each point, leading to a uniform bending. Spheres, with their constant curvature, are prime examples. On the other hand for surfaces in higher-dimensional spaces $\mathbb{E}^n$, $n > 3$ a flat normal bundle means that the normal spaces at each point can be parallel transported without rotation along the surface. This is equivalent to vanishing of normal curvature tensor of the surface.

The investigation of flat surfaces within Euclidean and Minkowski spaces plays a significant role in differential geometry, offering insights into both classical geometry and the geometry of relativity. In Euclidean space, $\mathbb{E}^n$, flat surfaces those with zero Gaussian curvature serves as foundational objects, often characterized by their simple geometric properties and applications in physical models. When extended to the Minkowski space, $\mathbb{E}^n_1$, flat surfaces take on additional complexity, given the indefinite metric signature. Recent studies, such as [5,6], have analyzed the curvature conditions that define flat surfaces in Minkowski space, exploring their causal properties and stability.

In [7], the classical concept of completeness was extended to flat fronts, and a representation formula was presented for a flat front that contained a non-empty compact singular set and has ended that were completed and immersed. As an application, it was demonstrated that such a flat front has properly embedded ends if and only if the image of its Gauss map constitutes a convex curve. Furthermore, the presence of at least four singular points, distinct from cuspidal edges, was established on a flat front with embedded ends, representing a variation of the classical four-vertex theorem for convex plane curves.

A pseudo-umbilical surface is a surface where the mean curvature vector and the normal vector at each point are aligned, but not necessarily all principal curvatures are equal. In a pseudo-umbilical surface, the mean curvature vector is collinear with the normal vector at every point, even if the individual principal curvatures are not equal. Unlike an umbilical surface, a pseudo-umbilical surface may have unequal principal curvatures, i.e., $k_1 \neq k_2$. However, the mean curvature vector is still aligned with the surface's normal direction. A minimal surface (where the mean curvature $H = 0$ everywhere) is a special case of a pseudo-umbilical surface, where the mean curvature vector is zero, and thus trivially aligned with the normal vector.

In [8], general rotational surfaces in $\mathbb{E}^4$ with meridian curves contained in two-dimensional planes were examined. First, all minimal general rotational surfaces were identified by solving the differential equation that characterizes such surfaces. Then, all pseudo-umbilical general rotational surfaces in $\mathbb{E}^4$ were determined.

A space-like surface S immersed in a 4-dimensional Lorentzian manifold is defined as umbilical along a direction N normal to S if the second fundamental form along N is proportional to the first fundamental form of S. Specifically, S is called pseudo-umbilical if it is umbilical along the mean curvature vector field H and (totally) umbilical if it is umbilical along all possible normal directions. The case in which the surface is umbilical along the unique normal direction orthogonal to H is called an "ortho-umbilical" surface. It has been proven that the necessary and sufficient condition for S to be umbilical along a normal direction is that two independent Weingarten operators (and consequently all of them)commute, or equivalently, that the shape tensor is diagonalizable on S. The umbilical direction is then uniquely defined. This is equivalent to a condition involving the normal curvature and the relevant part of the Riemann tensor of the space-time. In particular, for conformally flat space-times (including Lorentz space forms), the necessary and sufficient condition is that the normal curvature vanishes. Additional consequences are explored, and the extension of the main results to arbitrary signatures and higher dimensions was briefly discussed. These studies have been examined in [9].

In [10], it was proven that slant surfaces in a non-flat Lorentzian complex space form must be Lagrangian. By utilizing this result, a complete classification of pseudo-umbilical slant surfaces in Lorentzian complex space forms is provided. The classification results indicate that there are two families of pseudo-umbilical slant surfaces in the Lorentzian complex plane $\mathbb{C}^2_1$, three families in the complex projective plane $\mathbb{CP}^2_1$, and three families in the complex hyperbolic plane $\mathbb{CH}^2_1$.

This topic was examined in another study as follows [11]. A totally umbilical submanifold in pseudo-Riemannian manifolds is defined as one for which the second fundamental form is proportional to the metric, representing a fundamental concept and a generalization of a totally geodesic submanifold. In this paper, the classification of congruence classes of full totally umbilical submanifolds in non-flat pseudo-Riemannian space forms is presented, and the moduli spaces of such submanifolds are examined. As a result, it is found that some moduli spaces of isometric immersions between space forms with the same constant curvature are non-Hausdorff.

In [12], totally umbilical pseudo-slant submanifolds of Riemannian product manifolds were studied using the Riemannian curvature tensor, and a classification for these totally umbilical pseudo-slant submanifolds in Riemannian product manifolds was obtained.

An exhaustive classification of totally umbilical surfaces in unimodular and non-unimodular simply-connected 3-dimensional Lie groups equipped with arbitrary left-invariant Riemannian metrics was obtained. This completes the classification of totally umbilical surfaces in homogeneous Riemannian 3-manifolds. These studies were examined in [13].

The complex analytic structure of the complement of a non-singular hypersurface with a unitary flat normal bundle is investigated when the corresponding line bundle admits a Hermitian metric with semipositive curvature. These topics were addressed in a recent study [14]. In [15] the behavior of the second fundamental form of an isometric immersion of a space form with negative curvature into another space form, such that the extrinsic curvature is negative, is investigated. It has been proven that if the immersion has a flat normal bundle, the second fundamental form grows exponentially.

Given an immersed submanifold $M^n \subset R^{n+2}$, the vanishing of the normal curvature R_D at a point $P \in M$ is characterized in terms of the behavior of the asymptotic directions and the curvature locus at that point. The affine properties of codimension 2 submanifolds with a flat normal bundle are related to the conformal properties of hypersurfaces in Euclidean space. The semi umbilical, hyperspherical, and conformally flat submanifolds of codimension 2 are also characterized in terms of their curvature loci. These studies were examined [16].

In [17], some classes of rotational surfaces in the pseudo-Euclidean space E_t^4 with profile curves lying in 2-dimensional planes are studied. First, all such surfaces in the Minkowski 4-space E_1^4 with pointwise 1-type Gauss maps of the first and second kinds were determined. Then, rotational surfaces in E_2^4 with zero mean curvature and pointwise 1-type Gauss maps of the second kind are obtained.

In this work, by considering these studies, submanifolds lying on the degenerated hyper cylinder are going to be focused on $\mathbb{LC}^{n-2}$. In the first part of this study, a vector from the subspace M, a vector field normal to M, and another vector field normal to this vector field were considered. Here, a number of results were obtained using the Weingarten and Gauss formulas(Lemma1). There exists a global positively oriented orthonormal frame field $e1, e2$ of the tangent bundle of M such that Lemma 1. Building on this, especially by using (1) and (3) of Lemma 1, a new local coordinate system is obtained. This situation is addressed(Lemma2). In the second part of the study, the concepts of flat surfaces, flat normal bundle, marginally trapped, umbilical, and pseudo-umbilical were examined in M a subspace of $\mathbb{LC}^{n-2}$. Local classification theorems related to these concepts were obtained.

This thesis is contained in four sections and these are planned as follows. The opening section provides a concise overview of the historical development and foundational concepts concerning flat surfaces, umbilic surfaces, and surfaces with flat normal bundles, as well as a summary of the research progress made in this area to date. In the second section, essential notations are introduced, followed by the fundamental concepts related to submanifolds in Minkowski spaces. In the third section, space-like surfaces in $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$ are studied. Flat surfaces in $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$, surfaces with flat normal bundle in $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$ and pseudo-umbilical surfaces in $\mathscr{L}\mathscr{C}^2 \times \mathbb{R}$ have been studied. A number of results have been obtained on these topics. In the final section, conclusions are drawn, and suggestions for future research directions are provided.

2. PRELIMINARIES

In this section a brief summary of basic facts and definitions is given.

Let $\mathbb{E}_1^{n+2}$ denote the $n+2$-dimensional Minkowski space given by $\mathbb{E}_1^{n+2} = (\mathbb{R}^{n+1}, \langle \tilde{g} \rangle)$, where the metric $\tilde{g} = \langle , \rangle$ is the canonical Lorentzian metric defined by

$$\tilde{g} = -dx_1^2 + dx_2^2 + dx_3^2 + ... + dx_{n+2}^2$$

for a rectangular coordinate system $(x_1, x_2, ..., x_{n+2})$ on $\mathbb{R}^{n+2}$. The terminology of Minkowski space and Minkowski metric is also used to denote space and metric, respectively. In $n+1$ dimensional Minkowski space it is defined

$$\mathscr{LC}^{n+1} = \{(x_1, x_2, ...x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+2} : -x_1^2 + x_2^2 + x_3^2 + ... + x_{n+2}^2 = 0\}, \quad (2.0.1)$$

$$\mathscr{LC}^{n} \times \mathbb{R} = \{(x_1, x_2, ..., x_{n+1}, 0) \in \mathbb{R}^{n+2} : -x_1^2 + x_2^2 + ... + x_{n+1}^2 = 0\}. \quad (2.0.2)$$

A vector $v \in \mathbb{E}_1^{n+2}$ is said to be

spacelike if$\langle v, v \rangle > 0$ or $v = 0$, lightlike if $\langle v, v \rangle = 0$ or $v \neq 0$, timelike if $\langle v, v \rangle < 0$.

2.1 Submanifolds of Minkowski Spaces

Let M^n be a submanifold of $\mathbb{E}_r^{n+2}$ with Levi Civita connection ∇, second fundamental form h, shape operator A and normal connection $\nabla^\perp$. Then, the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi$$

are satisfied whenever X and Y are tangent to M and $\xi \in T^\perp M$, where $T^\perp M$ denotes the normal bundle of M. Note that the normal vector field ξ is said to be parallel if $\nabla_X^\perp \xi = 0$ whenever X is tangent to M. The curvature tensor of M is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For any tangent vector fields X, Y and Z on M. On the other hand, the mean curvature vector field H of M is defined by

$$H = \frac{1}{n}\text{trace}h. \tag{2.1.1}$$

M is said to be marginally trapped if H is lightlike at every point of M. Now, consider the case $n = 2$, that is, M is a surface. In this case the Gaussian curvature of M is defined by

$$K = \frac{R(X,Y,Y,X)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

Furthermore, the normal curvature of M is defined by

$$K^{\perp} = \langle R^{\perp}(e_1,e_2)e_3,e_4\rangle,$$

where e_1,e_2,e_3,e_4 is an orthonormal frame field on M. M is said to be flat if $K = 0$. On the other hand, M is said to have flat normal bundle if $K^{\perp} = 0$. Now, let M be a surface in $\mathbb{E}^n_r$ with the Levi- Civita connection ∇, h and A. The Codazzi and Gauss equations are expressed as follows

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) \tag{2.1.2}$$

and

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y \tag{2.1.3}$$

respectively, where the covariant derivative $(\overline{\nabla}_X h)(Y,Z)$ of the second fundamental form h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

On the other hand, the Ricci equation is

$$R^{\perp}(X,Y)\xi = h(X,A_{\xi}Y) - h(A_{\xi}X,Y) \tag{2.1.4}$$

whenever X, Y are tangent and ξ is normal to M. If there exists a smooth function f on M such that $\langle A_{\xi}X,Y\rangle = f\langle X,Y\rangle$, then M is said to be pseudo-umbilical along ξ. M is called totally umbilical if this is correct for every $\xi \in T^{\perp}M$. On the other hand, M is said to be pseudo-umbilical if it is pseudo-umbilical along it.

3. SPACELIKE SURFACES IN $\mathscr{LC}^2 \times \mathbb{R}$

In this section, space-like surfaces in $\mathscr{LC}^2 \times \mathbb{R}$ are going to be studied. The results obtained in this chapter are presented in [18].

Let M be a space-like surface in $\mathscr{LC}^2 \times \mathbb{R} \subset \mathbb{E}^4_1$ with the position vector

$$x = (x_1, x_2, x_3, x_4).$$

Then, the vector field $\phi = (x_1, x_2, x_3, 0)$ satisfies

$$\langle \phi, \phi \rangle = -x_1^2 + x_2^2 + x_3^2 = 0.$$

If $x(u,v)$ is a local parametrization near to $p \in M$, then implies

$$\begin{cases} \langle \dfrac{\partial x}{\partial u}, \phi \rangle = -x_1 \dfrac{\partial x_1}{\partial u} + x_2 \dfrac{\partial x_2}{\partial u} + x_3 \dfrac{\partial x_3}{\partial u} = 0, \\ \langle \dfrac{\partial x}{\partial v}, \phi \rangle = -x_1 \dfrac{\partial x_1}{\partial v} + x_2 \dfrac{\partial x_2}{\partial v} + x_3 \dfrac{\partial x_3}{\partial v} = 0. \end{cases} \tag{3.0.1}$$

Since $\left\{ \dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v} \right\}$ is a base for the tangent bundle of M, ϕ is normal to M. (3.0.1) yields that ϕ is light-like. So, there exists a normal vector field ξ such that

$$\langle \xi, \xi \rangle = 0, \quad \langle \phi, \xi \rangle = -1.$$

Thus, $\{\phi, \xi\}$ is a pseudo-orthonormal base for the normal bundle of M.

On the other hand, in order to construct an orthonormal base for the tangent bundle of M, the orthonormal decomposition

$$C_4 = (C_4)^T + (C_4)^\perp$$

is going to be considered.

Define a smooth function $\alpha : M \to \mathbb{R}$ by

$$\nabla^\perp_{C_4^T} \phi = \alpha \phi, \tag{3.0.2}$$

where $\nabla^{\perp}$ is the normal connection.

By a direct computation considering $\phi = x - x_4 C_4$, one can get

$$\tilde{\nabla}_X \phi = \tilde{\nabla}_X (x - x_4 C_4),$$

from which, along with the Weingarten formula, the equation

$$-A_\phi X + \nabla^{\perp}_X \phi = X - \langle X, (C_4)^T \rangle C_4$$

is obtained.

The tangential and normal part of the above equation gives

$$\begin{cases} A_\phi X &= -X + \left\langle X, (C_4)^T \right\rangle (C_4)^T, \\ \nabla^{\perp}_X \phi &= -\left\langle X, (C_4)^T \right\rangle (C_4)^{\perp}. \end{cases} \tag{3.0.3}$$

Note that the equation (3.0.3) for $X = (C_4)^T$ implies

$$\nabla^{\perp}_{C_4^T} \phi = -\langle (C_4)^T, (C_4)^T \rangle (C_4)^{\perp}.$$

By using (3.0.2) in this equation, one can get

$$\alpha \phi = -\|C_4^T\|^2 (C_4)^{\perp}.$$

Consequently, $(C_4)^{\perp}$ is lightlike, i.e., $\left\|(C_4)^{\perp}\right\| = 0$.

Therefore, $\left\|(C_4)^T\right\| = 1$ and

$$(C_4)^{\perp} = -\alpha \phi. \tag{3.0.4}$$

Next, a unit tangent vector field e_1 is defined by

$$e_1 = (C_4)^T \tag{3.0.5}$$

and let e_2 be a unit tangent vector field orthogonal to e_1. Note that (3.0.4) and (3.0.5) implies

$$C_4 = e_1 - \alpha \phi. \tag{3.0.6}$$

Furthermore, (3.0.3) for $X = e_1$ implies

$$A_\phi e_1 = -e_1 + \langle e_1, e_1 \rangle e_1 = 0, \tag{3.0.7}$$

$$\nabla^\perp_{e_1} \phi = \alpha\phi = -C_4^\perp \tag{3.0.8}$$

and (3.0.3) for $X = e_2$ gives

$$A_\phi e_2 = -e_2, \tag{3.0.9}$$

$$\nabla^\perp_{e_2} \phi = 0. \tag{3.0.10}$$

Next, one can get the covariant derivative of (3.0.6) along an $X \in TM$ to get

$$\tilde{\nabla}_X C_4 = \tilde{\nabla}_X e_1 - \tilde{\nabla}_X \alpha\phi, \tag{3.0.11}$$

$$0 = \nabla_X e_1 + h(X, e_1) - X(\alpha)\phi + \alpha A_\phi X - \alpha \nabla^\perp_X \phi. \tag{3.0.12}$$

The tangential and normal part of this equation give

$$\begin{cases} \nabla_X e_1 &= -\alpha A_\phi X, \\ h(X, e_1) &= X(\alpha)\phi + \alpha \nabla^\perp_X \phi. \end{cases} \tag{3.0.13}$$

Note that (3.0.13) for $X = e_1$ implies

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \tag{3.0.14}$$

$$h(e_1, e_1) = (e_1(\alpha) + \alpha^2)\phi \tag{3.0.15}$$

from which one can get

$$\langle h(e_1, e_1), \xi \rangle = -(e_1(\alpha) + \alpha^2) = \langle A_\xi e_1, e_1 \rangle. \tag{3.0.16}$$

Similarly, (3.0.13) for $X = e_2$ implies

$$\nabla_{e_2} e_1 = \alpha e_2, \tag{3.0.17}$$

$$h(e_1, e_2) = e_2(\alpha)\phi \tag{3.0.18}$$

and

$$\langle h(e_1, e_2), \xi \rangle = -e_2(\alpha) = \langle A_\xi e_1, e_2 \rangle. \tag{3.0.19}$$

By summing up (3.0.14)-(3.0.19), one can obtain the following lemma.

Lemma 3.1. *Let M be a space-like surface in $\mathscr{LC}^2 \times \mathbb{R} \subset \mathbb{E}^4_1$. Then, there exists an orthonormal base field e_1, e_2 for the tangent bundle of M and a pseudo-orthonormal base ϕ, ξ for the normal bundle of M such that, the followings are satisfied:*

1. $e_1 = (C_4)^T \quad , \quad -\alpha\phi = (C_4)^\perp.$
2. $\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_1 = \alpha e_2, \quad \nabla_{e_2} e_2 = -\alpha e_1.$
3. $A_\phi = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad A_\xi = \begin{pmatrix} -e_1(\alpha) - \alpha^2 & -e_2(\alpha) \\ -e_2(\alpha) & \beta \end{pmatrix}.$
4. $\nabla^\perp_{e_2} \phi = \nabla^\perp_{e_2} \xi = 0 \quad , \quad \nabla^\perp_{e_1} \phi = \alpha\phi \quad , \quad \nabla^\perp_{e_1} \xi = -\alpha\xi.$
5. $h(e_1, e_1) = (e_1(\alpha) + \alpha^2)\phi,$
   $h(e_1, e_2) = e_2(\alpha)\phi,$
   $h(e_2, e_2) = -\beta\phi + \xi.$

Lemma 3.2. *Let M be a surface in $\mathscr{LC}^2 \times \mathbb{R}$ and $p \in M$. Then, there exists a local coordinate system $(\mathscr{N}_p, (u, v))$ such that $p \in M$,*

$$e_1 = \partial_u, \; e_2 = \frac{1}{\gamma}\partial_v \tag{3.0.20}$$

such that

$$x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v), u) \tag{3.0.21}$$

where γ is a function satisfying

$$e_1(\gamma) = \alpha\gamma \tag{3.0.22}$$

and x_i satisfies

$$e_1(x_i) = \alpha x_i, \quad i = 1, 2, 3. \tag{3.0.23}$$

Proof. From (2) of Lemma 3.1 one can obtain that the equation

$$[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = -\alpha e_2 \tag{3.0.24}$$

is obtained. By a direct computation using (3.0.24) one can get

$$[e_1, \gamma e_2] = e_1(\gamma)e_2 - \gamma[e_1, e_2] = (e_1(\gamma) - \gamma\alpha)e_2.$$

Therefore, if γ satisfies (3.0.22), then $[e_1, \gamma e_2] = 0$. Therefore there exists a local coordinate system $(\mathscr{N}_p, (u, v))$ near to p such that (3.0.20) is satisfied. On the other hand from (1) of Lemma 3.1, one can get $\|C_4^T\| = 1$. Therefore, the position vector x of M satisfies

$$\begin{aligned} e_1 = x_u = (\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}, \frac{\partial x_4}{\partial u}), \\ \frac{\partial x_4}{\partial u} = 1 \end{aligned} \tag{3.0.25}$$

on $\mathscr{N}_p$. On the other hand, since e_2 satisfies

$$\langle e_2, C_4 \rangle = 0,$$

(3.0.20) implies

$$\begin{aligned} e_2 = x_v = (\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}, \frac{\partial x_4}{\partial v}), \\ \frac{\partial x_4}{\partial v} = 0. \end{aligned} \tag{3.0.26}$$

By combining (3.0.25) and (3.0.26) one can get (3.0.21). On the other hand, (3.0.23) follows from

$$\tilde{\nabla}_{e_1}\phi = -A_\phi e_1 + \alpha\phi = \alpha\phi,$$

which is obtained from (3) and (4) of Lemma 3.1. Hence the proof is completed. □

3.1 Flat Surfaces

In this subsection, flat surfaces lying in $\mathscr{LC}^2 \times \mathbb{R}$ are considered. First, the following example is constructed:

Proposition 3.1. *Let M be a surface in $\mathbb{E}^4_1$ parametrized by*

$$x(u, v) = (u + a(v))\eta(v) + u(C_4), \tag{3.1.1}$$

where η is arc-length parametrized curve such that $\langle \eta, \eta \rangle = 0, \quad \langle C_4, \eta \rangle = 0$. Then M lies on $\mathscr{LC}^2 \times \mathbb{R}$ and it is flat.

Proof. By a direct computation using (3.1.1), one can get

$$x_u = \eta(v) + C_4,$$

$$x_v = a'(v)\eta + (u + a(v))\eta'(v)$$

from which it is obtained that

$$E = \langle x_u, x_u \rangle = \langle \eta, \eta \rangle + 2\langle \eta, C_4 \rangle + \langle C_4, C_4 \rangle = 1,$$

$$F = \langle x_u, x_v \rangle = \langle \eta + C_4, a'(v)\eta + (u + a(v))\eta' \rangle = 0,$$

$$G = \langle x_v, x_v \rangle = \langle a'(v)\eta + (u + a(v))\eta'(v), a'(v)\eta + (u + a(v))\eta'(v) \rangle = (u + a(v))^2.$$

Therefore the vector fields e_1, e_2 given by

$$\begin{cases} e_1 = \dfrac{\partial}{\partial u} = \eta(v) + C_4, \\ e_2 = \dfrac{1}{u + a(v)} \dfrac{\partial}{\partial v} = \dfrac{a'}{u + a}\eta + \eta'. \end{cases} \tag{3.1.2}$$

By a direct computation, the equation $\langle h(e_1, e_1), h(e_2, e_2) \rangle - \langle h(e_1, e_2), h(e_1, e_2) \rangle = 0$ from which, along with the Gauss equation, one can obtain $K = 0$. Hence the proof is completed. □

In the following theorem, a local classification of flat surfaces lying on $\mathscr{LC}^2 \times \mathbb{R}$ is given. Note that, by considering the Gauss equation and (5) of the Lemma 3.1, it can be observed that being flat is equivalent to $h(e_1, e_1) = 0$.

Theorem 3.1. *Let M be a surface lying on $\mathscr{LC}^2 \times \mathbb{R}$. Then M is flat if and only if it is locally congruent to the surface described in Proposition 3.1.*

Proof. Let M be a surface in $\mathscr{LC}^2 \times \mathbb{R}$ and consider a local coordinate system satisfying the conditions given in Lemma 3.2. Note that (1)-(5) Lemma 3.1 hold. Now in order to prove the necessary condition assume that K=0. Then, (5) of Lemma 3.1 and the Gauss equation imply

$$\alpha_u + \alpha^2 = 0$$

which is equivalent to

$$\left(\frac{1}{\alpha(u)} \right)_u = 1.$$

By solving the above equation one can get

$$\alpha(u) = \frac{1}{u + a(v)}.$$

Therefore, (3.0.23) implies

$$\frac{\partial x_i}{\partial u} = \frac{1}{u + a(v)} x_i$$

whose solution is

$$x_i = (u + a(v))\eta_i(v). \tag{3.1.3}$$

By combining (3.1.3) and (3.0.21), and letting

$$\eta = (\eta_1, \eta_2, \eta_3, 0),$$

one can get (3.0.17). Hence, the proof of the necessary condition is completed. The proof of the sufficient condition is given in Proposition 3.1.

□

3.2 Surfaces with Flat Normal Bundle

Theorem 3.2. *Let M be a surface lying on $\mathscr{LC}^2 \times \mathbb{R}$. Then M has flat normal bundle if and only if it is locally congruent to the surface*

$$x(u,v) = \gamma(u)\eta(v) + uC_4, \tag{3.2.1}$$

where γ is a smooth function and η is an arc-length parametrized space-like curve such that $\langle \eta, \eta \rangle = \langle C_4, \eta \rangle = 0$.

Proof. Let M be a space-like surface in $\mathscr{LC}^2 \times \mathbb{R}$. Then, because of the Ricci equation,

$$K^{\perp} = 0,$$

if and only if

$$K^{\perp} = h(e_1, A_\phi e_2) - h(A_\phi e_1, e_2) = 0,$$

which is equivalent to

$$h(e_1, -e_2) - h(0, e_2) = e_2(\alpha)\phi = 0,$$

because of (3) of Lemma 3.1. Therefore, M has flat normal bundle if and only if

$$e_2(\alpha) = 0.$$

In order for the necessary condition to be proven it is assumed that $e_2(\alpha) = 0$ which implies

$$\alpha = \alpha(u).$$

Therefore, by considering Lemma 3.2, one can find

$$e_1(\gamma) = \alpha\gamma,$$

$$\frac{e_1(\gamma)}{\gamma} = \alpha(u),$$

which implies

$$\gamma = e^{\int \alpha(u)\, du}. \tag{3.2.2}$$

On the other hand, by combining Lemma 3.2 with (3.2.2), the equation

$$\frac{\partial x_i}{\partial u} = \alpha(u) x_i$$

is obtained.

By rewriting the differential equation to separate variables

$$\frac{1}{x_i}\frac{\partial x_i}{\partial u} = \alpha(u)$$

is obtained. Next, by integrated both sides with respect to $\{u\}$, one can get

$$\int \frac{1}{x_i}\frac{\partial x_i}{\partial u}\, du = \int \alpha(u)\, du,$$

$$\ln x_i = \int \alpha(u)\, du + C_i(v)$$

from which the equation

$$x_i = e^{C_i(v)} e^{\int \alpha(u)\, du}$$

is obtained. Using the equation (3.0.27), one can get

$$x_i = \gamma(u)\eta_i(v).$$

Hence, M can be parametrized as given in (3.2.1).

So, the proof of the necessary condition is completed. Conversely, assume that M is the surface in $\mathscr{LC}^2 \times \mathbb{R}$ parametrized by (3.2.1). Then, a direct computation implies

$$x_u = \gamma'(u)\eta(v) + C_4,$$

$$x_v = \gamma(u)\eta'(v).$$

From which it is obtained that

$$E = \langle x_u, x_u \rangle = \langle \gamma'(u)\eta(v) + C_4, \gamma'(u)\eta(v) + C_4 \rangle = 1,$$

$$F = \langle x_u, x_v \rangle = \langle \gamma'(u)\eta(v) + C_4, \gamma(u)\eta'(v) \rangle = 0,$$

$$G = \langle x_v, x_v \rangle = \langle \gamma(u)\eta'(v), \gamma(u)\eta'(v) \rangle = \gamma^2(u).$$

Therefore the vector fields e_1, e_2 given by

$$\begin{cases} e_1 = \dfrac{\partial}{\partial u} = \gamma'(u)\eta(v) + C_4, \\ e_2 = \dfrac{1}{\gamma(u)} \dfrac{\partial}{\partial v} = \eta'(v) \end{cases} \tag{3.2.3}$$

form an orthonormal base for the tangent bundle of M. By a direct computation, one can get

$$\tilde{\nabla}_{e_1} \phi = \gamma'(u)\eta(v) = \frac{\gamma'(u)}{\gamma(u)} \phi,$$

which implies

$$\alpha = \frac{\gamma'(u)}{\gamma(u)}.$$

Thus, α satisfies

$$e_2(\alpha) = 0.$$

Hence, M has flat normal bundle. □

3.3 Pseudo-Umbilical Surfaces

Lemma 3.3. *Let M be a surface lying on $\mathscr{LC}^2 \times \mathbb{R}$. M is pseudo-umbilical if and only if it is flat, marginally trapped and it has flat normal bundle.*

Proof. Let M be a surface in $\mathscr{LC}^2 \times \mathbb{R}$. As a consequence of (5) of Lemma 3.1, the mean curvature vector of M can be obtained as

$$H = \frac{1}{2}(e_1(\alpha) + \alpha^2 - \beta)\phi + \frac{1}{2}\xi. \tag{3.3.1}$$

Let M be a pseudo-umbilical surface. Then, there exist a smooth function f such that $\langle A_H X, Y\rangle = f\langle X, Y\rangle$ whenever X and Y are tangent to M. $X = e_1$ and $Y = e_2$ implies

$$\langle A_H e_1, e_2\rangle = 0, \tag{3.3.2}$$

while $X = Y = e_1$ and $X = Y = e_2$ give

$$\langle A_H e_1, e_1\rangle = \langle A_H e_2, e_2\rangle. \tag{3.3.3}$$

By a direct computation, the equation

$$A_H e_1 = \frac{e_1(\alpha) + \alpha^2 - \beta}{2} A_\phi e_1 + \frac{1}{2} A_\xi e_1,$$

from which one can get

$$A_H e_1 = \frac{-e_1(\alpha) - \alpha^2}{2} e_1 - \frac{e_2(\alpha)}{2} e_2.$$

Therefore, (3.3.1) implies

$$\langle A_H e_1, e_2\rangle = 0.$$

Thus, α satisfies

$$e_2(\alpha) = 0.$$

Hence, M has flat normal bundle. On the other hand, by a direct computation, the equation

$$A_H e_2 = \frac{e_1(\alpha) + \alpha^2 - \beta}{2} (A_\phi e_2) + \frac{1}{2} A_\xi e_2,$$
$$A_H e_2 = \frac{e_1(\alpha) + \alpha^2 - \beta}{2} (-e_2) + \frac{1}{2} \beta e_2.$$

Therefore, (3.3.2) implies

$$\frac{-e_1(\alpha) - \alpha^2}{2} = \frac{-e_1(\alpha) - \alpha^2 + 2\beta}{2}.$$

Thus, β satisfies

$$\beta = 0.$$

Now, consider the Codazzi Equations for $\beta = e_2(\alpha) = 0$. By a direct computation, one can obtain

$$\nabla^\perp_{e_1} h(e_2, e_2) - 2h(\nabla_{e_1} e_2, e_2) = 0 - \alpha h(e_2, e_2) + \alpha h(e_1, e_1)$$

which is obtained from (2) and (4) of Lemma 3.1. On the other hand, from (5) of Lemma 3.1, one can get

$$\nabla^{\perp}_{e_1}\xi = -\alpha\xi + \alpha(e_1(\alpha)+\alpha^2),$$

$$-\alpha\xi = -\alpha\xi + \alpha(e_1(\alpha)+\alpha^2).$$

Thus, α satisfies

$$\alpha = 0 \quad or \quad e_1(\alpha)+\alpha^2 = 0,$$

which implies

$$e_1(\alpha)+\alpha^2 = 0.$$

So M is flat. Furthermore, because of (3.3.1) one can get $H = \frac{1}{2}\xi$ which yields that M is marginally trapped. Conversely, assume that $K = K^{\perp} = 0$ and M is marginally trapped. In that case, $e_1(\alpha)+\alpha^2 = 0, \quad e_2(\alpha) = 0, \quad \beta = 0$. Therefore $H = \frac{1}{2}\xi$. So, $A_H = \frac{1}{2}A_\xi = 0$. Hence, the proof is completed. □

Theorem 3.3. *Let M be a surface lying on $\mathscr{LC}^2 \times \mathbb{R}$. Then M is pseudo-umbilical if and only if it can be parametrized by*

$$x(u,v) = (u+c)\eta(v) + uC_4, \tag{3.3.4}$$

$$\langle \eta,\eta\rangle = \langle C_4,\eta\rangle = 0.$$

Proof. By Lemma 3.3, M is pseudo-umbilical, then $K = K^{\perp} = 0$. Therefore, Theorem 3.1 implies that M is parametrized as given in (3.1.1). Furthermore, $K^{\perp} = 0$ implies $e_2(\alpha) = 0$. Let e_1, e_2 be tangent vector fields given by (3.1.2). By a direct computation, one can find

$$\tilde{\nabla}_{e_1}e_2 = h(e_1,e_2) = \frac{\partial}{\partial u}\left(\frac{a'}{u+a}\eta + \eta'\right) = \frac{-a'}{(u+a(v))^2}\eta$$

which implies $\alpha(u,v) = -\frac{a'(v)}{u+a(v)}$. Threfore, $e_2(\alpha) = 0$ implies $a'(v) = 0$ which yields $a(v) = c$ for a constant c. Consequently, (3.1.1) turns into (3.3.4). Converse follows from a direct computation. □

4. CONCLUSIONS AND RECOMMENDATIONS

This study provides a comprehensive analysis of the differential geometric properties of space-like surfaces in the $\mathscr{LC}^2 \times \mathbb{R}$ manifold. The fundamental structures of these surfaces, along with their curvature characteristics, have been investigated in detail.

It has been shown that surfaces with Gaussian curvature $K = 0$ in the $\mathscr{LC}^2 \times \mathbb{R}$ manifold can be locally parametrized as

$$x(u,v) = (u + a(v))\eta(v) + u(C_4), \tag{4.0.1}$$

where $\eta(v)$ is a light-like curve satisfying $\langle \eta, \eta \rangle = 0$. The geometric characterization of these flat surfaces is discussed, and the fundamental metric coefficients E, F, G are computed to confirm that the Gaussian curvature is zero.

The tangent and normal bundles of space-like surfaces have been analyzed. The tangent bundle e_1, e_2 forms an orthonormal basis, while the normal bundle ϕ, ξ forms a pseudo-orthonormal basis. Specifically, ϕ is defined as a light-like vector field, and ξ is orthogonal to ϕ, satisfying $\langle \phi, \xi \rangle = -1$.

The shape operators A_ϕ and A_ξ play a significant role in the geometric characterization of the surfaces. Using these operators, the expressions for the second fundamental form are derived, and the derivatives of the normal connection $\nabla^{\perp}_{e_i}$ are analyzed to describe the geometric behavior of ϕ and ξ.

Fundamental curvature quantities such as the Gaussian curvature K and the mean curvature H are related to the eigenvalues (λ_1, λ_2) of the shape operator. Classification problems for flat surfaces and flat normal bundle surfaces are resolved in Theorem 3.1 and Theorem 3.2, respectively.

These results provide a robust foundation for understanding the geometric and analytical properties of space-like surfaces in the $\mathscr{LC}^2 \times \mathbb{R}$ manifold. The findings serve as a fundamental framework for the differential geometry of light-like surfaces in Lorentz manifolds. Moreover, the analysis of light-cone structures highlights potential directions for future research, especially in contexts where such structures play significant roles in physical applications.

In the future, one can consider generalizing these results to n-dimensional submanifolds of the hypercylinder $\mathscr{LC}^n \times \mathbb{R}$ of $\mathbb{E}^{n+2}_1$. Flat and pseudo-umbilical submanifolds can be studied.

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